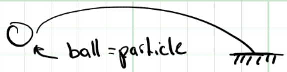
Chapter 2 – Kinematics of Particles

- motion of body without reference to the force causing motion
- position , s
  velocity , v
  acceleration , a
  time , t
  } particles and rigid bodies

→ <u>Kinetics</u>

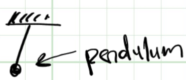
- study of unbalanced forces applied on a body
- forces , moments , torques , masses

→ Particle: no regard to dimensions



* dimensions are infinitely small
* entire mass is concentrated at a single point

→ Rigid body: pay regard to dimensions



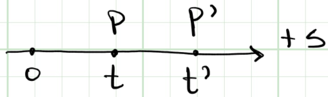
* every point on a pendulum has a different velocity and acceleration
* cannot neglect dimensions
* rigid body is assumed

Section 2.2 – Rectilinear Motion

* Rectilinear Motion : motion along a straight line
* For any kind of motion

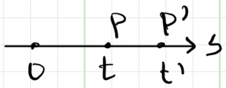
  1) Axis
  2) origin
  3) Direction

  } we need these



* For rectilinear motion, we also need to identify ...

(I) **Position**: vector from origin to particle



displacement $= s' - s = \Delta s$

(II) **Velocity** (v)

Average velocity $= \vec{v}_{av} = \dfrac{\Delta s}{\Delta t} = [m/s]$ $+, -$

Instantaneous velocity $= \boxed{\vec{v} = \dfrac{ds}{dt}}$ ← instant of time

(III) **Acceleration** (a)

Average acceleration $= \vec{a}_{av} = \dfrac{\Delta v}{\Delta t}$
- $\oplus$ ... $\vec{v}\uparrow$ speeding
- $\ominus$ ... $\vec{v}\downarrow$ slowing

Instantaneous acceleration $= \boxed{\vec{a} = \dfrac{dv}{dt} = \dot{v}}$ ← v-dot

→ Three Differential Equations

$$\boxed{v = \frac{ds}{dt}}^{1} \longrightarrow dt = \frac{ds}{v}$$

$$\boxed{a = \frac{dv}{dt}}^{2} \longrightarrow dt = \frac{dv}{a}$$

$$dt = dt$$

$$\frac{ds}{v} = \frac{dv}{a}$$

$$\boxed{a\,ds = v\,dv}^{3}$$

Kinematic Equations

* Consider 3 situations for acceleration

a
- $a(t)$ ~ cond. (I)
- $a(s)$ ~ cond. (II)
- $a(v)$ ~ cond. (III)

(I) —— $a = a(t)$

$a = \frac{dv}{dt}$ 1 $\longrightarrow a(t) = \frac{dv}{dt}$ 2 $\longrightarrow a(t)\,dt = dv$ 3 $\longrightarrow$ integrate

4 $\int_0^t a(t)\,dt = \int_{v_0}^{v} dv$

5 $\int_0^t a(t)\,dt = v - v_0$

6 $v(t) = v_0 + \int_0^t a(t)\,dt$

assuming a is constant

7 $\boxed{V(t) = V_0 + at}$ 1rst equation of motion

$v(t) = \frac{ds}{dt}$ 1 $\longrightarrow v(t)\,dt = ds$ 2 $\longrightarrow$ integrate

3 $\int_0^t \boxed{v(t)}\,dt = \int_{s_0}^{s} ds$

4 $\int_0^t \boxed{\left[\underbrace{v_0}_{\text{constant}} + \int_0^t a(t)\,dt\right]}\,dt = \int_{s_0}^{s} dx$

5 $v_0 + \int_0^t \left[\int_0^t a(t)\,dt\right] dt = s - s_0$

assuming a is constant

2nd eq of motion

$\boxed{s = s_0 + v_0 t + \frac{1}{2} a t^2}$

(II) —— $a = a(s)$

$a\,ds = v\,ds$ 1 $\longrightarrow a(s)\,ds = v\,dv$ 2 $\longrightarrow$ integrate

3 $\int_{s_0}^{s} a(s)\,ds = \int_{v_0}^{v} v\,dv$

4 $\int_{s_0}^{s} a(s)\,ds = \frac{v^2}{2} - \frac{v_0^2}{2}$

5 $v^2 = v_0^2 + 2\int_{s_0}^{s} a(s)\,ds$

3rd eq of motion

$\boxed{v(s) = v^2 = v_0^2 + 2a(s - s)}$

(III) —— $a = a(v)$

$$a(v) = \frac{dv}{dt} \rightarrow dt = \frac{dv}{a(v)} \xrightarrow{\text{integrate}} \int_0^t dt = \int_{v_0}^{v} \frac{dv}{a(v)}$$

$$\int_{v_0}^{v} \frac{dv}{a(t)} = t \xrightarrow[\text{const } a]{} \boxed{v = v_0 + at}$$ (same as eq.1)

Example 1: A freighter is moving at 4 m/s, Engine shut off. It decelerates at $a = -kv^2$. It takes 10 mins to reduce to 2m/s.

a) find $v(t)$
b) find distance travelled in those 10 mins.

Solution

$a = a(v)$

$$a = -kv^2 = \frac{dv}{dt}$$

$$-k\,dt = \frac{dv}{dv^2}$$

$$-k\int_0^t dt = \int_4^v \frac{dv}{v^2}$$

$$-kt = -\frac{1}{v}\Big|_4^v$$

$$-kt = -\frac{1}{v} + \frac{1}{4}$$

at $t = 10$ mins
$v = 2$ m/s

$$-k(600) = \frac{1}{(2)} + \frac{1}{4}$$

$$\therefore k = \frac{1}{2400}$$ ← plug this

$$\boxed{a)\ v = \frac{1}{\frac{1}{2400}t + \frac{1}{4}}}$$

We know that $v = \frac{ds}{dt} \rightarrow v\,dt = ds$, integrate velocity function

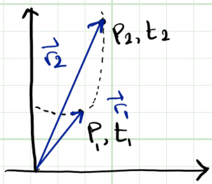
$$\int_0^{600} \frac{1}{\frac{1}{2400}t + \frac{1}{4}}\,dt = \int_0^s ds$$

$$2400\int_0^{600} \frac{dt}{t+600} = s$$

$$s = 2400\ln(t+600)\Big|_0^{600} = 1663 \text{ m}$$

$$\boxed{\text{b) distance} = 1663\text{m}}$$

Section 2.3 – Curvilinear Motion



displacement $= \Delta r = \vec{r}_2 - \vec{r}_1$

velocity $= \dfrac{d\vec{r}}{dt} = \dot{\vec{r}} = \vec{v}$ ← direction: always tangent to the displacement.

speed $= v = |\vec{v}| = \dfrac{dr}{dt}$

* velocity is a vector and always tangent to the path of motion.
* position: vector that starts from origin to the particle.
* acceleration: cannot predict direction

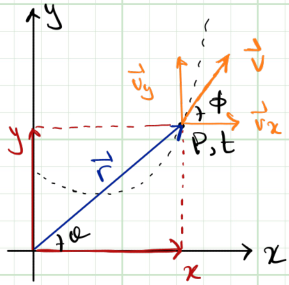
→ Things to remember

(i) If it was a position vector, it is simply a line from the origin to the particle

(ii) If it was a velocity vector, it would be tangent lines along the path of motion

(iii) For acceleration vectors, we don't know the direction.

Section 2.4 – 2D Rectangular Coordinates



x-direction → unit vector $\hat{i}$

y-direction → unit vector $\hat{j}$

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{x}\hat{i} + \dot{y}\hat{j}$$

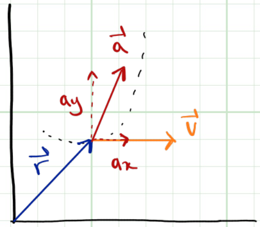
$$= v_x\hat{i} + v_y\hat{j}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\phi = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

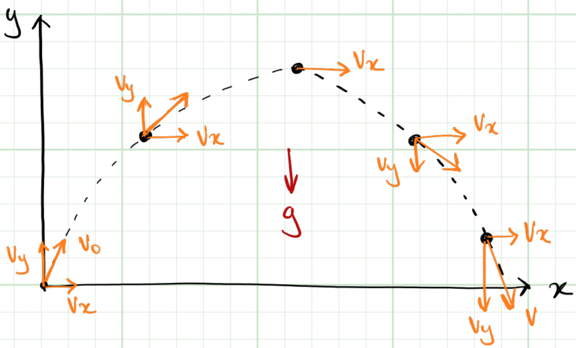
$$|\vec{v}| = \sqrt{v_x^2 + v_y^2}$$

Week 0 : Lec 3



$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} = a_x\hat{i} + a_y\hat{j}$$

Projectile Motion



x-direction : $\vec{a_x} = 0 \longrightarrow$ No force acting along x

$\Rightarrow V_x = \text{constant} = V_0\cos\theta$

$\Rightarrow x = x_0 + (V_0)_x t + \cancel{\frac{1}{2}a_x t^2}^{\,0}$ } distance travelled in x

$x = x_0 + (V_0\cos\theta)t$

y-direction :

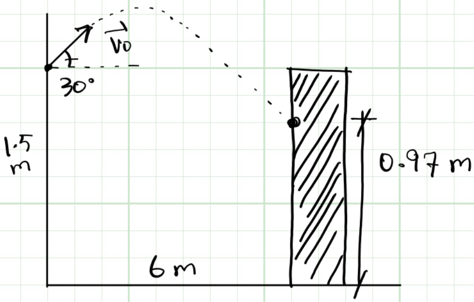
$\Rightarrow y = y_0 + (V_0)_y t - \frac{1}{2}g t^2$ } distance travelled in y

$\Rightarrow \vec{V_y} = (\vec{V_0})_y + \vec{a_y}\,t \longrightarrow \vec{V_y} = V_0\sin\theta - gt$ } V_y at given instant in time

$\Rightarrow V_y^2 = (V_0)_y^2 + 2a_y(y - y_0)$

$V_y^2 = (V_0\sin\theta)^2 - 2g(y - y_0)$ } y-veloty w.r.t. y and acceleration.

Example 1: Find $\vec{v}_0$



x-direction

$$x = x_0 + (v_0)_x t \rightarrow \text{eq. 1}$$

$$6 = 0 + (v_0 \cos 30)\, t$$

$$t = \frac{6}{v_0 \cos 30^\circ}$$

y-direction

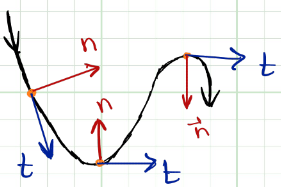
$$y = y_0 + (v_0)_y t - \frac{1}{2} g t^2$$

$$0.97 = 1.5 + (v_0 \sin 30^\circ)\, t - 4.9\, t^2$$

$$|v_0| = 14.5 \text{ m/s}$$

$$\vec{v}_0 = (14.5 \cos 30^\circ)\hat{i} + (14.5 \sin 30^\circ)\hat{j}$$

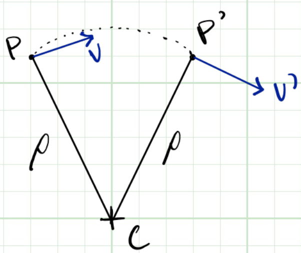
Section 2.5 — Normal & Tangential Coordinates



t: always tangent to the path

(+)ve : direction of motion

unit vector: $\hat{e}_t$ direction not constant



n: always ⊥ to tangential

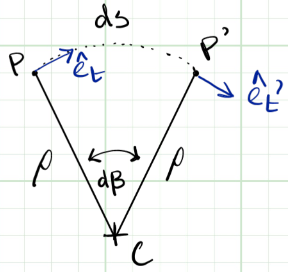
(+)ve : towards centre of curvature

unit vector : $\hat{e}_n$ direction not constant

no position vector since no origin

Week 1 : Lec 1

Equations For T & N Coordinates



Velocity

$$v = \frac{ds}{dt} \leftarrow ds = \rho\, d\beta$$

$$\Rightarrow v = \frac{\rho\, d\beta}{dt} = \rho\dot{\beta}$$

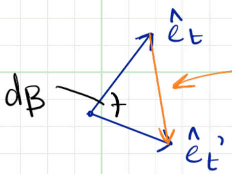
no normal component to velocity

$$\Rightarrow v = (\rho\dot{\beta})\,\hat{e}_t + (0)\,\hat{e}_n$$

Acceleration

$\vec{v} = v\hat{e}_t$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v\hat{e}_t) = \dot{v}\hat{e}_t + v\dot{\hat{e}}_t$$



Magnitude

$$d\hat{e}_t \longrightarrow |d\hat{e}_t| = \underbrace{|\hat{e}_t|}_{=1}\,d\beta = d\beta$$

$$d\hat{e}_t = d\beta\,\hat{e}_n \leftarrow \text{direction}$$

$$\frac{d\hat{e}_t}{dt} = \frac{d\beta}{dt}\,\hat{e}_n = \dot{\beta}\,\hat{e}_n$$

$$\therefore\ \vec{a} = \dot{v}\hat{e}_t + v\dot{\beta}\hat{e}_n$$

Since $v = \rho\dot{\beta} \rightarrow \dot{\beta} = \dfrac{v}{\rho}$

$$\Rightarrow \vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

Summary

$$\vec{v} = r\dot{\theta}\hat{e}_t$$

$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{r}\hat{e}_n$$

Special Case: Circular Motion

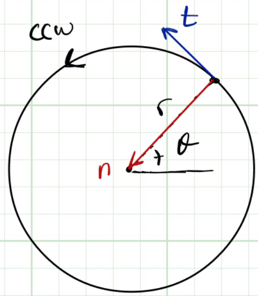
$r = \text{constant}$

$\dot{r} = \ddot{r} = 0$

$\vec{v} = r\dot{\theta}\hat{e}_t$

$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{r}\hat{e}_n$

$$\underbrace{\boxed{\vec{a} = r\ddot{\theta}\hat{e}_t + \frac{v^2}{r}\hat{e}_n}}_{\text{one way of showing } \vec{a}}$$



$\dot{v} = r\ddot{\theta} + \cancel{\dot{r}}^{\,0}\theta$ since r is constant

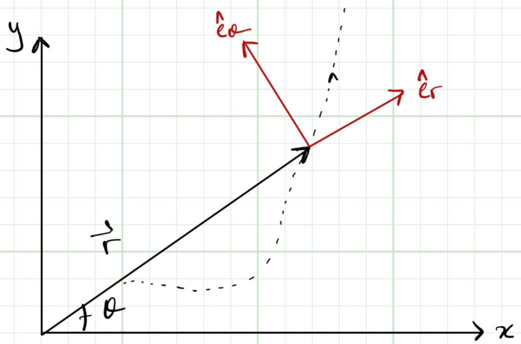
$\dot{v} = r\ddot{\theta}$

But we also know

$v = r\dot{\theta} \longrightarrow \frac{v^2}{r} = \frac{r^{\cancel{2}}\dot{\theta}^2}{\cancel{r}} = r\dot{\theta}^2$

$$\boxed{\vec{a} = r\ddot{\theta}\hat{e}_t + r\dot{\theta}^2\hat{e}_n}$$ another way to show $\vec{a}$

Section 2.6 – Polar Coordinates ($R-\theta$)



$\hat{e}_r$: from particle away from origin parallel to $\vec{r}$

$\hat{e}_\theta$: $\perp$ to $\hat{e}_r$

+ve direction of rotation

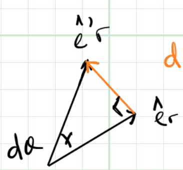
$+\theta$ $\curvearrowleft$ $\longrightarrow +x$

→ Position : $\vec{r} = r\hat{e}_r$

→ Velocity : $\vec{v} = \frac{d\vec{r}}{dt}$

$= \dot{r}\hat{e}_r + r\dot{\hat{e}}_r$

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$



$|d\hat{e}_r| = \underbrace{|\hat{e}_r|}_{=1} d\theta = d\theta$

$d\hat{e}_r = \underbrace{d\theta}_{\text{magnitude}}\ \underbrace{\hat{e}_\theta}_{\text{direction}}$

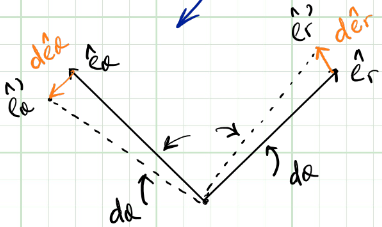
$\dot{\hat{e}}_r = \frac{d\hat{e}_r}{dt} = \frac{d\theta\,\hat{e}_\theta}{dt} = \dot{\theta}\hat{e}_\theta$

$$\vec{v}\begin{cases} v_r = \dot{r} \sim \text{time rate of change of position along radial direction} \\ v_\theta = r\dot{\theta} \sim \text{time rate of change of position along the curvature at a given } r \end{cases}$$

→ Acceleration : take derivative of $\vec{v}$

$$\vec{a} = \frac{d\vec{v}}{dt} = \underset{\checkmark}{\ddot{r}\hat{e}_r} + \underset{?}{\dot{r}\dot{\hat{e}}_r} + \underset{\checkmark}{\dot{r}\dot{\theta}\hat{e}_\theta} + \underset{\checkmark}{r\ddot{\theta}\hat{e}_\theta} + \underset{?}{r\dot{\theta}\dot{\hat{e}}_\theta}$$

$$\dot{\hat{e}}_r = \frac{d\hat{e}_r}{dt} = \boxed{\dot{\theta}\hat{e}_\theta}$$



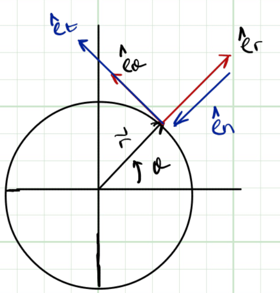
$$|d\hat{e}_\theta| = |\hat{e}_\theta|\,d\theta = d\theta$$

$$d\hat{e}_\theta = \underbrace{d\theta}_{\text{magnitude}}\underbrace{(-\hat{e}_r)}_{\text{direction}}$$

$$\therefore \dot{\hat{e}}_\theta = \frac{d\hat{e}_\theta}{dt} = \boxed{-\dot{\theta}\hat{e}_r}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

Circular Motion



$r = \text{const}$

$\dot{r} = \ddot{r} = 0$

Convention

$\hat{e}_r = -\hat{e}_n$

$\hat{e}_\theta = \hat{e}_t$

$$\vec{v} = \cancelto{0}{\dot{r}\hat{e}_r} + r\dot{\theta}\hat{e}_\theta = r\dot{\theta}\hat{e}_\theta \; (\hat{e}_t)$$

$$\vec{v} = v\hat{e}_t = r\dot{\theta}\hat{e}_\theta$$

$$\vec{a} = (\cancelto{0}{\ddot{r}} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\cancelto{0}{\dot{r}}\dot{\theta})\hat{e}_\theta$$

$$\vec{a} = -r\dot{\theta}^2\hat{e}_r + r\ddot{\theta}\hat{e}_\theta$$

$$\vec{a} = \frac{v^2}{r}\hat{e}_n + \dot{v}\hat{e}_t$$

Week 1: Lec 3

Normal and Tangential coordinates

If you know the path of motion as $f(x,y)$

$$\Rightarrow \boxed{\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left[d^2y/d^2x\right]}}$$

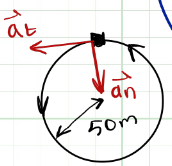
radius of curvature

2nd derivative

path: $y = 3x^2 + 2x + 7$

$$\Rightarrow \rho = \frac{\left[1 + (6x+2)^2\right]^{3/2}}{6}$$

Example: A car on a circular track with a speed of 20 m/s. It slows down with a rate of $0.7\ m/s^2$. Find $\vec{a}$ after 8 s.



$|\vec{v}| = 20\ m/s$

$\dot{v} = -0.7\ m/s^2$

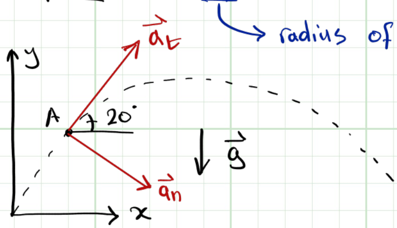
$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{r}\hat{e}_n$$

t-direction: $V = V_0 + a_t t = 20 - 0.7(8) = 14.4\ m/s$

$$\left.\begin{aligned} a_t &= \dot{v} = -0.7\ m/s^2 \\ a_n &= \frac{v^2}{r} = \frac{14.4^2}{50} = 4.15\ m/s^2 \end{aligned}\right\} |\vec{a}| = \sqrt{0.7^2 + 4.15^2} = \boxed{4.2\ m/s^2}$$

Example: Find ρ of the rock projectile at A if $V_A = 50\ m/s$

radius of curvature, use T & N coordinates



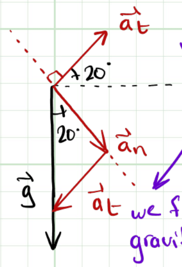
radius of curvature

$\vec{a} = \vec{g}$

we know a_n and a_t must add up $\vec{g}$.

$$a_n = \frac{v^2}{\rho}$$

$$a_t = \dot{v}$$



translate $\vec{a}_t$ down

we flip $\vec{a}_t$ because gravity is going down

$a_n = 9.81\cos 20 = 9.22\ m/s^2$

$$\Rightarrow \rho = \frac{v^2}{a_n} = \frac{50^2}{9.22} = \boxed{271\ m}$$

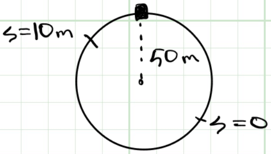
$|a_t| = 9.81\sin 20 = 3.355\ m/s^2$

$\therefore a_t = -3.36\ m/s^2$

$\boxed{\therefore \rho \text{ is } 271\ m}$

Example: Car starting from $s=0$, $v_0 = 4$ m/s. If $\dot{v} = (0.05)s$ m/s^2

find $\vec{v}, \vec{a}$ at $s = 10$ m   we only care about a_t



$$a_t = \dot{v} = a(s) = 0.05s$$

$$a_t ds = v dv \Rightarrow \int_0^{10} a_t ds = \int_4^v v dv$$

$$\int_0^{10} 0.05 s ds = \int_4^v v dv \qquad \therefore \boxed{\vec{v} = 4.58 \hat{e}_t \text{ m/s}}$$

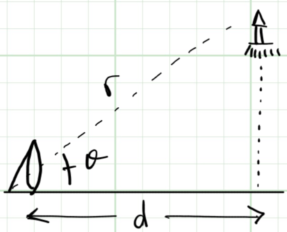
$$\frac{0.05}{2} s^2 \Big|_0^{10} = \frac{v^2}{2} \Big|_4^v \Rightarrow v = 4.58 \text{ m/s}$$

$$a_t = \dot{v} = 0.05s = 0.05(10) = 0.5 \text{ m/s}^2$$

$$a_n = \frac{v^2}{r} = \frac{4.58^2}{50 \text{ m}} = 0.42 \text{ m/s}^2$$

$$\boxed{\begin{aligned} \vec{a} &= 0.5\hat{e}_t + 0.42\hat{e}_n \\ |\vec{a}| &= 0.65 \text{ m/s} \end{aligned}}$$

Example: Fired rocket tracked by radar. At $\theta = 60°$, find $\vec{a}$.



$$\ddot{r} = 21 \text{ m/s}^2$$

$$\dot{\theta} = 0.02 \text{ rad/s}$$

$$r = 9 \text{ km}$$

$$\vec{a} = (\underset{\checkmark}{\ddot{r}} - \underset{\checkmark}{r}\underset{\checkmark}{\dot{\theta}^2})\hat{e}_r + (\underset{\checkmark}{r}\underset{?}{\ddot{\theta}} + 2\underset{?}{\dot{r}}\underset{\checkmark}{\dot{\theta}})\hat{e}_n$$

$$r\cos\theta = d \Rightarrow 9000\cos 60 = 4500 \text{m}$$

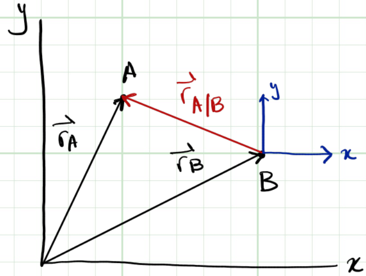
$$\boxed{r\cos\theta = 4500}$$

$$\frac{d}{dt} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \Rightarrow \dot{r} = 320 \text{ m/s}$$

$$\frac{d^2}{d^2t} = \ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta = 0$$

$$\ddot{\theta} =$$

Section 2.8 – Relative Motion



$\vec{r}_{A/B}$: standing at B, looking at A

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

$$\boxed{\vec{r}_A = \vec{r}_{A/B} + \vec{r}_B}$$

$\frac{d}{dt}$

$$\boxed{\vec{v}_A = \vec{v}_{A/B} + \vec{v}_B}$$

$\frac{d}{dt}$

$$\boxed{\vec{a}_A = \vec{a}_{A/B} + \vec{a}_B}$$

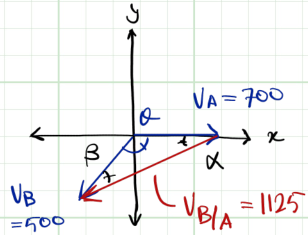
$\vec{r}_{B/A}$: standing at A, looking at B

→ Eg 1: 2 airplanes, A and B

$\vec{v}_A = 700$ km/h → East

$\vec{v}_B = 500$ km/h → southwest (angle ?)

$\vec{v}_{B/A} = 1125$ km/h → Find direction

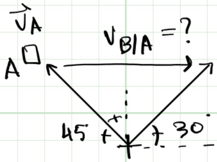


$$|\vec{v}_{B/A}|^2 = |\vec{v}_A|^2 + |\vec{v}_B|^2 - 2|\vec{v}_A||\vec{v}_B|\cos\theta$$

$$1125^2 = 700^2 + 500^2 - 2(700)(500)\cos\theta \quad , \quad 90 < \theta < 180$$

$$\therefore \theta = 139°$$

→ Eg 2: Find velocity of car B w.r.t. A, $v_A = 5$ m/s, $v_B = 3$ m/s



$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} \Rightarrow \vec{v}_{B/A} = \vec{v}_B - \vec{v}_A$$

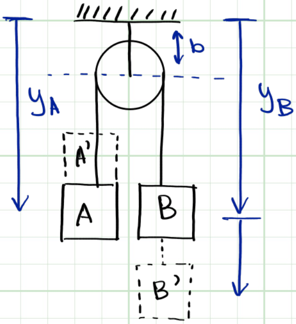
$\vec{v}_{Ax} = -5\cos 45°$ $\qquad$ $\vec{v}_{Bx} = 3\cos 30°$

$\vec{v}_{Ay} = 5\cos 45°$ $\qquad$ $\vec{v}_{By} = 3\sin 30°$

$$\vec{v}_{(B/A)x} = \vec{v}_{Bx} - \vec{v}_{Ax} \qquad |\vec{v}_{B/A}| = \sqrt{v_{(B/A)x} + v_{(B/A)y}}$$

$$\vec{v}_{(B/A)y} = \vec{v}_{By} - \vec{v}_{Ay} \qquad \theta = \tan^{-1}\left(\frac{v_{(B/A)y}}{v_{(B/A)x}}\right)$$

Section 2.9 – Constrained Motion



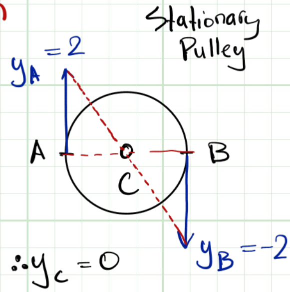
$y_A = -y_B$

$\downarrow \frac{d}{dt}$

$v_A = -v_B$

$\downarrow \frac{d}{dt}$

$a_A = -a_B$

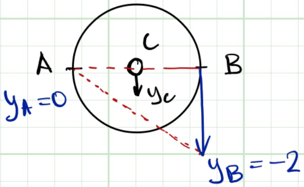


$\therefore y_c = 0$

* Method 1 : displacement of the centre is average dispacement of the 2 ends. $\Big\}$ $\boxed{y_c = \dfrac{y_A + y_B}{2}}$

$$y_c = \frac{y_A + y_B}{2} = \frac{2 + -2}{2} = \frac{0}{2} = 0$$

<u>Non-stationary Pulley</u>



$$y_c = \frac{y_A + y_B}{2} = \frac{0 - 2}{2} = -1$$

<u>Method 2</u>

L = constant length of the rope

$$L = y_A - \overset{\text{const}}{b} + y_B - \overset{\text{const}}{b} + \overset{\text{const}}{\pi r}$$

$$L = y_A + y_B + \text{constant}$$

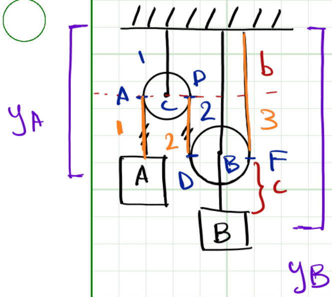
$$\dot{L}^{\,0} = \dot{y}_A + \dot{y}_B \longrightarrow 0 = v_A + v_B$$

$v_A = -v_B$

$a_A = -a_B$

Week 3: Lec 2

→ **Eg 1 :** Find a_B in terms of a_A



Method 1

0 since pulley 1 is stationary

Pulley 1: $\cancel{v_C} = \dfrac{v_A + v_P}{2} \implies v_A = -v_P$

Pulley 2: $v_B = \dfrac{v_P + \cancel{v_F}}{2}$ 0 because point F is connected to the top, which is stationary

$$v_B = \frac{-v_A}{2}$$

$$\boxed{a_B = \frac{-a_A}{2}}$$

Method 2

$$L = \underbrace{y_A - b}_{1} + \underbrace{y_B - c - b}_{2} + \underbrace{y_B - c}_{3}$$

$$L = y_A + y_B + y_B + \text{con}$$

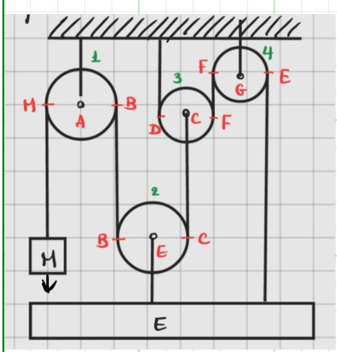
$$= y_A + 2y_B + \text{constant}$$

$$\dot{L} = v_A + 2v_B + 0$$

$$v_B = -\frac{v_A}{2}$$

$$\boxed{a_B = -\frac{a_A}{2}}$$

→ **Eg 2:** $\vec{v}_M = -2$ m/s , find $\vec{v}_E = ?$



Pulley 1: $\cancel{v_A} = \dfrac{v_H + v_B}{2} = \dfrac{v_M + v_B}{2}$

$$v_B = -v_M = -(-2)$$

$$\underline{\underline{v_B = 2 \text{ m/s}}}$$

Pulley 2: $v_E = \dfrac{v_B + v_C}{2}$

$$2v_E = v_B + v_C$$

$$2v_E = 2 + v_C$$

$$\underline{\underline{v_C = 2v_E - 2}}$$

Pulley 3: $V_c = \dfrac{\cancel{V_D}^{\,0} + V_F}{2}$

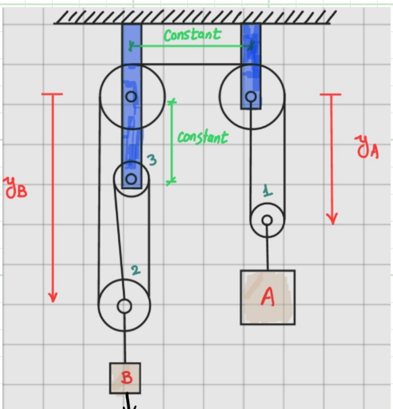
$$2V_c = V_F$$
$$2(2V_E - 2) = V_F$$
$$\underline{\underline{V_F = 4V_E - 4}}$$

Pulley 4: $\cancel{V_G}^{\,0} = \dfrac{V_F + V_E}{2}$

$$\underline{\underline{V_F = -V_E}}$$

$$-V_E = 4V_E - 4$$
$$5V_E = 4$$
$$\boxed{V_E = 0.8 \text{ m/s}}$$

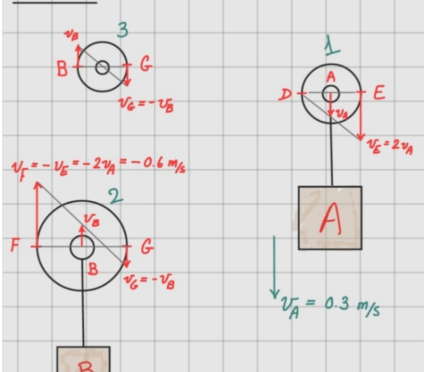
Eg 3: Mass A has downward velocity of 0.3 m/s. Find the velocity of B.



Method 2

$$L = y_B + y_B + y_B + y_A + y_A + C$$
$$L = 3y_B + 2y_A + C$$
$$\dot{L} = 3V_B + 2V_A$$
$$\boxed{\therefore V_B = -0.2 \text{ m/s}}$$
$$0 = 3V_B + 2V_A$$
$$V_B = \frac{-2V_A}{3} = -\frac{2}{3}(0.3) = -0.2$$

Method 1 :



* Pulley 1 : $v_A = \dfrac{\cancel{v_D}^{\,0} + v_E}{2}$ (0.3)

$\Rightarrow v_E = 0.6$ m/s

* Pulley 2 : $v_E = -v_F$

→ Connected through a rope, equal in magnitud opposite in direction

$\Rightarrow v_B = -v_G$

$\Rightarrow v_B = \dfrac{v_F + v_G}{2}$ (-0.6, $-v_B$)

$\Rightarrow 3v_B = -0.6 \frac{m}{s}$ $\Rightarrow$ $\boxed{v_B = -0.2 \text{ m/s}}$

Chapter 3 – Kinetics of Particles

Kinetics : study of unbalanced forces and their resulting changes

Newton's 2nd Law of Motion

- a particle will accelerate if there is a net force applied on it.
- the acceleration is in the same direction as the unbalanced force.

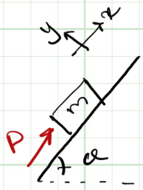
$$\sum \vec{F} = m\vec{a}$$

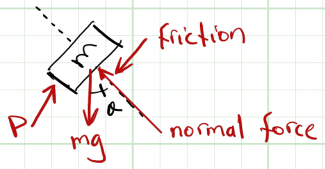
$$\sum \vec{F_x} = m\vec{a_x} \qquad \sum \vec{F_n} = m\vec{a_n} \qquad \sum \vec{F} = m\vec{a_\alpha}$$

$$\sum \vec{F_y} = m\vec{a_y} \qquad \sum \vec{F_t} = m\vec{a_t}$$

Section 3.4 — Rectilinear Motion

1. Draw an FBD
2. Choose a coordinate system (assign (+)ve direction)
3. Draw all the forces
4. Show the direction of each force (for unknown forces, assume positive direction)





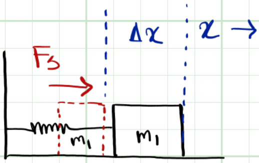
3 Types of Common Forces

(I) Gravitational Force :

$g = 9.81\ m/s^2$

$$F_g = \frac{G m_1 m_2}{r^2} = \frac{G M_{earth}}{R_{earth}^2} m_2 = g m_2 \qquad \boxed{\therefore \vec{F_g} = m\vec{g}}$$

(II) Spring Forces



k = spring constant [N/m]

l_0 = relaxed length of spring

$$\boxed{F_s = -kx} \longleftarrow \text{Hookes Law}$$

(III) Friction Force $\mu_s > \mu_k$

a) Static Friction

No slippage : $|\vec{F}_{static}| \leq \mu_s N$

coefficient of static friction

normal force

$$\underbrace{(\vec{F}_{static})_{max} = \mu_s N}_{\text{slippage is about to occur}}$$

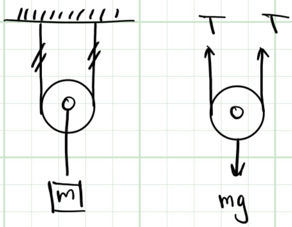
$\sum F = 0$ only for static friction

b) Kinetic Friction

Motion is initiated

$$|\vec{F}_{kinetic}| = \mu_k N$$

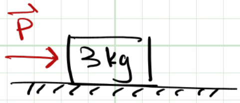
Pulleys

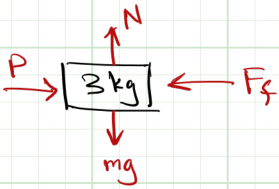


$2T = mg$

$T = \frac{mg}{2}$

Example : $\vec{P}$ increasing from 0, $\mu_s = 0.3$, $\mu_k = 0.25$





$\sum F_y = 0 = N - mg$

$N = mg$

$= 3(9.81)$

$\sum F_x = 0 = P - F_f$

$P = F_f = \mu_s N$

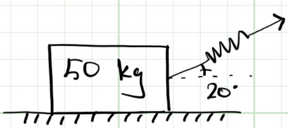
$= 8.83 \text{ N}$

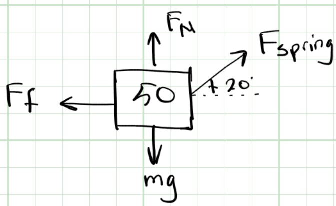
$\therefore$ for $P \geq 8.83 \longrightarrow$ slippage occurs

$\sum F_x = ma_x \longrightarrow \underset{8.83}{P} - \underbrace{F_{f\,kinetic}}_{\mu_k N} = ma_x \Rightarrow \boxed{a_x = 0.49 \text{ m/s}^2}$

Section 3.2 — 3.4

$\longrightarrow$ Eg 1 $\mu_s = 0.4$, $\mu_k = 0.25$, relaxed spring length $= 3$m, sliding happens when spring reached length of 5 m. Find k





$\sum F_y = F_N - mg + F_{sp}\sin\theta$

$F_N = mg - F_{sp}\sin\theta$

$\sum F_x = 0 = F_{sp}\cos\theta - F_f$

$\mu_s F_N = F_{sp}\cos\theta$

$$\mu_s F_N = F_{sp}\cos\theta$$

$$\mu_s (mg - F_{sp}\sin\theta) = F_{sp}\cos\theta$$

$$\mu_s mg - \mu_s F_{sp}\sin\theta = F_{sp}\cos\theta$$

$$F_{sp}\cos\theta + \mu_s F_{sp}\sin\theta = \mu_s mg$$

$$F_{sp}(\cos\theta + \mu_s\sin\theta) = \mu_s mg$$

$$F_{sp} = \frac{\mu_s mg}{\cos\theta + \mu_s\sin\theta} = \frac{(0.4)(50)(9.81)}{\cos 20^\circ + 0.4\sin 20^\circ} = 182.257\ N$$

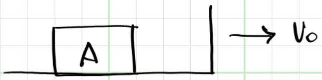
$$F_{sp} = k\,\Delta x$$

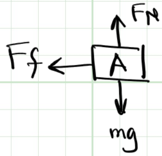
$$k = \frac{F_{sp}}{\Delta x} = \frac{182.257\ N}{5-3\ m}$$

$$\boxed{\therefore k = 91.13\ N/m}$$

→ Eg 2: truck driving, driver hits brakes, find min stopping distance and time for box not sliding forward?

$\mu_s = 0.35$, $V_0 = 100\ km/h = 27.78\ m/s$





$$\sum F_x = ma = -F_f$$

$$\cancel{m}a = -\mu_s \cancel{m}g$$

$$a = -\mu_s g$$

$$V_f = V_0 + at$$

$$t = \frac{V_f - V_0}{a} = \frac{-V_0}{a}$$

$$t = \frac{-27.78}{-0.35\times 9.81} = 8.1\ s$$

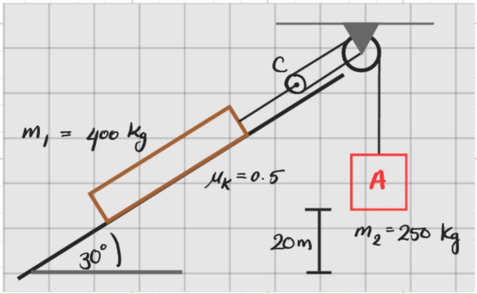
$$V_f^2 = V_0^2 + 2a\Delta S$$

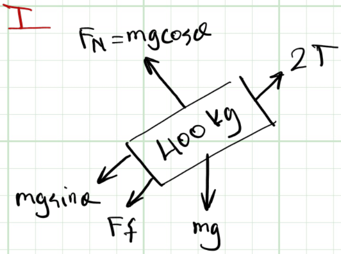
$$\Delta S = \frac{V_f^2 - V_0^2}{2a} = \frac{-(V_0)^2}{2a}$$

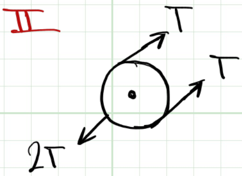
$$\Delta S = \frac{-(27.78)^2}{2(-0.35\times 9.81)} = 112.38\ m$$

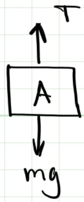
$$\boxed{\begin{aligned}\therefore\ \Delta S &= 112.4\ m\\ t &= 8.1\ s\end{aligned}}$$

Eg 3: Find velocity of mass A as it hits the ground.









III

$$\sum F_y =$$

$$T = (250)(9.81)$$

$$T = 2452.5$$

$$\sum F_x = ma$$

$$ma = 2T - mg\sin\theta - \mu_k F_N$$

$$ma = 2T - mg\sin\theta - \mu_k mg\cos\theta$$

$$400a = 2T - mg(\sin\theta + \mu_k\cos\theta)$$

Section 3.5 — Curvilinear Motion

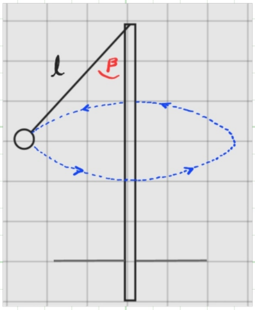
$$x\text{-}y \begin{cases} \sum F_x = ma_x = m\ddot{x} \\ \sum F_y = ma_y = m\ddot{y} \end{cases}$$

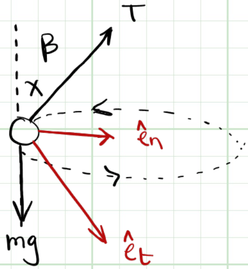
$$r\text{-}\theta \begin{cases} \sum F_r = ma_r = m(\ddot{r} - r\dot{\theta}^2) \\ \sum F_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \end{cases}$$

$$n\text{-}t \begin{cases} \sum F_n = ma_n = m\dfrac{v^2}{r} \\ \sum F_t = ma_t = m\dot{v} \end{cases}$$

Eg 1: Ball whirls around pole w/ constant v. Find β and T?

$v = 4\,m/s$, $\ell = 1.8\,m$, $m = 0.45\,kg$





$$\sum F_n = m a_n$$

$$T \sin\beta = m \frac{v^2}{r} \quad \text{— eq 2}$$

$$r \hookrightarrow \ell \sin\beta$$

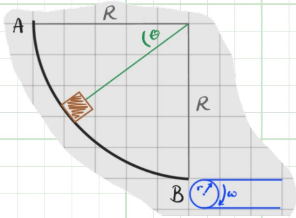
$$\sum F_{vertical} = 0$$

$$T \cos\beta - mg = 0 \quad \text{— eq 1}$$

eq 1 — $T \cos\beta = mg$

eq 2 —

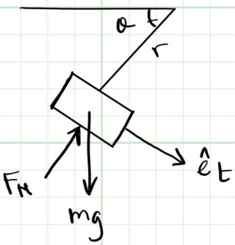
Eg 2 : Small object released from rest at A, sliding down the smooth circular surface of radius R to conveyor B.



normal force

a) Expression for N in terms of θ ?

b) correct ω for no slipping



$$\sum F_t = ma_t$$

$$mg\cos\theta = ma_t$$

$$a_t = g\cos\theta$$

$$\int_0^v v\,dv = \int_0^\theta a\,ds$$

$R\,d\theta$

$g\cos\theta$

$$v^2 = 2gR\sin\theta$$

$$\sum F_n = ma_n$$

$$N - mg\sin\theta = m\frac{v^2}{R}$$

$$\boxed{N = mg\sin\theta}$$

at $\theta = \frac{\pi}{2} \longrightarrow v^2 = 2gR\sin\left(\frac{\pi}{2}\right)$

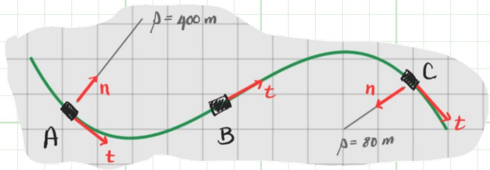
$$v^2 = 2gR$$

$$v = \sqrt{2gR}$$

$$\omega_{belt} = \frac{v_{belt}}{r} = \frac{\sqrt{2gR}}{r}$$

$$\boxed{\therefore \omega_{belt} = \frac{\sqrt{2gR}}{r}}$$

Eg 3: Determine the horizontal force (friction) exerted by the road on the tires at A, B, C.

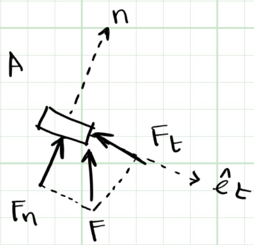


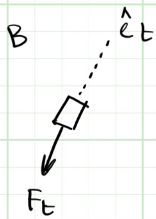
Car mass $= 1500$ kg

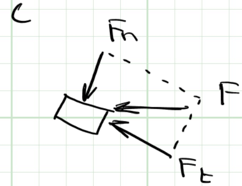
Speed at A $= 100$ km/h

Speed at C $= 50$ km/h

Distance from A — B $= 200$m







$$\sum F_t = \underbrace{m}_{1500} a_t^{?} \qquad \sum F_n = \overset{1500}{m} \overset{v^2/r}{a_n}$$

$$a_n = \frac{v^2}{r}$$

$$V_c^2 = V_A^2 + 2a\,\Delta s$$

$$\left(\frac{50}{3.6}\right)^2 = \left(\frac{100}{3.6}\right)^2 + 2a_t(200)$$

at A: $a_n = \dfrac{\left(\frac{100}{3.6}\right)^2}{400} = 1.929 \text{ m/s}^2$

$$a_t = 1.447 \text{ m/s}^2$$

at B: X

at C: $a_n = \dfrac{\left(\frac{50}{3.6}\right)^2}{80} = 2.41 \text{ m/s}^2$

$$\sum F_t = ma_t = 1500(1.447) = 2170 \text{ N}$$

$$\sum F_n = ma_n$$

↳ At A: $F_n = 1500(1.929) = 2890$ N

↳ At B: $F_n = 0$

↳ At C: $F_n = 1500(2.41) = 3620$ N

get magnitude

At A $\rightarrow$ $F = \sqrt{F_t^2 + F_n^2} = 3620\ N$

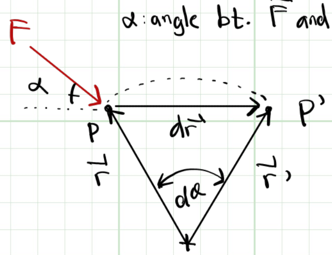
At B $\rightarrow$ $F = F_t = 2170$

At C $\rightarrow$ $F = \sqrt{F_t^2 + F_n^2} = 4220\ N$

Section 3.6 — Principles of Work & Energy

Kinetics
- (i) integration of a force w/respect to displacement of a particle — trip travelled by particle $\rightarrow$ work & energy
- (ii) integration of a force w/respect to time application of force $\rightarrow$ how long the force is applied $\rightarrow$ impulse & momentum

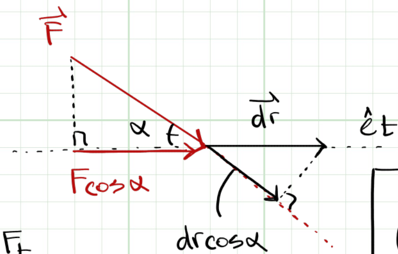
α: angle bt. $\vec{F}$ and $\vec{dr}$



$|\vec{dr}| = ds$

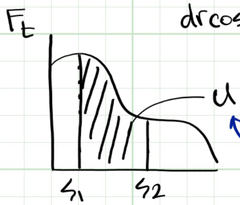
$$dU = \vec{F} \cdot \vec{dr}$$

alligning both vectors together or multiplying by cos α



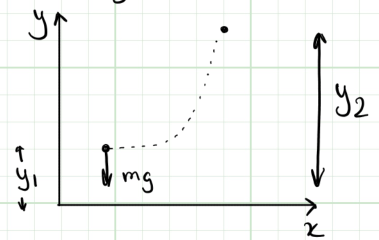
$$dU = \underbrace{F ds \cos\alpha}$$

$$\boxed{U = \int_1^2 \vec{F} \cdot \vec{ds} = \int_1^2 \overbrace{F\cos\alpha}^{F_t}\, ds = \int_1^2 F_t\, ds}$$



work done

(i) Weight



Gravitational force $= F_g$

$$F_g = -mg\hat{j}$$

$$U_g = \int_1^2 F_g \cdot dr = \int_{y_1}^{y_2} -mg\,dy$$

$$= -mg(y_2 - y_1)$$

$$\boxed{\therefore U_g = -mg\Delta y}$$

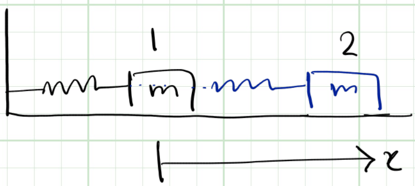
raising object

$y_2 > y_1 \longrightarrow U_g < 0$

$y_1 > y_2 \longrightarrow U_g > 0$

dropping object

(ii) Spring



$$F_s = -kx \; ; \; |d\vec{r}| = dx$$

$$U_s = \int_1^2 \vec{F_s} \cdot \vec{dr} = \int_1^2 -kx\,dx$$

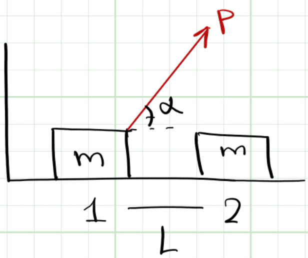
$$= -\frac{1}{2}k(x_2^2 - x_1^2)$$

$$\boxed{U_s = -\frac{1}{2}k(x_2^2 - x_1^2)}$$

if $|x_2| > |x_1| \longrightarrow U_s < 0$

$|x_2| < |x_1| \longrightarrow U_s > 0$

(iii) External Forces



$$U = \int_1^2 \vec{F} \cdot \vec{dr} = \int_1^2 P\cos\alpha\,dx = P\cos\alpha L$$

$$\boxed{U = P\cos\alpha L}$$

Friction Forces

1) Static Friction $\longrightarrow$ no motion, $U=0$

2) Kinetic Friction $\longrightarrow$ $F_f = \mu_k F_N$, $U = -\mu_k F_N (x_2 - x_1)$

↳ Friction forces are typically dissapated into heat and sound energy, so friction is not stored as energy

↳ Non-conservative Force

Principle of Work & Energy

$F_t = m a_t$

$a_t ds = v dv$

$$U = \int_1^2 \vec{F} \cdot d\vec{r} = \int_{s_1}^{s_2} F_t \, ds$$

$$U = \int_{s_1}^{s_2} m a_t \, ds$$

$$\boxed{U = \int_{v_1}^{v_2} m v \, dv = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2}$$

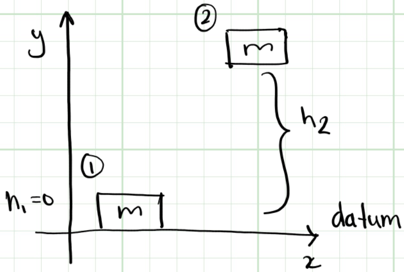
if $v_1 = 0$, then $\boxed{U = \frac{1}{2} m v^2}$ equation for kinetic energy if $v_1 = 0$

Notes: kinetic energy $\rightarrow$ "T"

$$U = T_2 - T_1$$

$$T_2 = T_1 + \underbrace{U_{1\to 2}}_{\text{work done by conservative forces only}}$$

Section 3.7 — Potential Energy



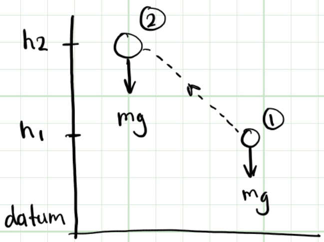
$$V_g = -mg(h_2 - h_1)$$

conservative forces → Work is independent of the path taken

Nonconservative forces → dissapated forces → Work done is dependent on path taken

Potential Energy (V_g)

V_g ~ gravitational potential energy



$$\left.\begin{aligned} V_{g_1} &= mgh_1 \\ V_{g_2} &= mgh_2 \end{aligned}\right\} \Delta V_g = V_{g_2} - V_{g_1}$$

$$= mg(h_2 - h_1)$$

work done → $-\Delta U_g = \Delta V_g$ ← change in energy

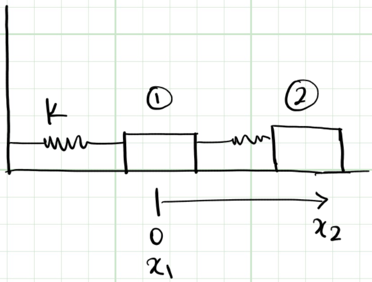
→ Consider as h_1 as new datum

$$V_{g_1} = mgh_1 = 0$$

$$V_{g_2} = mg(h_2 - h_1)$$

Spring Force

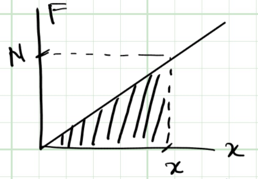
$V_E \sim$ elastic potential energy



x - distance from datum

Most convenient datum is the relaxed length of spring

$\Delta V_E = ?$



$$\text{Area} = \frac{1}{2} x F \quad \leftarrow kx$$

$$= \frac{1}{2} k x^2$$

$$\Delta V_E = \frac{1}{2} k \left(x_2^2 - x_1^2\right)$$

$$\Delta U_E = -\frac{1}{2} k \left(x_2^2 - x_1^2\right)$$

Overall Equation

$$T_2 = T_1 + U_{1-2}$$

conservative

non-conservative $\rightarrow$ U'_{1-2}

$$U_{cons_{1-2}} = -\Delta V$$

$-\Delta V_E$ $-\Delta V_g$

work done by nonconservative forces

$$T_2 = T_1 + U_{1-2}$$

$$T_2 = T_1 \underbrace{- \Delta V_g}_{V_{g_2} - V_{g_1}} - \underbrace{\Delta V_E}_{V_{E_2} - V_{E_1}} + U'_{1-2}$$

$$\boxed{T_2 + V_2 = T_1 + V_1 + U_{1-2}}$$

$$\boxed{T_2 + V_{g_2} + V_{E_2} = T_1 + V_{g_1} + V_{E_1} + U'_{1-2}}$$

T = Kinetic Energy $= \frac{1}{2}mv^2$

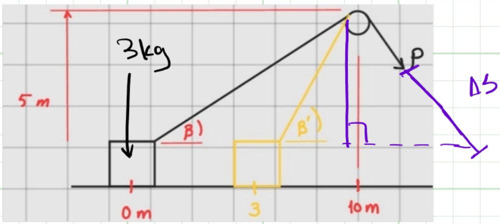
V_g = Gravitation Potential Energy $= mgh$

V_E = Elastic Potential Energy $= \frac{1}{2}kx^2$

U'_{1-2} = Work done by external working forces, but not gravity or spring forces

Section 3.6 - 3.7 Examples

Eg 1: $P = 10\,N$, $\vec{v}_i = 4\,m/s\ \hat{i}$, find v when block is at $x = 3m$



Position 1

$T_1 = \frac{1}{2}mv_i^2 = \frac{1}{2}(3)(4)^2$

$V_{g_1} = 0$

$V_{E_1} = 0$

$U'_{1\to 2} = \int_1^2 \vec{P} \cdot \vec{dx}$

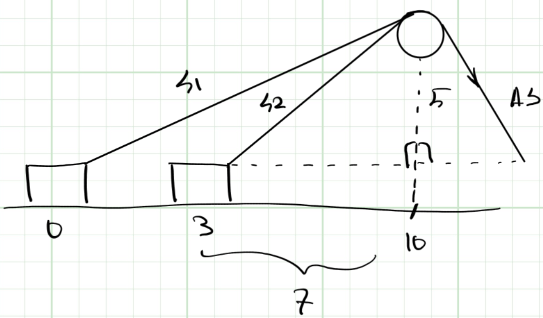
$P\cos\beta$

Position 2

$T_2 = \frac{1}{2}mv_f^2 = \frac{1}{2}(3)(v_f)^2$

$V_{g_2} = 0$

$V_{E_2} = 0$



$\Delta s = s_1 - s_2$

$\Delta s = \sqrt{10^2 + 5^2} - \sqrt{7^2 + 5^2}$

$\Delta s = 2.58\,m$

$U'_{1-2} = P\Delta s = 10(2.58) = 25.8\,J$

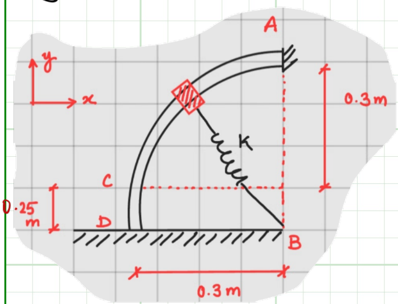
$$T_2 = T_1 + U'_{1-2}$$

$$\frac{1}{2}(3)v_f^2 = \frac{1}{2}(3)(4)^2 + 25.8\,J$$

$$\boxed{\therefore v_f = 5.76\,\hat{i}\ m/s}$$

$$v_f = 5.76\ m/s$$

Eg 2: Slider moves from $A \rightarrow D$. Find $\vec{v}$ right before C?



$m = 2.55$ kg

$k = 400$ N/m

$\vec{v}_A = 1.5$ m/s

$l_0 = 0.3$ m

Choose point C as datum

Position 1 : A

Position 2 : C

$$T_1 = \frac{1}{2}mv_i^2 = \frac{1}{2}(2.55)(1.5)^2 \qquad U'_{1\to 2} = 0$$

$$V_{g_1} = mgh = (2.55)(9.81)(0.3)$$

$$V_{E_1} = \frac{1}{2}kx^2 = \frac{1}{2}(400)(0.55-0.3)^2 = \frac{1}{2}(400)(0.25)^2$$

$$T_2 = \frac{1}{2}mv_f^2$$

spring length : $\sqrt{0.25^2 + 0.3^2} = 0.3905$

$$V_{g_2} = 0$$

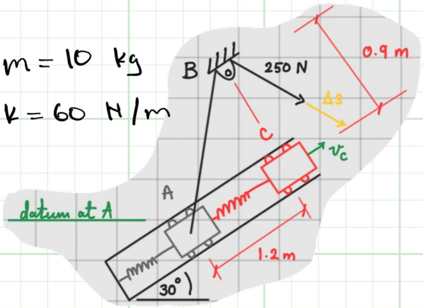
$$V_{E2} = \frac{1}{2}kx^2 = \frac{1}{2}(400)(0.3905-0.3)^2 =$$

$$T_2 + \cancel{V_{g2}}^{\,0} + V_{E2} = T_1 + V_{G1} + V_{E1} \implies \boxed{v_c = -4.05\,\hat{j}\ m/s}$$

Eg 3: @A → Spring stretched by 0.6m
@A → at rest } Find V_c

$m = 10$ kg

$k = 60$ N/m



Position 1 → A

Position 2 → C

$$\Delta s = BA - BC$$
$$= \sqrt{1.2^2 + 0.9^2} - 0.9$$
$$= 0.6 \text{ m}$$

$T_1 = 0$

$V_{g_1} = 0$

$V_{E_1} = \frac{1}{2}kx^2 = \frac{1}{2}(60)(0.6)^2$

$T_2 = \frac{1}{2}(10)V_f^2$

$V_{g_2} = (10)(9.81)1.2\sin 30^\circ$

$V_{E_2} = \frac{1}{2}(60)(0.6+1.2)^2$

$U'_{1\to2} = F\Delta s = 250(0.6)$

$\Rightarrow$ Plug into $T_2 + V_{g2} + V_{E2} = \cancel{T_1}^{0} + \cancel{V_{g1}}^{0} + V_{E1} + U'_{1\to2}$

$$\boxed{\therefore V_c = 0.974 \text{ m/s}}$$

Section 3.8 - 3.9 – Principle of Linear Momentum and Impulse

$$\sum\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} \longrightarrow \sum\vec{F}dt = md\vec{v}$$

integrate

$$\int_{t_1}^{t_2} \sum F dt = \int_{V_1}^{V_2} m d\vec{v}$$

$$m\vec{v}_2 = m\vec{v}_1 + \underbrace{\int_{t_1}^{t_2} \sum \vec{F}\,dt}_{\text{linear impulse}}$$

linear momentum

typically applied over short period of time "collisions"

Linear Momentum : $\vec{G} = m\vec{v}$ measurement

Linear Impulse : changes the linear momentum of moving body

$$\boxed{\vec{G}_2 = \vec{G}_1 + \int_{t_1}^{t_2} \sum \vec{F}\,dt}$$

principle of impulse and momentum

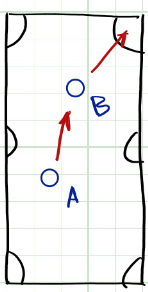
units $\left[kg \cdot \frac{m}{s}\right]$ ← unit for momentum

$$\boxed{T_2 = T_1 + \int_1^2 \sum \vec{F} \cdot dr}$$

principle of work and energy

Conservation of Momentum

$$\underbrace{\sum \vec{F}}_{\text{external}} = 0 \longrightarrow \vec{G}_2 = \vec{G}_1 \longrightarrow \Delta \vec{G} = 0$$



V_{A_1} V_{B_1} → before impact

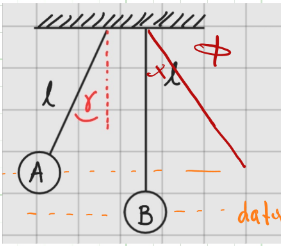
V_{A_2} V_{B_2} → after impact

assuming all of A's momentum transfered to B

$$m_A v_{A_1} + m_B \cancelto{0}{v_{B_1}} = m_A \cancelto{0}{v_{A_2}} + m_B v_{B_2}$$

$$\boxed{m_A v_{A_1} = m_B v_{B_2}}$$

Eg 1



A is released from rest at an angle γ.
It hits B and stops after the collision.
Find the rebound angle ϕ by ball B?

$m_A = 1.5$ kg
$m_B = 3$ kg
$\gamma = 70°$

1. A goes down
2. collision

$$1 \to 2: \quad T_{A_2} + \cancel{V_{A_2}}^{\,0} = \cancel{T_{A_1}}^{\,0} + V_{A_1} + \cancel{U'_{1\to2}}^{\,0}$$

$$\frac{1}{2} m_A v_A^{\,2} = m_A g (l - l\cos\gamma)$$

$$v_{A_2} = 1.47\sqrt{gl}$$

$$2 \to 3: \quad m_A v_{A_2} + \cancel{m_B v_{B_2}}^{\,0} = \cancel{m_A v_{A_3}}^{\,0} + m_A v_{B_3}$$

$$v_{B_3} = 0.574\sqrt{gl}$$

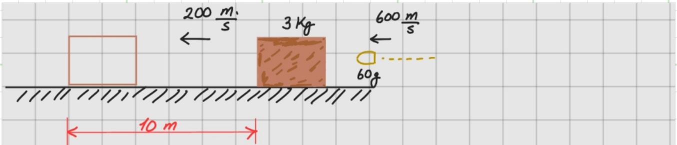
$$3 \to 4: \quad \cancel{T_{B_4}}^{\,0} + V_{B_4} = T_{B_3} + \cancel{V_{B_3}}^{\,0} + \cancel{U'_{3\to4}}^{\,0}$$

$$m_B g (l - l\cos\phi) = \frac{1}{2} m_B v_{B_3}^{\,2}$$

$$m_B \cancel{g} l (1 - \cos\phi) = \frac{1}{2} m_B (0.574)^2 \cancel{g} l$$

$$\boxed{\phi = 33.3°}$$

→ Eg 2:

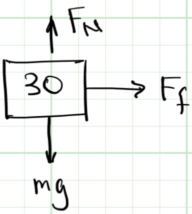


Bullet fired horizontally at $v_1 = 600\,\frac{m}{s}$ into the wooden block initially at rest. The bullet emerges from the block at the velocity $v_2 = 200\,\frac{m}{s}$, and the block slides a distance 10 m before coming to rest. Determine the coefficient of kinetic friction, μ_k between the block and the horizontal surface

$V_{b_1} = 600$

$V_{b_2} = 200$

$\mu_k = ?$



$$\sum F_y = F_N - mg$$

$$F_N = mg = 29.43\ N$$

$$T_2 = T_1 + U'_{1-2}$$

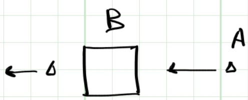
$-\mu_k F_N (10)$

$$\frac{1}{2} m \cancel{v_2}^2 = \frac{1}{2} m v_1^2 + U'_{1-2}$$

0

$$\frac{1}{2}(3)\, v_1^2 = \mu_k (29.43)(10)$$

need to find this



$V_{1A} = 600$

$V_{2A} = 200$

$V_{1B} = 0$

$V_{2B} = ?$

0

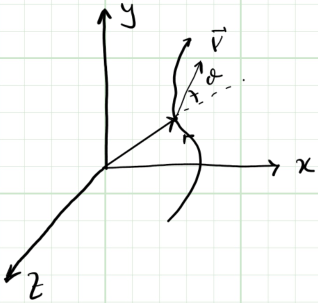
$$m_B \cancel{V_{1B}} + m_A V_{1A} = m_B V_{2B} + m_A V_{2A}$$

$$0.06(600) = 0.06(200)\ 3 V_{2B}$$

$$V_{2B} = -8\ \frac{m}{s}$$

$$\boxed{\therefore \mu_k = 0.326}$$

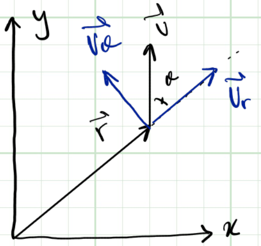
Section 3.10 — Angular Impulse and Momentum



Angular momentum $= \vec{H}_o$

$\vec{H}_o = \vec{r} \times m\vec{v}$

- $\vec{H}$ is measured w.r.t. a reference pt.
- $\vec{H}$ is a vector → right hand rule
- 2D problems : $|\vec{H}_o| = rmv\sin\theta$



$|\vec{v}_r| = |\vec{v}|\cos\theta$

$|\vec{v}_\theta| = |\vec{v}|\sin\theta$

$$|\vec{H}_o| = m|\vec{r}|\underbrace{|\vec{v}|\sin\theta}_{v_\theta}$$

- What causes change in $\vec{H}_o$?

$\vec{H}_o = \vec{r} \times m\vec{v} \longrightarrow \dfrac{d\vec{H}_o}{dt} = \dfrac{d}{dt}(\vec{r} \times m\vec{v})$ ← product rule

$\vec{v} \times \vec{v} = 0$

$$\dot{\vec{H}}_o = \cancel{\underbrace{\dot{\vec{r}}}_{\vec{v}} \times m\vec{v}} + \vec{r} \times m\underbrace{\dot{\vec{v}}}_{\vec{a}}$$

$$= 0 + \vec{r} \times \underbrace{m\vec{a}}_{\Sigma F}$$

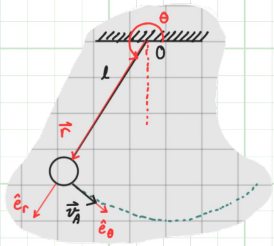
$$\frac{d\vec{H}_o}{dt} = \vec{r} \times \Sigma F = \Sigma M$$

$$\int_{\vec{H}_{o1}}^{\vec{H}_{o2}} d\vec{H}_o = \underbrace{\int_{t_1}^{t_2} \Sigma M \, dt}_{\text{angular impulse}}$$

Principle of Angular Impulse and Momentum

$$\boxed{(\vec{H}_o)_2 = (\vec{H}_o)_1 + \int_{t_1}^{t_2} \Sigma M}$$

→ Eg 1: $m_A = 2\,kg$, $l = 3.5\,m$, $\theta = 200°$, $\vec{v}_A = 4\,\frac{m}{s}$ ccw



a) $\dot{\theta}$ at instant shown?

$$\vec{v}_r = 0$$

$$\vec{v}_A = \vec{v}_r = r\dot{\theta} \quad \rightarrow \quad \dot{\theta} = \frac{\vec{v}_A}{r} = \frac{4}{3.5} = 1.14$$

$$\boxed{\therefore \dot{\theta} = 1.14 \text{ rad/s}}$$

b) Angular momentum of A about O

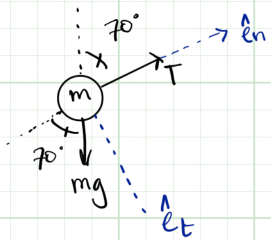
$$\vec{H}_o = \vec{r} \times m\vec{v}$$

$$\boxed{\therefore H_o = 28 \text{ kg}\cdot\text{m}^2/\text{s}\ \hat{k}}$$

$$|\vec{H}_o| = r m v_A \sin\alpha$$

$$= (3.5)(2)(4)\sin 90 = 28\ \frac{kg \cdot m^2}{s}$$

c) Tension in rope?



$$\sum F_n = ma_n = T - mg\cos 70°$$

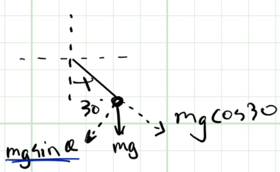
$$m\frac{v^2}{r} = T - mg\cos 70°$$

$$T = m\frac{v^2}{r} + mg\cos 70°$$

$$\boxed{\therefore T = 15.85\ N}$$

d) rate of change of angular momentum when $\theta = 300°$

$$\dot{\vec{H}}_o = \sum \vec{M}_o = \vec{r} \times \sum F$$

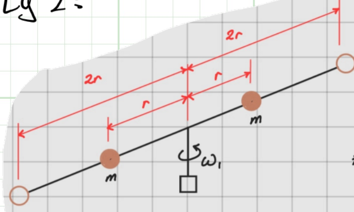


$$|\sum \vec{M}_o| = |mg\sin 30°|\, l$$

$$= 34.33\ N\cdot m$$

direction $= -\hat{k}$ ↻ cw

→ Eg 2:



The spheres slide along the rotating rod. Initial position at distance r from centre rotating freely at ω.

* Determine the new angular velocity ω_2 after the spheres are released and reach the ends of the rods at distance 2r
* Find the change in Kinetic energy

$r = 20\,mm$; $\omega_1 = 20\ rad/s$; $m = 2\,kg$

- No angular impulse $\rightarrow \Delta \vec{H_o} = 0$
- $(\sum \vec{r} \times m\vec{v})_2 - (\sum \vec{r} \times m\vec{v})_1 = 0$

$\underbrace{\sum \vec{r} \times m\vec{v}}_{\sin 90^\circ = 1}$ for circular motion

$\Longrightarrow \sum r m \underbrace{v}_{r\omega} = \sum r^2 m \omega$

$$|\Delta \vec{H_o}| = 2\left[(r^2 m \omega)_2 - (r^2 m \omega)_1\right]$$

$$0 = 2\left[(40)^2(2)\omega_2 - (20)^2(2)(20)\right]$$

$$\boxed{\therefore \omega_2 = 5\ \frac{rad}{s}}$$

b) $\Delta T = \frac{1}{2}(2)m\,V_2^2 - \frac{1}{2}(2)mV_1^2$

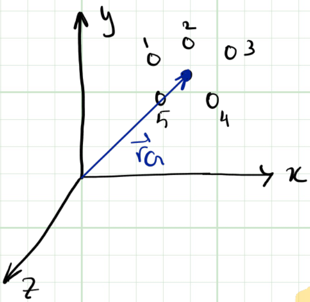
$= \cancel{\frac{1}{2}}(\cancel{2})m(V_2^2 - V_1^2)$

$= m\left[(r\omega)_2^2 - (r\omega)_1^2\right]$

$\boxed{= -0.24\ J}$

Section 4.1 – 4.4 : Kinetics to System of Particles

- for any system of particles, you need to know G centre of mass



$$\vec{r}_G = \frac{\sum_{i=1}^{N} m_i r_i}{\sum_{i=1}^{N} m_i} \xrightarrow{\frac{d}{dt}} \vec{V}_G = \frac{\sum m_i v_i}{\sum m_i} \xrightarrow{\frac{d}{dt}} \vec{a}_G = \frac{\sum m_i a_i}{\sum m_i}$$

- for a particle i : $\vec{F}_i + \vec{f}_i = m_i a_i$ ⟶ for all particles

$$\sum_{i=1}^{N} \vec{F}_i + \cancel{\sum_{i=1}^{N} \vec{f}_i} = \sum m_i a_i$$

cancel each other

$$\Rightarrow \sum_{i=1}^{N} \vec{F}_i = m\vec{a}_G$$

Newton's 2nd Law for a system of particles

Section 4.3 – Work & Energy

- for a particle : $(T_2)_i = (T_1)_i + (U'_{1-2})_i$

$$\vdots \qquad \vdots \qquad \vdots$$

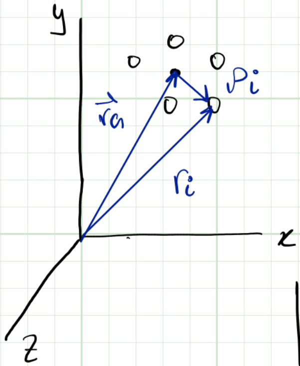
$$(T_2)_N = (T_1)_N + (U'_{1-2})_N$$

- for N particles : $\sum_{i=1}^{N} (T_2)_i = (T_2)_i + (U'_{1-2})_i$

work & energy for systems of particles

Section 4.4 — Impulse, Momentum For System of Particles

(a) Linear impulse & momentum



$$\vec{\rho}_i = \vec{r}_i - \vec{r}_G$$

$$\boxed{\vec{r}_i = \vec{r}_G + \vec{\rho}_i}$$

$$\boxed{\vec{v}_i = \vec{v}_G + \dot{\vec{\rho}}_i} \quad \dot{\vec{\rho}}_i = \vec{v}_{i/G}$$

$$\boxed{\vec{a}_i = \vec{a}_G + \ddot{\vec{\rho}}_i} \quad \ddot{\vec{\rho}}_i = \vec{a}_{i/G}$$

$$\vec{r}_G = \frac{\sum m_i r_i}{m_i} = \frac{\sum m_i(\vec{r}_G + \vec{\rho}_i)}{\sum m_i} = \frac{\sum m_i \vec{r}_G}{\sum m_i} + \frac{\sum m_i \vec{\rho}_i}{\sum m_i} \rightarrow 0$$

$$\vec{r}_G \frac{\cancel{\sum m_i}}{\cancel{\sum m_i}}$$

$$\sum m_i \dot{\vec{\rho}}_i = 0$$

Linear Momentum

for N particles: $\vec{G} = \sum_{i=1}^{N} m_i \vec{v}_i = \sum m_i \vec{v}_G + \cancel{\sum m_i \dot{\vec{\rho}}_i}^{\,0}$

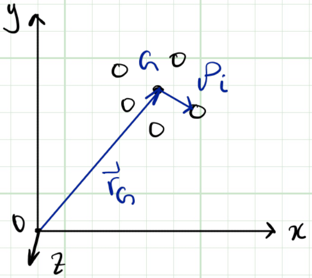
$$\boxed{\therefore \vec{G} = m\vec{v}_G}$$

← equation for lin. momentum for sys. of part.

$$\boxed{\underset{m(\vec{v}_G)_2}{\vec{G}_2} = \underset{m(\vec{v}_G)_1}{\vec{G}_1} + \sum_{i=1}^{N} \int \vec{F}_i \, dt}$$

← linear impulse & momentum for sys. of particles.

(b) Angular Impulse & Momentum



(i) Reference point is fixed point O

$\Rightarrow$ for N particles

$$\vec{H}_0 = \sum_{i=1}^{N} (H_0)_i = \sum_{i=1}^{N} (\vec{r}_i \times m_i \vec{v}_i) \qquad \dot{\vec{r}} = \vec{v}_i$$

$$\dot{\vec{H}}_0 = \frac{d}{dt} \sum (\vec{r}_i \times m_i \vec{v}_i) = \underbrace{\cancel{\sum \dot{\vec{r}} \times m_i \vec{v}_i}}_{\vec{v} \times \vec{v} = 0} + \overbrace{\sum \vec{r} \times \underbrace{m_i \dot{\vec{v}}_i}_{m\vec{a}_i}}^{\sum M_{oi}}$$

$$\frac{d}{dt}\vec{H}_0 = \dot{\vec{H}}_0 = \sum_{i=1}^{N} \underbrace{(M_0)_i}_{\text{moment}}$$

integrate

$$\boxed{(\vec{H}_0)_2 = (\vec{H}_0)_1 + \sum_{i=1}^{N} \int_{t_1}^{t_2} (M_0)_i \, dt}$$

principle of angular impulse and momentum for N particles w.r.t a fixed pt.

(ii) Reference point G

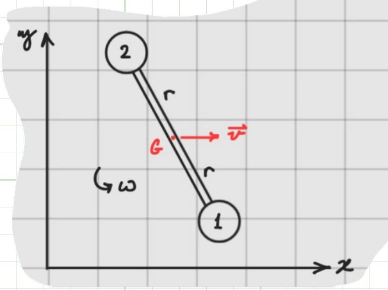
*check LecNote 17 for full derivation

$$\dot{\rho}_i = \vec{v}_{i/G}$$

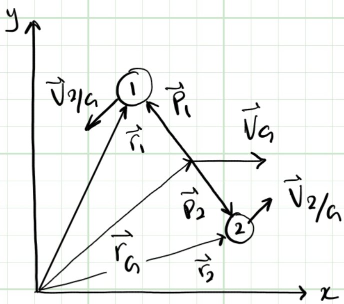
$$\boxed{\vec{H}_G = \sum \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i}$$

$$\boxed{(\vec{H}_G)_2 = (\vec{H}_G)_1 + \sum_{i=1}^{N} \int_{t_1}^{t_2} (M_G)_i \, dt}$$

Eg 1



Two spheres with mass m are connected with a rod. Mass of rod is negligible. Rod is rotating with respect to G at angular velocity $\omega = \dot{\theta}$. G is moving to the right with $\vec{v}$. Find $\vec{H}_o$ of the system.



We know the following

- $\vec{r}_i = \vec{r}_G + \vec{\rho}_i$
- $\vec{V}_i = \vec{V}_G + \dot{\vec{\rho}}_i$
- $\vec{V}_{i/G} = r\omega = \dot{\vec{\rho}}_i$
- $\vec{r}_G = x\hat{i} + y\hat{j}$
- $\vec{v}_G = y\hat{i}$

$$\vec{H}_o = \sum_{i=1}^{2} \vec{r}_i \times m_i \vec{V}_i$$

$$= \sum_{i=1}^{2} (\vec{r}_G + \vec{\rho}_i) \times m_i (\vec{V}_G + \dot{\vec{\rho}}_i)$$

$$= \sum_{i=1}^{2} \underbrace{\vec{r}_G \times m_i \vec{V}_G}_{1} + \underbrace{\cancelto{0}{\vec{r}_G \times m_i \dot{\vec{\rho}}_i}}_{2} + \underbrace{\cancelto{0}{\vec{\rho}_i \times m_i \vec{V}_G}}_{3} + \underbrace{\vec{\rho}_i \times m_i \dot{\vec{\rho}}_i}_{4}$$

$$\hat{i} \times \hat{i} = 0$$
$$\hat{j} \times \hat{i} = -\hat{k}$$

1: $(x\hat{i} + y\hat{j}) \times m(y\hat{i}) \times 2 = \underline{\underline{-2mv_y\hat{k}}}$

4: $(\rho_1 \times mr\omega + \rho_2 \times mr\omega)\hat{k}$

$= (mr^2\omega + mr^2\omega)\hat{k} = 2mr^2\omega\hat{k}$

$\therefore \vec{H}_o = 2mr^2\omega\hat{k} - 2mv_y\hat{k}$

$$\boxed{\vec{H}_o = 2m(r^2\omega - v_y)\hat{k}}$$

Chapter 5 - Plane Kinematics of Rigid Bodies

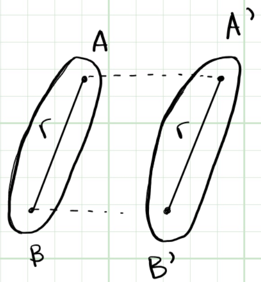
Section 5.1 – 5.3

* **Rigid Body**: a system of particles and distances between particles and shape remains constant

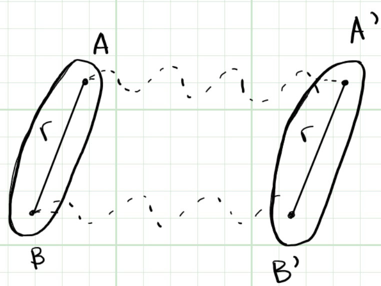
* **Plane motion**: no out-of-plane motion (2D motion)
  a) Pure translation
  b) Pure rotation
  c) General Plane motion

Pure Translation (no rotation)

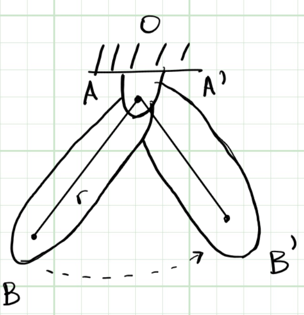
* rectilinear motion



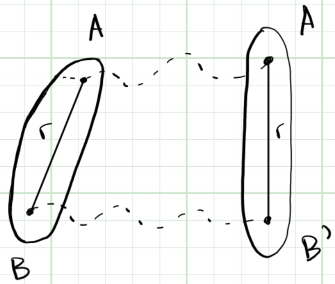
* curvilinear motion



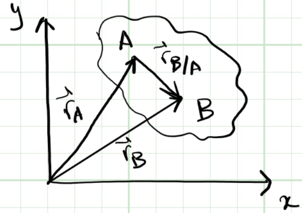
Pure Rotation (fixed axis rotation)



General Plane Motion



* Translation



$$\vec r_B = \vec r_A + \vec r_{B/A}$$

constant because in rigid bodies, dist b/t particles stay same

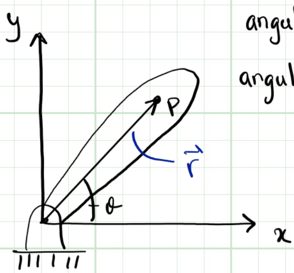
$$\frac{d}{dt} \quad \vec v_B = \vec v_A$$

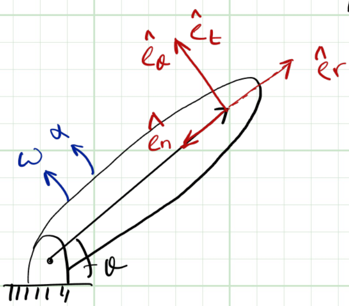
$$\frac{d}{dt} \quad \vec a_B = \vec a_A$$

* Fixed Axis Rotation

angular velocity of whole body $= \dot\theta = \omega \left[\frac{rad}{s}\right]$

angular acceleration " " $= \ddot\theta = \alpha \left[\frac{rad}{s^2}\right]$





$$\hat e_r = -\hat e_n$$

$$\hat e_\theta = \pm \hat e_t$$

depending on direction of motion

$$\omega = \dot{\theta} = \frac{d\theta}{dt} \longrightarrow \omega = \frac{d\theta}{dt}$$

$$\alpha = \ddot{\theta} = \frac{d\omega}{dt} \longrightarrow \alpha = \frac{d\omega}{dt}$$

$$\boxed{\alpha\, d\theta = \omega\, d\omega}$$

Equations Summary

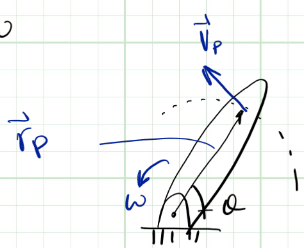
Translation	Rotation
s	θ
$v = \dot{s}$	$\omega = \dot{\theta}$
$a = \dot{v} = \ddot{s}$	$\alpha = \dot{\omega} = \ddot{\theta}$
$\downarrow$ const a	$\downarrow$ α constant
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$s = s_0 + v_0 t + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(s - s_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

* Remember velocity equation in r-θ coordinates

$\vec{v} = \cancel{\dot{r}}^{\,0}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$ since $|\vec{r}|$ not changing : $\dot{\vec{r}} = \ddot{\vec{r}} = 0$

$\vec{v} = r\dot{\theta}\hat{e}_\theta = r\omega\hat{e}_\theta$

$|\vec{v}| = r\omega$



$\vec{v}_P = \vec{\omega} \times \vec{r}_P$

$\vec{v}_P \perp \vec{r}_P$

Let $\vec{\omega} = \omega\hat{k}$

$\vec{r}_P = x\hat{i} + y\hat{j}$

$$\vec{v}_P = \vec{\omega} \times \vec{r}_P = \omega\hat{k} \times (x\hat{i} + y\hat{j}) = \omega x\hat{j} - \omega y\hat{i}$$

Summary Velocity :

CCW Rotation: $\vec{v}_P = \omega x \hat{j} - \omega y \hat{i}$

CW Rotation : $\vec{v}_P = -\omega x \hat{j} + \omega y \hat{i}$

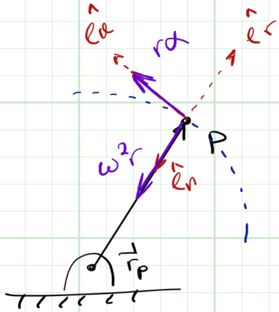
Acceleration

$$\vec{a}_P = (\cancel{\ddot{r}}^{\,0} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\cancel{\dot{r}\dot{\theta}}^{\,0})\,\hat{e}_\theta$$

$$\vec{a}_P = -r\dot{\theta}^2\,\hat{e}_r + r\ddot{\theta}\,\hat{e}_\theta = -r\omega^2\,\hat{e}_r + r\alpha\,\hat{e}_\theta$$

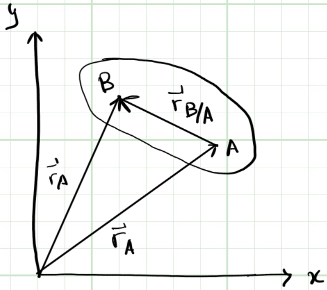
$$\Rightarrow \vec{a}_P = \vec{\omega} \times \underbrace{(\vec{\omega} \times \vec{r})}_{\vec{v}} + \vec{\alpha} \times \vec{r}$$

$$\Rightarrow \vec{a}_P = \underbrace{\vec{\omega} \times \vec{v}}_{\vec{a}_n} + \underbrace{\vec{\alpha} \times \vec{r}}_{\vec{a}_t}$$



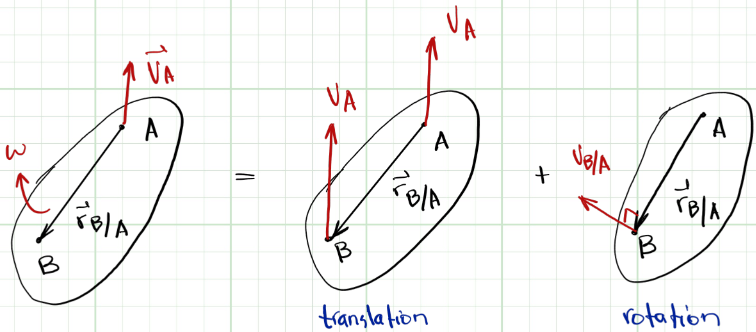
Section 5.4 – Relative Velocity

- <u>General Plane Motion</u>

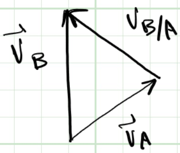


$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

$$\vec{v}_B = \underbrace{\vec{v}_A}_{\text{translation}} + \underbrace{\vec{v}_{B/A}}_{\text{rotation}}$$



$$|\vec{v}_{B/A}| \perp |\vec{r}_{B/A}| \quad , \quad |\vec{v}_{B/A}| = \underbrace{|\vec{\omega}|}_{\text{constant for RB}} |\vec{r}_{B/A}|$$



since $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$

Guidelines

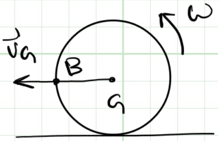
On a rigid body, you can find $\vec{v}$ for any point B if you know

1) Geometry of the body $\longrightarrow \vec{r}_{B/A}$
2) Velocity of another point A $\longrightarrow \vec{V}_A$
3) Angular velocity $\vec{\omega}$ of RB

* sometimes, $\vec{\omega}$ is unknown

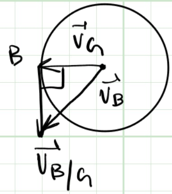
$$\boxed{|\vec{\omega}| = \frac{|\vec{V}_A - \vec{V}_B|}{|\vec{r}_{A/B}|}}$$

→ **Eg 1**



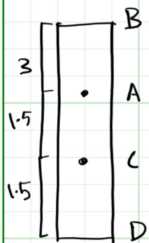
$$\left.\begin{array}{l} \vec{v}_G = -20\,\hat{i}\ m/s \\ r = 35\ cm \\ \omega = 57.14\ rad/s \end{array}\right\} \text{Find } \vec{V}_B = ?$$

$$\vec{V}_B = \vec{V}_G + \vec{V}_{B/G}$$
$$= -20\hat{i} + (-0.35)(57.14)\hat{j}$$



$$\vec{V}_B = -20\hat{i} - 20\hat{i}$$

→ **Eg 2**

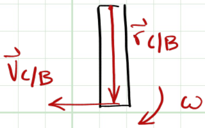


$$\left.\begin{array}{l} \vec{V}_B = 4\hat{i} + 3\hat{j}\ m/s \\ \vec{V}_A = 2\hat{i} + 3\hat{j}\ m/s \end{array}\right\} \text{Find } \omega, \vec{V}_C, \vec{V}_D$$

$$\vec{\omega} = \frac{|2\hat{i} + 3\hat{j} - 4\hat{i} - 3\hat{j}|}{|-3\hat{j}|} = \frac{|-2\hat{i}|}{|-3\hat{j}|} = 0.667$$

$$\vec{\omega} = -0.667\,\hat{k}\ rad/s \quad \text{(Right hand rule)}$$

$\vec{v}_C = ?$



$$\vec{V}_C = \vec{V}_B + \vec{V}_{C/B}$$

$$|\vec{v}_{C/B}| = |\vec{\omega}||\vec{r}_{C/B}| = (0.667)(6) = 4 \text{ m/s} \; -\hat{i} \; = -4\hat{i} \text{ m/s}$$

$$\vec{V}_C = \vec{V}_B + \vec{V}_{C/B}$$

$$= 4\hat{i} + 3\hat{j} - 4\hat{i} = 3\hat{j}$$

$\therefore \vec{V}_C = 3\hat{j}$ m/s

$\vec{V}_D = ?$

$$\vec{v}_D = \vec{v}_C + \vec{v}_{D/C}$$

$$|\vec{v}_D| = |\vec{\omega}||\vec{r}_{D/C}| = 0.667(1.5) = 1 \text{ m/s}$$

$$\vec{V}_D = 3\hat{j} + 1\hat{i}$$

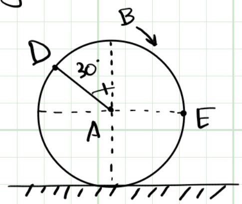
$\therefore \vec{v}_D = 1\hat{i} + 3\hat{j}$

(IC)

Section 5.5 – Instaneous Centre of Zero Velocity

- The point of contact is instananeously at rest.

<u>Eg: 1</u>



Wheel rolling w/o slipping, $\omega = 90.91 \frac{\text{rad}}{s}$

$$\left.\begin{aligned} \vec{V}_A &= 25\hat{i} \text{ m/s} \\ R_{wheel} &= 0.275 \end{aligned}\right\} \vec{V}_D$$

$$\vec{V}_A = \cancel{\vec{V}_C}^{\,0} + \vec{V}_{A/C}$$

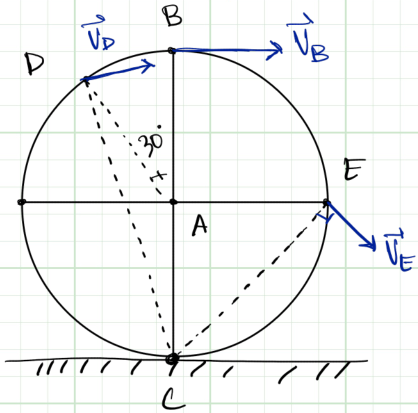
$$\vec{v}_D = \vec{V}_A + \vec{V}_{D/A}$$

$R_{wheel} = 25$ m/s

$$\vec{V}_A = \vec{V}_{A/C}$$

$$\vec{v}_D = 25\hat{i} + 25\cos 30\hat{i} + 25\sin 30\hat{j}$$

$$25 = R_{wheel} = |\vec{r}_{A/C}||\vec{\omega}|$$

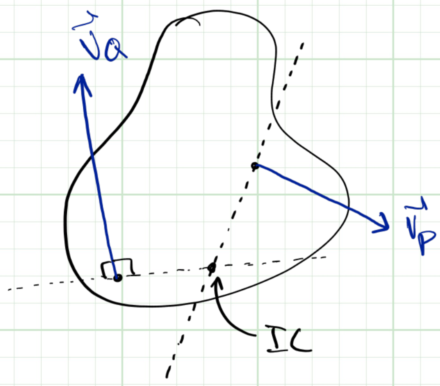


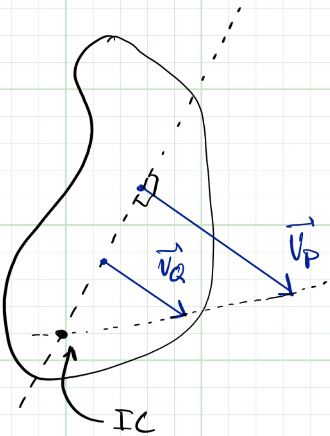
→ How to Find IC?

1) Find 2 points on the body with known velocity directions

2) Draw lines ⊥ to these velocities

3) Find point of intersection ⟶ IC



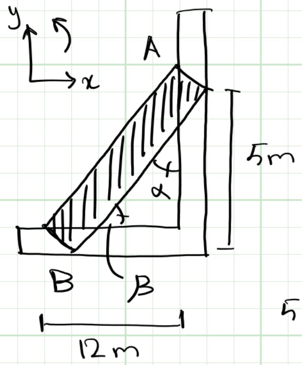


$$\frac{\vec{V}_P}{\vec{V}_Q} = \frac{\vec{r}_{P/IC}}{\vec{r}_{Q/IC}}$$ ⟵ divide each other

$$\begin{cases} \vec{V}_P = \cancel{\vec{V}_{IC}}^{\,0} + \vec{V}_{P/IC} = \vec{\omega}\,\vec{r}_{P/IC} \\ \vec{V}_Q = \vec{V}_{IC} + V_{Q/IC} = \vec{\omega}\,\vec{r}_{Q/IC} \end{cases}$$

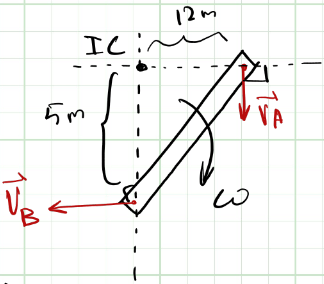
when $\vec{V}$ are parallel, u need both magnitude and dir.

Eg2:



A moves vertically
B moves horizontally
$\vec{V}_A = -8\hat{j}\ \frac{m}{s}$

} Find ω ?



$\vec{V}_A = \cancel{V_{IC}}^{\,0} + \vec{V}_{A/IC}$

$\vec{V}_A = \vec{V}_{A/IC}$

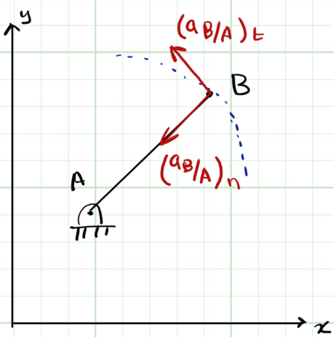
$\vec{V}_A = \vec{\omega} \times \vec{r}_{A/IC}$

$8 = 12\,\omega_{AB}$

$\omega_{AB} = \frac{8}{12} = \boxed{0.667\ \frac{rad}{s}}$

$\therefore\ \vec{\omega} = -0.667\,\hat{k}\ rad/s$

Section 5.6 — Relative Acceleration



Fixed axis rotation

$\Rightarrow \vec{a}_B = \cancel{\vec{a}_A}^{\,0} + \underbrace{\vec{a}_{B/A}}$

$(\vec{a}_{B/A})_t + (\vec{a}_{B/A})_n$

$$\vec{a}_B = \underbrace{(\vec{a}_{B/A})_t}_{\vec{\alpha} \times \vec{r}_{B/A}} + \underbrace{(\vec{a}_{B/A})_n}_{\vec{\omega} \times \underbrace{(\vec{\omega} \times \vec{r}_{B/A})}_{\vec{V}_{B/A}}}$$

$$(\vec{a}_{B/A})_t = \pm\, \vec{r}_{B/A}\,\alpha$$

$$(\vec{a}_{B/A})_n = \vec{r}_{B/A}\,\omega^2$$

$$\vec{a}_B = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$$

$$\vec{a}_B = (\pm r\alpha)\,\hat{e}_t + (r\omega^2)\,\hat{e}_n$$

General Plane Motion

not necessarily zero

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

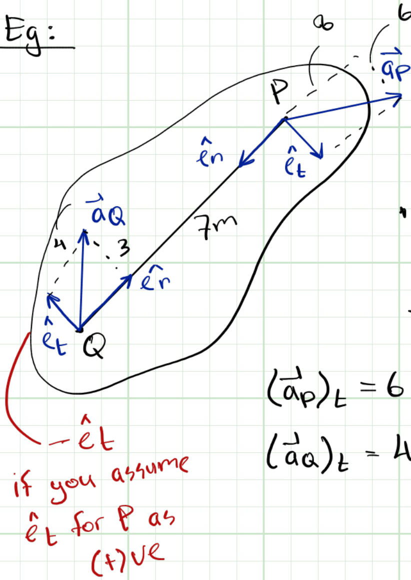
$$\vec{a}_B = \vec{a}_A + (\vec{a}_{B/A})_t + (\vec{a}_{B/A})_n$$

$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})$$

$$(\vec{a}_{B/A})_t = \vec{\alpha} \times \vec{r}_{B/A} = (\vec{a}_B)_t - (\vec{a}_A)_t$$

$$\boxed{|\vec{\alpha}| = \frac{|\vec{a}_B - \vec{a}_A|_t}{|\vec{r}_{B/A}|}}$$

Eg:



$\vec{a}_P = 10$ m/s
$\vec{a}_Q = 5$ m/s

Find α

• only look at tangential component bec normal comp. doesn't contribute to the rotation

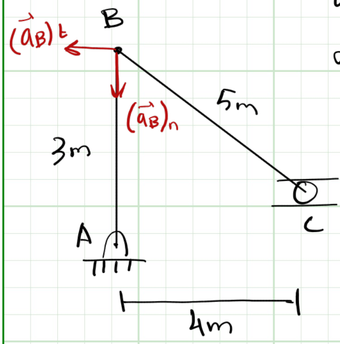
$(\vec{a}_P)_t = 6\ \frac{m}{s^2}$

$(\vec{a}_Q)_t = 4\ \frac{m}{s^2}$

$$|\vec{\alpha}| = \frac{|6-(-3)|}{7} = \frac{9}{7}$$

$$\therefore\ \vec{\alpha} = -\frac{9}{7}\,\hat{k}\ \frac{rad}{s^2}$$

→ Eg 2:



$$\left.\begin{aligned} \omega_{A/B} &= -3 \text{ rad/s} \\ \alpha_{A/B} &= 2 \text{ rad/s}^2 \end{aligned}\right\} \text{Find } \vec{a}_B,\ \vec{a}_C$$

$$\vec{a}_B = \cancelto{0}{\vec{a}_A} + \vec{a}_{B/A}$$

$$\vec{a}_B = \vec{a}_{B/A} = \underbrace{(\vec{a}_{B/A})_t}_{\alpha r} + \underbrace{(\vec{a}_{B/A})_n}_{r\omega^2}$$

$$\vec{a}_B = \underbrace{2(3)}_{-\hat{i}} + \underbrace{(-3)^2(3)}_{-\hat{j}} = -6\hat{i} - 27\hat{j} \ \frac{m}{s^2}$$

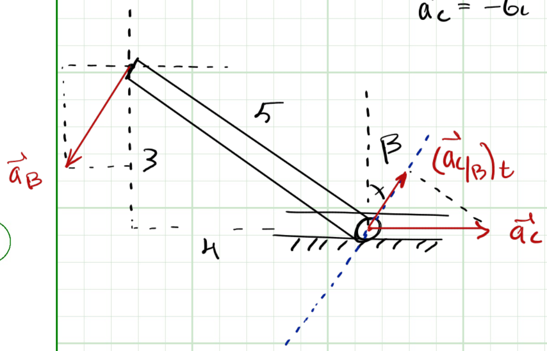
$$\vec{a}_C = \vec{a}_B + (\vec{a}_{C/B})_t + (\vec{a}_{C/B})_n$$

$$\underbrace{\vec{V}_C}_{\hat{i}} = \underbrace{\vec{V}_B}_{\hat{i}} + \underbrace{\cancelto{0}{\vec{V}_{C/B}}}_{\hat{i}+\hat{j}} \qquad \vec{V}_{C/B} = 0 = \cancelto{0\text{, does not rotate}}{\vec{\omega}_{CB}} \times \vec{r}_{C/B}$$

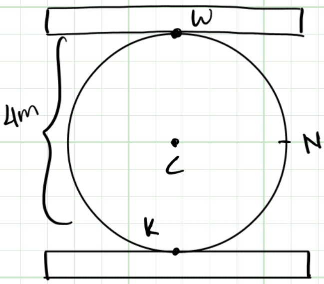
$$\vec{a}_C = \vec{a}_B + (\vec{a}_{C/B})_t + \cancelto{0\text{, since } \omega = 0}{(\vec{a}_{C/B})_n}$$

$$\vec{a}_C = \vec{a}_B + \vec{\alpha} \times \vec{r}_{C/B}$$

$$\vec{a}_C = -6\hat{i} - 27\hat{j} + 5\alpha\sin\beta\,\hat{i} + 5\alpha\cos\beta\,\hat{j}$$



Eg: Wheel squeezed b/n 2 plates rolling w/o slipping } Find v, a for W, C, N, K



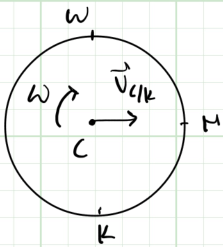
$v = 5\ \frac{m}{s} \rightarrow$

$a = 8\ \frac{m}{s^2} \leftarrow$

$v = 2 m/s \leftarrow$

$a = 7 m/s^2 \rightarrow$

(a) Finding Velocities



$$\vec{v}_C = \underbrace{\vec{v}_K}_{-2\hat{i}} + \underbrace{\vec{v}_{C/K}}_{\vec{\omega} \times \vec{r}_{C/K}}$$

$$\vec{\omega} \times \vec{r}_{C/K} = \vec{\omega} \times 2\hat{j}$$

$$\left.\begin{aligned} \vec{v}_W &= 5\hat{i} \\ \vec{v}_K &= -2\hat{i} \end{aligned}\right\} |\vec{\omega}| = \left| \frac{5-(-2)}{4} \right| = 1.75 \text{ rad/s}$$

$$\therefore \vec{\omega} = -1.75\hat{k}$$

$$\vec{v}_C = \vec{v}_K + \vec{v}_{C/K}$$

$$= -2\hat{i} + (-1.75\hat{k} \times 2\hat{j})$$

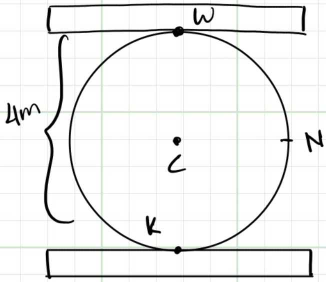
$$= -2\hat{i} + (1.75)(2)\hat{i} = 1.5\hat{i}\ \frac{m}{s} = \vec{v}_C$$

$$\vec{v}_M = \vec{v}_C + \underbrace{\vec{v}_{M/C}}_{\vec{\omega} \times \vec{r}_{M/C}} = 1.5\hat{i} - 3.5\hat{j}\ m/s = \vec{v}_M$$

$$-1.75\hat{k} \times -2\hat{i} = -3.5\hat{j}$$

(b) Finding Accelerations

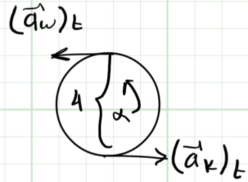
* Find α first



$a = 8\ m/s^2 \leftarrow$

tangential accelerations only

$a = 7\ m/s^2 \rightarrow$



$(\vec{a}_w)_t = -8\ m/s^2$

$(\vec{a}_k)_t = 7\ m/s^2$

$$|\vec{\alpha}| = \left|\frac{-8-7}{4}\right| = 3.75\ rad/s^2 \longrightarrow \underline{\underline{\vec{\alpha} = 3.75\,\hat{k}}}$$

$$\vec{a}_c = \vec{a}_w + \vec{a}_{c/w}$$

$$\vec{a}_c = \underset{-\hat{i}}{(\vec{a}_w)_t} + \underset{-\hat{j}}{(\vec{a}_w)_n} + \underset{\hat{i}}{(\vec{a}_{c/w})_t} + \underset{\hat{j}}{(\vec{a}_{c/w})_n}$$

$\hat{i} = -\hat{i} \quad -\hat{j} \quad \hat{i} \quad \hat{j}$

$\therefore \vec{a}_c = -0.5\,\hat{i}$

$$\vec{a}_c = (\vec{a}_w)_t + (\vec{a}_{c/w})_t = -8\hat{i} + (3.75)(2)\hat{i} = -0.5\ \frac{m}{s^2}\ \hat{i}$$

$$\longrightarrow \vec{\omega} \times \vec{r}_{c/w} = \hat{k} \times -\hat{j} = \hat{i}$$

$$\vec{a}_w = \vec{a}_c + (\vec{a}_{w/c})_t + (\vec{a}_{w/c})_n$$

$$= -0.5\hat{i} - (3.75)(2)\hat{i} + (-\hat{j})(1.75)^2(2)$$

$\vec{a}_w = -8\hat{i} - 6.125\hat{j}$

$$\vec{a}_k = \vec{a}_c + (\vec{a}_{k/c})_t + (\vec{a}_{k/c})_n$$

$$= -0.5\hat{i} + (\hat{i})(3.75)(2) + (1.75^2)(2)(\hat{j})$$

$\vec{a}_k = 7\hat{i} + 6.125\hat{j}$

$$\vec{a}_N = \vec{a}_C + (\vec{a}_{N/C})_t + (\vec{a}_{N/C})_n$$

$$= -0.5\hat{i} + (\hat{j})(3.75)(2) + (-\hat{i})(1.75^2)(2)$$

$$\vec{a}_N = -6.625\hat{i} + 7.5\hat{j}$$

Chapter 6: Kinetics of Rigid Bodies

- **Rigid Body**: a great # of particles glued together
- since # of particles $\to \infty$, we use integration rather Σ

$$\vec{r}_G = \frac{\int r\,dm}{\int dm}$$

dm → differential mass element

- Translation:

One particle: $\Sigma \vec{F} = m\vec{a}$

N particles: $\Sigma \vec{F} = m\,\vec{a}_G$

m → total mass of body

$$\vec{r}_G = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i} \longrightarrow \vec{a}_G = \ddot{\vec{a}}_G$$

- Fixed Axis Rotation, "O":

→ N particles: $\dot{\vec{H}}_O = \Sigma \vec{M}_O$, $\dot{\vec{H}}_G = \Sigma \vec{M}_G$

A rigid body: N particles with a mass Δm_i each

$$\vec{H}_O = \vec{r} \times m\vec{v}$$

$$\vec{H}_O = \Sigma\, \vec{r}_i \times m_i \vec{v}_i \quad \longrightarrow \quad \vec{H}_O = \sum_{i=1}^{N} (r_i)(\Delta m_i)(r_i)(\omega)$$

$\vec{v}_i = \vec{r}_i \omega$

tak out

$$\vec{H_o} = \omega \sum_{i=1}^{N} \Delta m_i r_i^2 \longrightarrow I_o \sim \text{mass moment of inertia}$$

* for solid body, $I_o = \int r^2 \, dm$

Mass Moment of Inertia

resistance to change

(i) Mass moment of inertia $\longrightarrow$ resistance to rotational motion

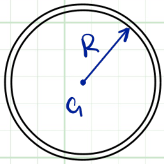
(ii) Area moment of inertia $\longrightarrow$ resistance to bending

(iii) Polar moment of inertia $\longrightarrow$ resistance to torsion

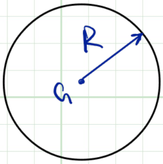
$$\vec{H_o} = I_o \omega$$

$$\dot{\vec{H_o}} = I_o \alpha = \sum \vec{M_o} \quad , \quad \dot{\vec{H_G}} = I_G \alpha$$

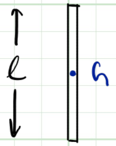
Common Shapes



$I_G = mR^2 \quad [kg \cdot m^2]$ hollow thin ring



$I_G = \frac{1}{2} mR^2$ Solid disk



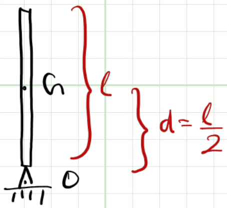
$I_G = \frac{1}{12} m \ell^2$ thin stick

Parallel Axis Theorem

* can be used if you want to calculate I about another point rather than G.

$$\Rightarrow \boxed{I_O = I_G + md^2}$$

d: ⊥ distance b/w O and G

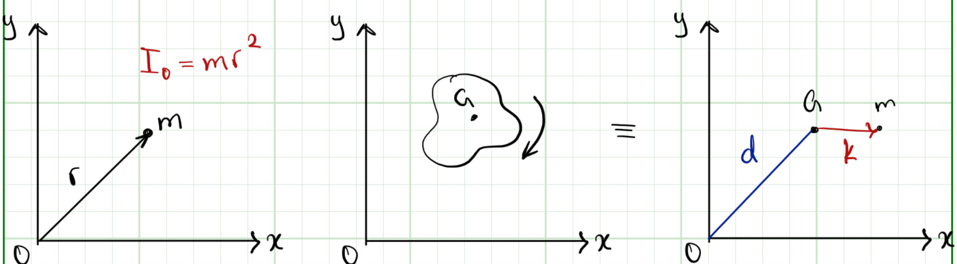


$$I_G = \frac{1}{12} m\ell^2$$

$$I_O = \frac{1}{12} m\ell^2 + m\left(\frac{\ell}{2}\right)^2 = \frac{1}{3} m\ell^2$$

Radius of Gyration

* $k \rightarrow$ radius that can be used to calculate I for irregular shapes



$$I_G = mk^2$$

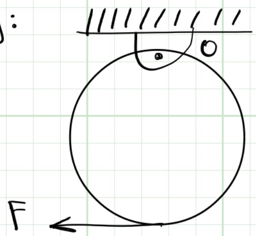
Summary: $\Sigma \vec{F} = m\vec{a}_G$

O is fixed pt: $\Sigma \vec{M}_O = I_O \vec{\alpha}$

G is centre of mass: $\Sigma M_G = I_G \vec{\alpha}$

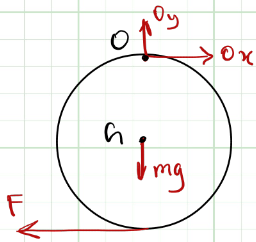
$$I_O = I_G + md^2$$
$$= mk^2 + md^2$$

→ Eg:



$m = 6\ kg$
$R = 0.16m$
$\vec{F} = -20i\ N$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \vec{\alpha} = ?$



$$\Sigma \vec{F_x} = m(\vec{a_x})$$

$$\Sigma M_o = I_o \vec{\alpha}$$

$$-F(2R) = (I_G + md^2)\vec{\alpha}$$

$$-F(2R) = \left(\tfrac{1}{2}mR^2 + mR^2\right)\vec{\alpha}$$

$$\vec{\alpha} = \frac{-2FR}{\frac{3}{2}mR^2} = -27.8\ rad/s^2$$

Section 6.5 (see Lecture notes 25)

- General Plane Motion <u>RB</u>

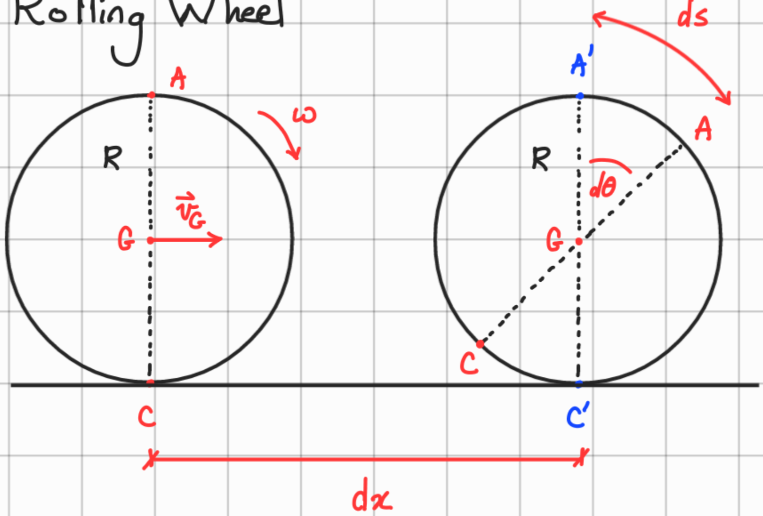
Lecture 25

Section 6.5 - General Plane Motion

In Kinetics of Rigid bodies

* General Plane Motion

↳ Rolling Wheel



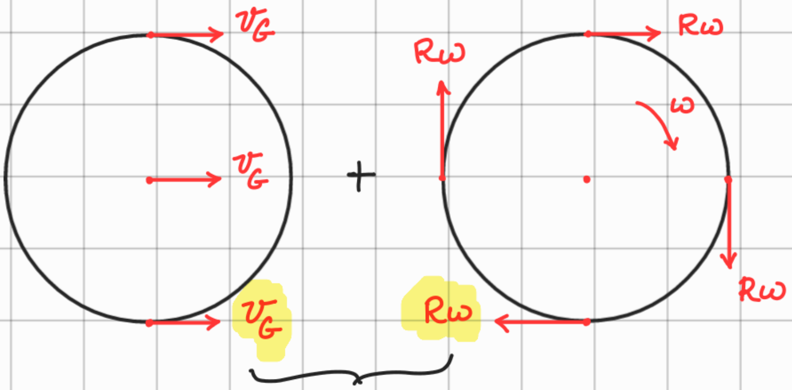
$ds = R\,d\theta$

$v_G = \dfrac{dx}{dt} \Rightarrow dx = v_G\,dt$

That is everything we know about the system so far

⇒ We understand that rolling is a type of general plane motion

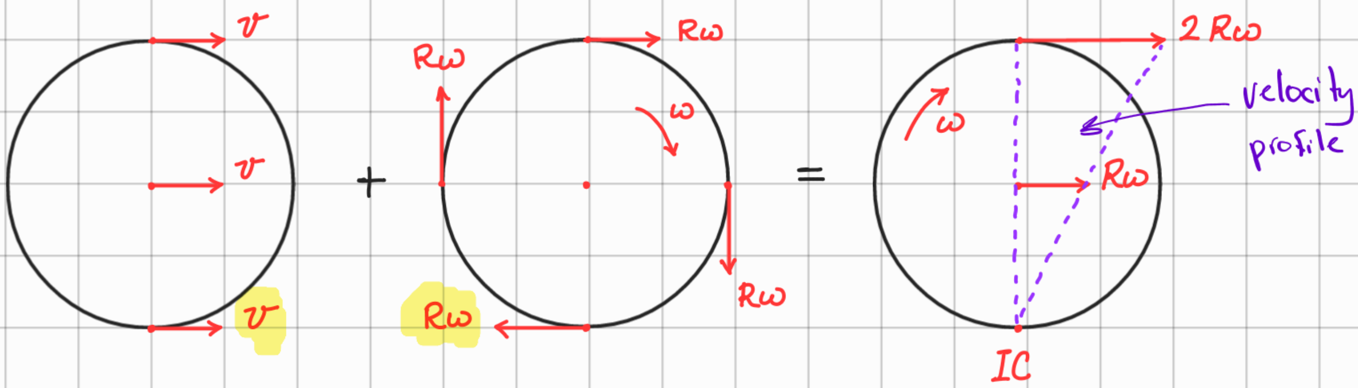
⇒ Rolling = Translation + Rotation



These vectors are counteracting one another. Their magnitudes will determine whether this motion involves slipping or not

(i) Rolling without slipping $\Rightarrow dx = ds \Rightarrow v\,dt = R\,d\theta \Rightarrow v = R\dfrac{d\theta}{dt} = R\omega$

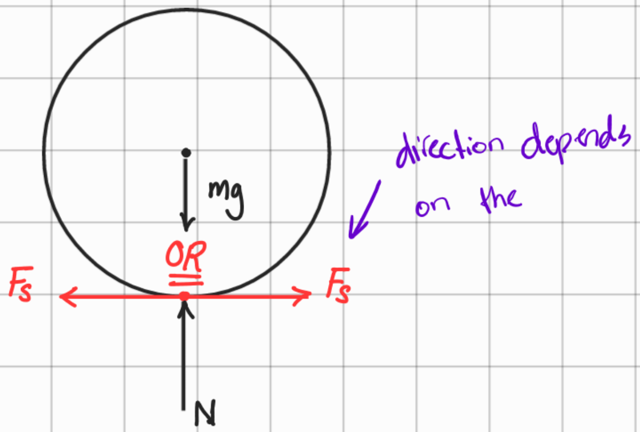
$\hookrightarrow v = v_G$ for simplicity



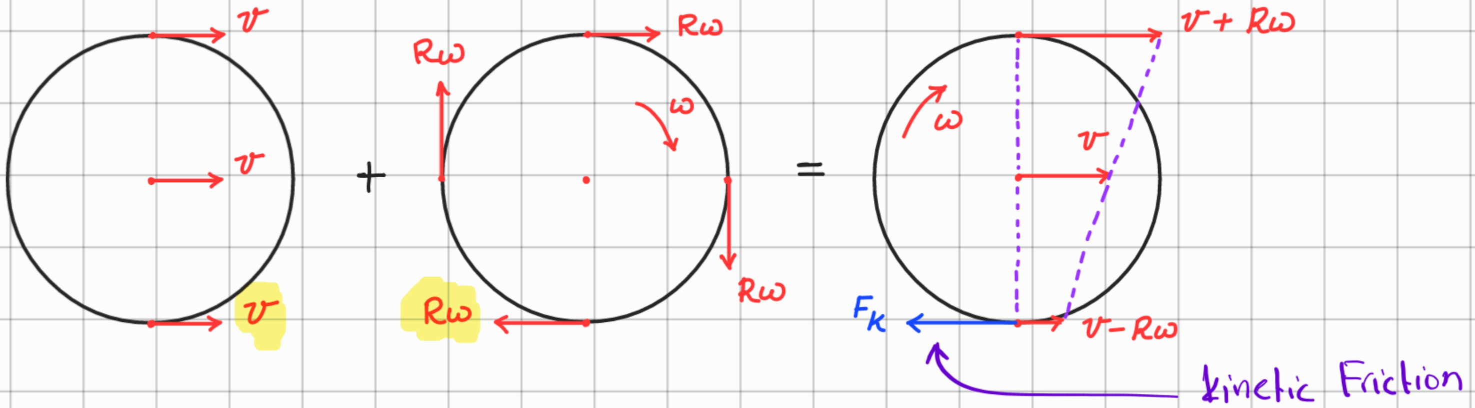
$\Rightarrow$ No slipping $\longrightarrow$ Static friction

$\Rightarrow$ Magnitude and direction of F_s

$\hookrightarrow \Sigma F, \Sigma M \rightarrow \Sigma M = I\alpha$



(ii) Forward slipping $\Rightarrow dx > ds \Rightarrow v > R\omega$

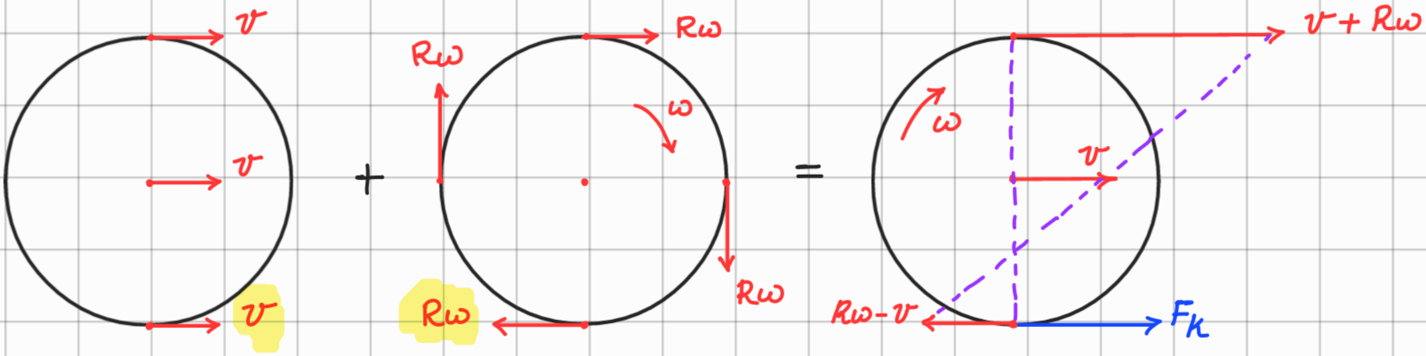


$\Rightarrow$ In this scenario, there is extra motion forward at the point of contact.

$\Rightarrow F_k$ is counteracting this motion

$\Rightarrow F_k$ is supporting (in the same direction of) rotational motion

(iii) Backward slipping $\Rightarrow dx < ds \Rightarrow v < R\omega$



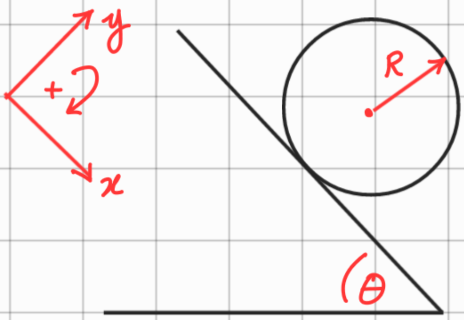
$\Rightarrow$ In this scenario, the contact point has a velocity against the motion

$\Rightarrow$ F_K is counteracting this velocity

$\Rightarrow$ F_K is <u>not</u> supporting the rotational motion

---

Example:



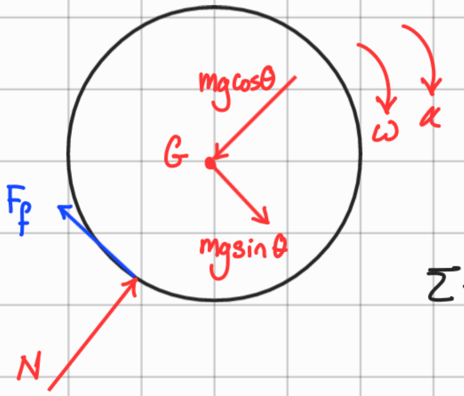
Given: $R = 15\ cm$ $\quad \mu_s = 0.3$

$m = 0.5\ Kg$ $\quad \mu_K = 0.2$

$\theta = 40°$

A homogenous wheel is released from rest

↳ Find $\vec{a}_G$ and $\vec{F}_f$?



$\Rightarrow$ We don't know if the wheel is slipping or not

$\Rightarrow$ Assume no slipping

$$\Sigma F_y = 0 \Rightarrow N = mg\cos\theta$$

$$\left.\begin{aligned} &\Sigma F_x = m a_G \Rightarrow mg\sin\theta - F_f = m a_G \\ &\Sigma M_G = I_G \alpha \Rightarrow F_f (R) = \frac{1}{2} m R^2 \alpha \end{aligned}\right\} \begin{array}{l} 2 \text{ equations} \\ 3 \text{ unknowns} \end{array}$$

$\Rightarrow$ In this case, you can resort to Kinematics to complement your equations

$$\Rightarrow \vec{a}_G = \cancelto{0}{\vec{a}_{IC}} + \underbrace{\left(\vec{a}_{G/IC}\right)_t}_{= R\alpha} + \cancelto{0}{\left(\vec{a}_{G/IC}\right)_n}$$

$$\Rightarrow \vec{a}_G = R\alpha\, \hat{i}$$

$$\left.\begin{array}{l} \Sigma F_x = m a_G \Rightarrow mg\sin\theta - F_f = mR\alpha \\ \Sigma M_G = I_G\alpha \Rightarrow F_f(R) = \frac{1}{2} mR^2\alpha \end{array}\right\} \begin{array}{l} F_f = 1.05\ N \\ \alpha = 28\ rad/s^2 \end{array}$$

$\Rightarrow$ Check Assumption : $F_{s,max} = \mu_s N = \mu_s mg\cos\theta = 1.128\,N > 1.05\ N \rightarrow$ Valid assumption

$$\Rightarrow \vec{a}_G = R\alpha = (0.15)(28) = 4.2\,\hat{i}\ \frac{m}{s^2}$$

* What if our assumption was wrong?

$$\Rightarrow F_f = \mu_k N = \mu_k mg\cos\theta$$

$$\Rightarrow a_G \neq R\alpha \longrightarrow \begin{array}{l} \Sigma F_x = m a_G \\ \Sigma M_G = I_G\alpha \end{array}$$

General Plane Motion — Equations for 2D Kinetics :

$\Sigma\vec{F} = m\vec{a}_G$

$\Sigma\vec{M}_G = I_G\vec{\alpha}$

$\Sigma\vec{M}_O = I_O\vec{\alpha}$

IC is also a fixed point $\Rightarrow \Sigma M_{IC} = I_{IC}\vec{\alpha}$

K : Radius of gyration $\rightarrow I = mK^2$

Parallel-axis theorem $\rightarrow I_O = I_G + md^2$

$\vec{v}_A = \vec{v}_B + \vec{v}_{A/B}$; $|\vec{v}_{A/B}| = |\vec{r}_{A/B}|\,\omega$

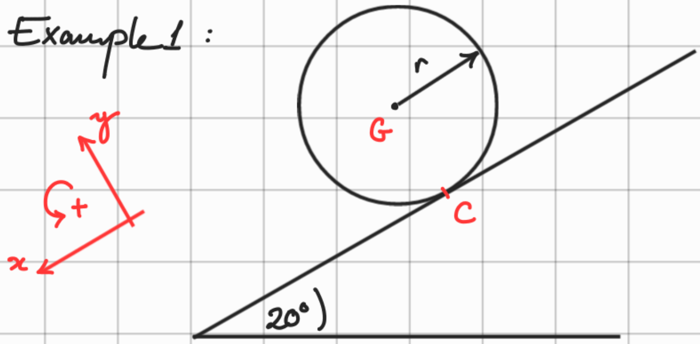
$$|\vec{\omega}| = \frac{|\vec{v}_A - \vec{v}_B|}{|\vec{r}_{A/B}|}$$

$$\vec{a}_A = \vec{a}_B + \overbrace{\underbrace{\left(a_{A/B}\right)_t}_{\perp \vec{r}_{A/B}}}^{\pm r\alpha\,\hat{e}_t} + \overbrace{\underbrace{\left(\vec{a}_{A/B}\right)_n}_{\text{from } A \text{ to } B}}^{r\omega^2\,\hat{e}_n}$$

Lecture 26

Continue Section 6.5 – General Plane Motion in Kinetics of Rigid Bodies

Example 1 :



A hoop released from rest.

$\mu_s = 0.15$; $\mu_k = 0.12$; $r = 60$ cm

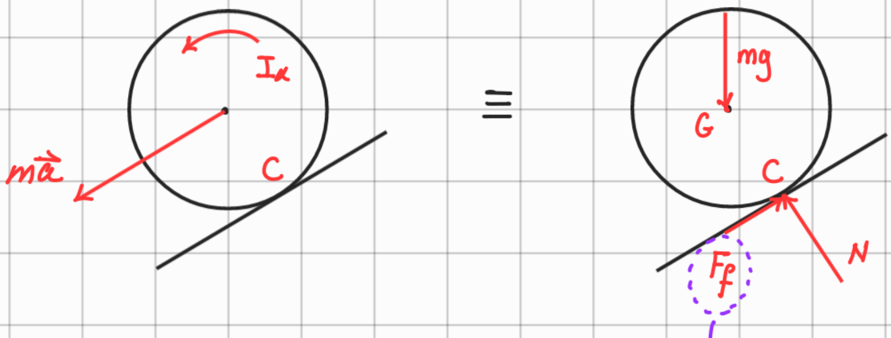
Find α and the time, t for the hoop to move a distance 10 m down the incline

$\Rightarrow$ Assume the hoop rolls without slipping $\Rightarrow \vec{a} = r\alpha$

$\Rightarrow$ α is counterclockwise

$\Rightarrow$ Draw free body diagram

Friction causes moment and ∴ induces α.



This direction of F_f is the only direction to create CCW for α

$\Rightarrow \Sigma F_x = m\vec{a}_x$

$mg\sin 20° - F_f = m\vec{a} \rightarrow$ eq 1

$\Rightarrow \Sigma M_G = I\vec{\alpha}$ $\quad a = r\alpha$

$F_f r = mr^2\alpha$

$F_f = mr\alpha = m\vec{a} \rightarrow$ eq 2

Add both equations

$\Rightarrow \cancel{m}g\sin 20° = 2\cancel{m}\vec{a}$

$\Rightarrow \vec{a} = \dfrac{g\sin 20°}{2}$

$\Rightarrow$ Plug the value for $\vec{a}$ in eq 1 : $F_f = mg\sin 20° - m\dfrac{g\sin 20°}{2} = 0.171\,mg$

$\Rightarrow \Sigma F_y = m\vec{a}_y = 0 \longrightarrow N - mg\cos 20° = 0 \longrightarrow N = mg\cos 20° = 0.94\,mg$

$\Rightarrow F_{f\,max} = \mu_s N \rightarrow F_{f\,max} = 0.15\,(0.94\,mg) = 0.141\,mg$

⇒ Since the evaluated $F_f > F_{f\,max}$ → Slippage occurs → $\vec{a} \neq r\vec{\alpha}$

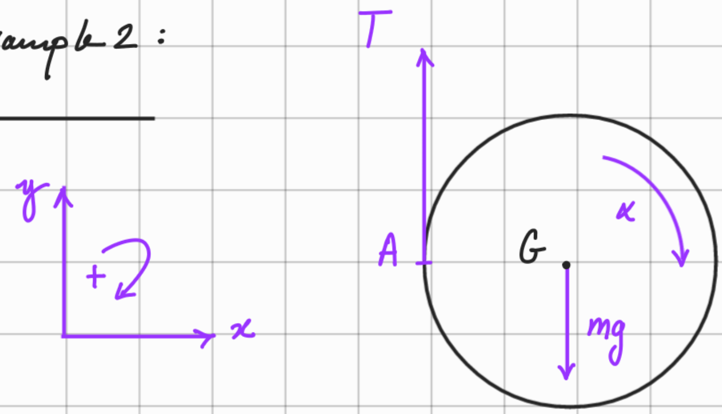
⇒ F_f then becomes the Kinetic value → $F_f = \mu_k N = 0.12(0.94\,mg) = 0.1128\,mg$

⇒ $\Sigma F_x = m\vec{a}_x$ → $mg\sin 20° - 0.1128\,mg = m\vec{a}$ → $a = ✓$

⇒ $\Sigma M_G = I\alpha$ → $F_f\, r = mr^2\alpha$ → $0.1128\,mg\,r = mr^2\alpha$ → $\alpha = ✓$

⇒ $x = \cancel{x_0}^{\,0} + \cancel{v_0}^{\,0} t + \frac{1}{2}at^2$ → $t = \sqrt{\frac{2x}{a}} = ✓$

Example 2 :



→ Uniform disk ; mass = m
Radius = R

→ Rope unwound around disk, rope is being lifted upwards with tension T and speed v

→ v is increasing by 3 m/s²

→ Find α and T ?

⇒ A is not IC (Consider the disk rolling horizontally on a surface, but the floor is moving in the opposite direction beneath it)

⇒ $\Sigma M_G = I_G\alpha$ → $\boxed{TR = \frac{1}{2}mR^2\alpha}$ → eq 1

when you run out of options, you use kinematics

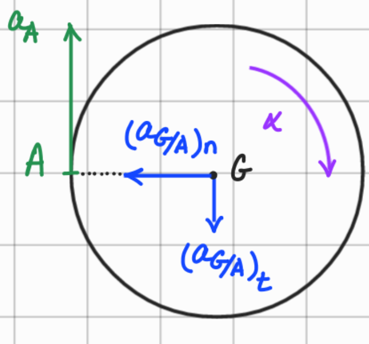
⇒ $\Sigma F_y = m(a_G)_y$ → $\boxed{T - mg = m(a_G)_y}$ → eq 2

path of G is horizontal, so radius of curvature is ∞

because G is not rotating

$3\hat{j}$ m/s² $R\alpha$

⇒ $\underbrace{\vec{a}_G}_{\hat{j}} = \underbrace{\vec{a}_A}_{\hat{j}} + \underbrace{(\vec{a}_{G/A})_t}_{-\hat{j}} + \cancel{(\vec{a}_{G/A})_n}^{\,0}$ → $\boxed{\vec{a}_G = 3\hat{j} - R\alpha\hat{j}}$ → eq 3



⇒ Solve 3 equations → $T = \frac{m(3+g)}{3}$; $\alpha = \frac{2(3+g)}{3R}$

Section 6.6 – Work/Energy Relations

1. Pure Translation : $T = \frac{1}{2} m V_G^2$

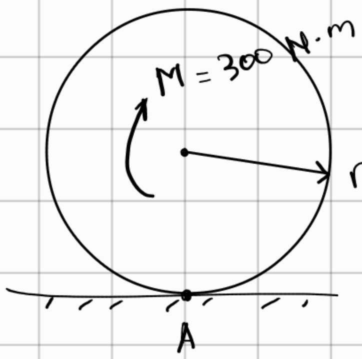
2. Pure Rotation : a) about mass centre (G) — $T = \frac{1}{2} I_G \omega^2$
b) about point O — $T = \frac{1}{2} I_O \omega^2$

3. General Plane Motion : $T = \frac{1}{2} m V_G^2 + \frac{1}{2} I_G \omega^2$

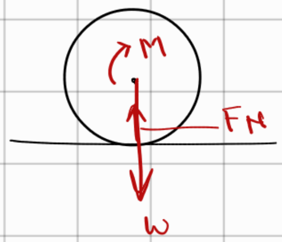
Section 6.8 – Impulse and Momentum

- linear momentum of rigid body : $\vec{G} = m\vec{V}_G$
- angular momentum of rigid body : $\vec{H}_G = I_G \omega$ or $\vec{H}_O = I_O \omega$

→ Eg: $m = 300$ kg, $k_O = 0.4$ m, $M = 300$ N·m, velocity after 6 s = ?



$\underline{\underline{I_O = m k_O^2}}$



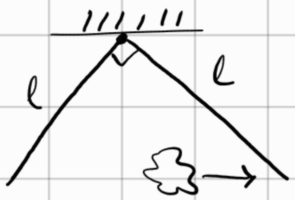
$$(H_O)_2 = \cancel{(H_O)_1}^{\,0} + \int_{t_1}^{t_2} \Sigma M_O \, dt$$

$$I_A \omega = M_O \Delta t$$
$$I_G \omega + m V_G r_G = M_O \Delta t$$
$$I_G \omega + m \omega r_G^2 = M_O \Delta t$$
$$m k_O^2 \omega + m \omega r_G^2 = M_O \Delta t$$

$$\omega m (k_O^2 + r_G^2) = M_O \Delta t$$
$$\omega = \frac{M_O \Delta t}{m(k_O^2 + r_G^2)} = 11.5 \text{ rad/s}$$

Eg: $m = 1\,kg$, $\ell = 1\,m$, cheese $= 0.1\,kg$, $V_{cheese} = 10\,m/s$

Find ω after impact (cheese sticks)



Before Impact

$$I_o = 2(I_G + md^2)$$
$$I_o = 2\left(\frac{1}{12}m\ell^2 + m\left(\frac{\ell}{2}\right)^2\right)$$
$$I_o = 2\left(\frac{1}{3}m\ell^2\right)$$
$$(I_o)_1 = \frac{2}{3}m\ell^2$$

After Impact

$$I_o = (I_o)_{rod} + (I_o)_{rod}$$
$$I_o = \frac{2}{3}m\ell^2 + m_{cheese}r^2$$
$$I_o = \frac{2}{3}m\ell^2 + m_{chc}\ell^2$$
$$(I_o)_2 = \ell^2\left(\frac{2}{3}m + m_{ch}\right)$$

Conservation of Angular Momentum

$$(H_o)_1 = (H_o)_2$$
$$m_{chc}\, r V_{\ell} = I_o\,\omega^2$$
$$\omega = \sqrt{\frac{m_{ch}\, r V_{\ell}}{I_o}}$$

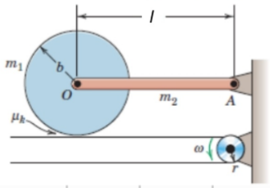
$V_{\ell} = V\cos 48^\circ$

Eg 2:

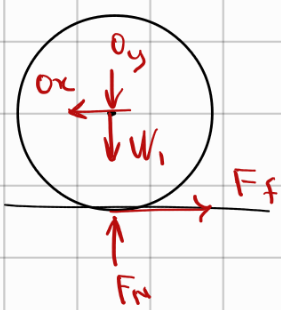
The initially stationary uniform disk of mass m_1 and radius b is allowed to drop onto the moving belt from a very small elevation. Determine the time t required for the disk to acquire its steady-state angular speed. The belt drive pulley rotates with a constant **clockwise** (opposite to what is shown in the figure) angular velocity ω. Ignore any sag in the belt.

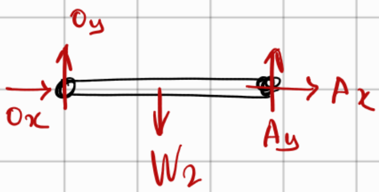
m_1, m_2

$r = b$, $\omega = 0$, $I_o = \frac{1}{3}m r^2$



steady state, so $\frac{d}{dt} = 0$ $\quad \frac{d\omega}{dt} = \alpha = 0$, M is constant





$$+\circlearrowleft \Sigma M_A = 0$$
$$0 = \frac{\ell}{2}W_2 - O_y\ell$$
$$O_y = \frac{W_2}{2}$$

$$+\uparrow \Sigma F_y = 0 - O_y - W_1 + F_N$$
$$F_N = O_y + W_1 = \frac{W_2}{2} + W_1$$

$$\rightarrow F_f = \mu_k(O_y + W_1) = \mu_k\left(\frac{W_2}{2} + W_1\right)$$

Apply Angular Momentum / Impulse

$$\cancelto{0}{(H_o)_1} + \int_{t_1}^{t_2} M_o\,dt = (H_o)_2$$

$$M\Delta t = I_o\omega_2$$

$$\mu_k\left(\frac{w_2}{2}+w_1\right) r\,\Delta t = I_o\omega_2$$

$$\Delta t = \frac{I_o\omega_2}{\mu_k\left(\frac{W_2}{2}+W_1\right) r}$$

$$\Delta t = \frac{\frac{1}{2}mr\,\omega_2}{\mu_k\left(\frac{m_2 g}{2}+m_1 g\right)}$$

Section 8.2 – Free Vibrations of Particles

Vibrations
- Free Vibrations (only gravity or elastic)
  - damped
  - undamped
- Forced Vibrations (externa force)
  - damped
  - undamped

Free Vibrations — Undamped

$$\Sigma F_x = ma_x$$

$$-kx = m\ddot{x}$$

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega_n^2 x = 0$$

Solutions: $\ddot{x} + \omega_n^2 x = 0$

$$x = A\cos\omega_n t + B\sin\omega_n t$$

$$x = C\sin(\omega_n t + \psi)$$

$$x_0 = A = C\sin\psi$$

$$\dot{x}_0 = B\omega_n = C\omega_n\cos\psi$$

$$C = \sqrt{A^2+B^2}$$

$$\psi = \arctan\left(\frac{A}{B}\right)$$

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$$

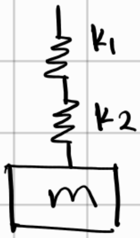
$$\boxed{T = \frac{2\pi}{\omega_n}}$$

- Period $\boxed{T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}}}$
- Frequency $\boxed{f_n = \frac{1}{T}}$

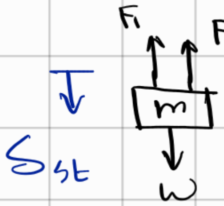
Eg: Determine equivalent stiffness of each scenario

(a)

(b)



(a) Parallel



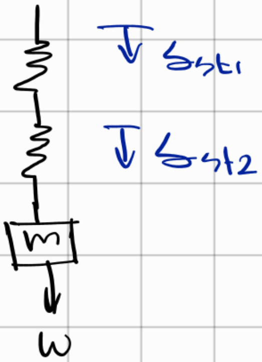
$F_{eq} = F_1 + F_2$ $\qquad F_s = k\,\delta_{st}$

$k_{eq}\,\cancel{\delta_{st}} = k_1\,\cancel{\delta_{st}} + k_2\,\cancel{\delta_{st}}$

$k_{eq} = k_1 + k_2$

$\therefore\ \omega_n = \sqrt{\dfrac{k_1+k_2}{m}}$

(b)



$F_{eq} = F_1 = F_2 = F$ $\qquad \delta_{st} = \dfrac{F}{k}$

total $\delta_{st} = \delta_{st_1} + \delta_{st_1}$

$$\frac{\cancel{F}}{k_{eq}} = \frac{\cancel{F}}{k_1} + \frac{\cancel{F}}{k_2}$$

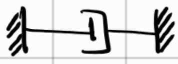
$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2}$$

$\therefore\ \omega_n = \sqrt{\dfrac{k_{eq}}{m}}$

Free Vibrations — Damped

Viscous Damper Sketch :

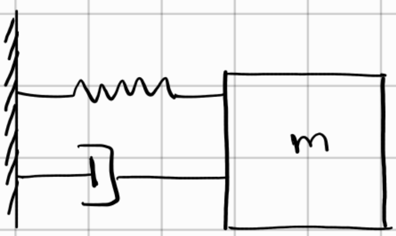


$\Sigma F_x = ma$

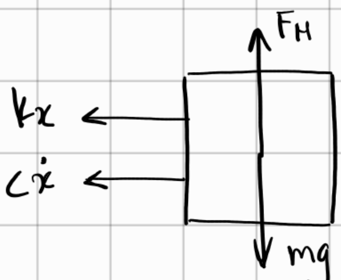
$m\ddot{x} = -kx - c\dot{x}$

$m\ddot{x} + c\dot{x} + kx = 0$

Viscous Damping Force : $F_c = c\dot{x}$



$\Longrightarrow$



Standard Form $\longrightarrow$ $m\ddot{x} + c\dot{x} + kx = 0$

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0}$$

$$\boxed{\zeta = \frac{c}{2m\omega_n}} \qquad \boxed{\omega_n = \sqrt{\frac{k}{m}}} \qquad \boxed{\text{critical damping coefficient}: C_c = 2m\omega_n}$$

Char. Poly. Roots

$$\boxed{\begin{aligned} \lambda_1 &= \omega_n\left(-\zeta + \sqrt{\zeta^2-1}\right) \\ \lambda_2 &= \omega_n\left(-\zeta - \sqrt{\zeta^2-1}\right) \end{aligned}}$$

General Solution

$$\boxed{x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}}$$

Solution Types

- overdamped, $\zeta > 1$
- critically damped, $\zeta = 0$
- underdamped, $\zeta < 1$

Overdamped ($\zeta > 0$)

- λ_1 and λ_2 real, distinct, negative

Underdamped ($\zeta < 1$)

- λ_1 and λ_2 imaginary

$$\boxed{x = C\sin(\omega_d t + \psi)e^{-\zeta\omega_n t}}$$

Critically Damped ($\zeta = 1$)

$$\boxed{\begin{aligned} &\cdot\ \lambda_1 = \lambda_2 = -\omega_n \\ &\cdot\ x = (A_1 + A_2 t)e^{-\omega_n t} \end{aligned}}$$

- damped natural frequency $\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$
- damped period

$$\boxed{\tau_d = \frac{2\pi}{\omega_d}}$$

8.4 — Vibration of Rigid Bodies

replace

x — α

$\dot{x}$ — $\dot{\theta}$

$\ddot{x}$ — $\ddot{\theta}$

m — I_o

For particle : $\Sigma F_x = ma_x$

For rigid bodies: $\Sigma M_o = I_o\alpha$

$\Sigma M_o = I_o\ddot{\theta}$

8.3 — Forced Vibrations