MAT291 Main Concepts

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Part 1A: Limits and Continuity

Limits and Limit Laws

- $\boxed{\lim_{(x,y)\to(a,b)} f(x,y)}$ exists if limit is the same along any path in the domain. Otherwise, limit DNE

- What is a path?
  1) $y = h(x)$, f is continuous and $h(a) = b$
  2) $x = g(y)$, g is continuous and $g(a) = b$
  3) $(x,y) = (f(t), g(t))$ and $(a,b) = (f(t_0), g(t_0))$

- If $\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = L}$ exist, then for any path,

  $\rightarrow y = h(x) \implies \lim_{x\to a} f(x, h(x)) = L$

  $\rightarrow x = g(y) \implies \lim_{y\to b} f(g(y), y) = L$

  $\rightarrow (x,y) = (g(t), h(t)) \implies \lim_{t\to t_0} f(g(t_0), h(t_0)) = L$

- To show limit exists at a point
  - plug in the values if defined
  - basic limit laws (split limits by sum/product)
  - algebraic manipulation (multiply by conjugate, factoring)
  - change of variables

- To show limit DNE at a point
  - find a path along which limit DNE
  - find two paths along which limits exist but are not equal

Continuity

A function $f(x,y)$ is continuous at a point (a,b) if
1) f is defined at (a,b)
2) $\lim_{(x,y)\to(a,b)} f(x,y)$ exists
3) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

- continuity means no weird jumps in the graph
- there are infinite paths to consider in multivariables, so continuity in multivariables is more strict than single variables

- Properties of continuous functions
  1) Sums and differences of cont. func^{n}s are cont.
  2) Products of cont. func^{n}s are cont.
  3) Quotients of cont. func^{n}s are cont. at all pnts. where denominator is not zero.
  4) Compositions of cont. func^{n}s are cont.
  5) Projection functions are cont. $f_x(x,y) = x$ $\quad f_y(x,y) = y$

Part 1B: Differentiation

Partial Derivatives

- Given $f(x,y)$:

$$\boxed{\frac{\partial}{\partial x} f(a,b) = \lim_{t\to 0} \frac{f(a+t,b) - f(a,b)}{t}} = f_x(a,b)$$

$$\boxed{\frac{\partial}{\partial y} f(a,b) = \lim_{t\to 0} \frac{f(a,b+t) - f(a,b)}{t}} = f_y(a,b)$$

- The partial is change in f with change in one variable while other variable(s) are held constant

Implicit Differentiation

- Given an implicit function $F(x,y)=0$: $\boxed{\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}}$

Higher Order Partials

pure partials

$$\frac{\partial f}{\partial x} \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial f}{\partial y} \Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

mixed partials

Mixed Partials

- Clariot's Theorem: If f is continuous and 2^{nd} order partials of f are continuous in a neighborhood of (a,b), then order of mixed partial does not matter

$$\frac{\partial^2}{\partial x \partial y} f(a,b) = \frac{\partial^2}{\partial y \partial x} f(a,b)$$

Tangent Planes

- $F(x,y,z)=0$: $F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(z-c) = 0$
- $z = f(x,y)$: $z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$

Differentiabity

We say $f(x,y)$ is differentiable at (a,b) if the tangent plane is a good approx. of f around (a,b):

1) the partials at (a,b) exist : $f_x(a,b)$ and $f_y(a,b)$ exist
2) $$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L_{(a,b)}(x,y)}{\|(x,y)-(a,b)\|} = 0$$

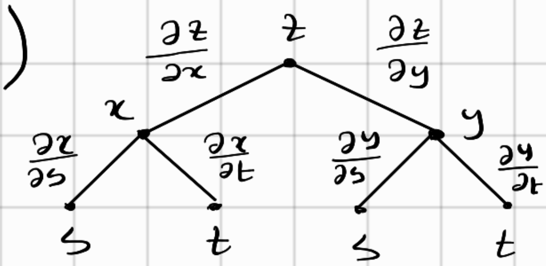
where $L_{(a,b)}(x,y) = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + f(a,b)$

- if partials of f exist in a neighborhood of (a,b) and are continuous on that neighborhood, then f is differentiable at (a,b)
- differentiability implies continuity, but not the other way

Chain Rule

- One indep. var.: $z(x(t), y(t)) = \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

- several indep. var.: $z\big(x(s,t), y(s,t)\big)$



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Directional Derivative

: R.O.C as we move along arbitrary dir. $\vec{u}$

$$\boxed{D_{\vec{u}} f(x_0, y_0) = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}}$$

where direction $\vec{u} = \langle u_1, u_2 \rangle$ and $x(s) = x_0 + su_1$

unit vector $\qquad\qquad\qquad\qquad$ $y(s) = y_0 + su_2$

$$\boxed{D_{\vec{u}} f(x_0, y_0) = \nabla f \Big|_{(x_0,y_0)} \cdot \vec{u} = \left\langle \frac{\partial}{\partial x} f(x_0,y_0), \frac{\partial}{\partial y} f(x_0,y_0) \right\rangle \cdot \langle u_1, u_2 \rangle}$$

- directional derivative is dot product b/n gradient vector and $\vec{u}$

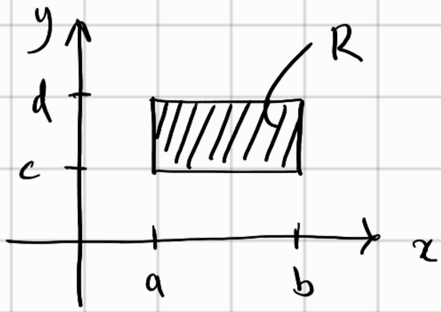
Gradient

The gradient of f at (x,y) is the vector-valued function : $\boxed{\nabla f(x,y) = \left\langle \frac{\partial}{\partial x} f(x,y), \frac{\partial}{\partial y} f(x,y) \right\rangle}$

- Directions of Change : f diff. @ (a,b), $\nabla f(a,b) \neq 0$
  1) gradient is direction of steepest ascent (maximal increase)
  2) gradient is opposite direction of steepest descent (maximal decrease)
  3) gradient is orthogonal to level curve

Part 1C: Integration in Cartesian Coordinates

Double Integrals : Rectangles



Let domain of integration be:

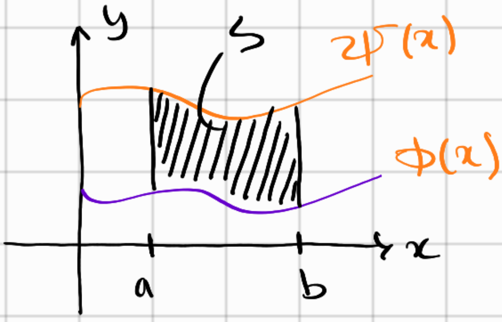
$R = [a,b] \times [c,d]$

Fubini's for Rectangles :

$$\boxed{\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy}$$

Double Integrals : General

A simple region $S \subseteq \mathbb{R}^2$: $S = \{(x,y) : a \leq x \leq b,\ \phi(x) \leq y \leq \psi(x)\}$



$$\boxed{\iint_S f(x,y)\, dA = \int_a^b \int_{\phi(x)}^{\psi(x)} f(x,y)\, dy\, dx}$$

Triple Integrals : Rectangles

$R = [a,b] \times [c,d] \times [e,f]$

$$\iiint_V f(x,y,z)\, dV = \int_a^b \int_c^d \int_e^f f(x,y,z)\, dz\, dy\, dx$$

Triple Integrals : General

$S \subseteq \mathbb{R}^3 : S = \{(x,y,z) : a \leq x \leq b,\ g(x) \leq y \leq h(x),\ \phi(x,y) \leq z \leq \psi(x,y)\}$

$$\iiint_V f(x,y,z)\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{\phi(x,y)}^{\psi(x,y)} f(x,y,z)\, dz\, dy\, dx$$

Part 1D: Integration in Transformed Coordinate System

Change of Variables

Let R be region in xy-plane and T be region in uv-plane.

Let $(x,y) = (g_1(u,v), g_2(u,v))$ where $g(u,v) : T \mapsto R$

$$\boxed{\iint_R f(x,y)\, dx\,dy = \iint_T f(u,v) \left\| \operatorname{Det} \begin{bmatrix} \frac{\partial}{\partial x} g_1 & \frac{\partial}{\partial y} g_1 \\ \frac{\partial}{\partial x} g_2 & \frac{\partial}{\partial y} g_2 \end{bmatrix} \right\| du\,dv}$$

Steps

1) sketch the region in xy plane (initial region)
2) Find the limits of integration for the new integral based on the given transformation formula for u and v, then sketch the new region
3) Calculate the absolute value of Jacobian
4) Change variables and evaluate the new integral

To find a substitution formula

Look at the bounded points/lines/planes

⇒ if they are the same and only shifted w.r.t. each other, then use them as transformation formula

⇒ if the points are representing parallelogram, then derive the equations and use them.

Polar Coordinates $(x,y) = (r\cos\theta, r\sin\theta)$

$$\iint_R f(x,y)\, dA = \iint_{\tilde{R}} f(r,\theta)\, dA \quad \text{where } dA = dx\,dy = r\,dr\,d\theta$$

Cylindrical Coordinates $(x,y,z) = (r\cos\theta, r\sin\theta, z)$

$$\iiint_V f(x,y,z)\, dV = \iiint_V f(r,\theta,z)\, dV \quad \text{where } dV = dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Spherical Coordinates $(x,y,z) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$

$$\iiint_V f(x,y,z)\, dV = \iiint_V f(\rho,\theta,\phi)\, dV \quad \text{where } dV = dx\,dy\,dz = \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

Part 2A: Line Integrals

Curves

a 1D object (string) embedded in 2D or 3D

1) Parametrized: $\left.\begin{aligned}\vec{r}(t) &= \langle x(t), y(t)\rangle \\ \vec{r}(t) &= \langle x(t), y(t), z(t)\rangle\end{aligned}\right\} \quad a \le t \le b$

2) Explicit: $\left.\begin{aligned}\vec{r}(x) &= \langle x, y(x)\rangle \\ \vec{r}(x) &= \langle x, y(x), z(x)\rangle\end{aligned}\right\} \quad a \le x \le b$

3) Level curve: $F(x,y) = C$ ex// $F(x,y) = x^2 + y^2 = 1$

circle radius 1 @ origin

Line Integrals of Scalar Functions

1) Parametrize the curve C: $\vec{r}(t) = \langle x(t), y(t)\rangle$ $a \le t \le b$
2) Compute $ds = \|\vec{r}'(t)\| dt = \sqrt{x'(t)^2 + y'(t)^2}\, dt$
3) Integral:

$$\boxed{\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \|\vec{r}'(t)\|\, dt}$$

Part 2B: Surface Integrals

Surfaces

a 2D object embedded in 3D

1) Parametrized: $\vec{r}(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$ $\begin{aligned} a \le s \le b \\ c \le t \le d\end{aligned}$

2) Explicit: $\vec{r}(x,y) = \langle x, y, z(x,y)\rangle$ $\begin{aligned} a \le x \le b \\ c \le y \le d\end{aligned}$

3) Level curve: $F(x,y,z) = C$ ex// $F(x,y,z) = x^2 + y^2 + z^2 = 1$

sphere radius 1 @ origin

Surface Integrals of Scalar Functions

1) Parametrize the surface S: $\vec{r}(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$

2) Compute: $dS = \left\| \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} \right\| ds\,dt$ $\quad a \le s \le b$, $c \le t \le d$

3) Integral:

$$\boxed{\iint_S f(x,y,z)\, dS = \int_a^b \int_c^d f\big(x(s,t), y(s,t), z(s,t)\big) \left\| \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} \right\| ds\,dt}$$

Part 2C: Vectors

Tangent and Normal Vectors for Curves

Given a curve $C: \vec{r}(t) = \langle x(t), y(t) \rangle$ (proper oriented)

$$\boxed{\text{tangent vector}: \hat{T} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}}$$

normal vector: if $\hat{T} = \langle a, b \rangle$, $\hat{n} = \pm \langle -b, a \rangle$
given level curve $f(x,y) = C$,
then $\hat{n} = \nabla f(x_0, y_0)$ at $\vec{p} = (x_0, y_0)$

Normal Vectors for Surfaces

Given a surface $S: \vec{r}(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$

$$\boxed{\hat{n} = \frac{\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}}{\left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\|}}$$

or

$\hat{n} = \frac{\nabla f}{\|\nabla f\|}$ if S given as a level curve $f(x,y,z) = C$

Part 2D: Circulation along Contour

1) Parametrize the curve C: $\vec{r}(t) = \langle x(t), y(t) \rangle$ $a \leq t \leq b$
2) Compute $\hat{T} ds = \vec{r}'(t)\, dt = \langle x'(t), y'(t) \rangle dt$
3) Integral :

$$\boxed{\int_C \vec{F} \cdot \hat{T} ds = \int_a^b \vec{F}\langle x(t), y(t) \rangle \cdot \vec{r}'(t)\, dt}$$

Part 2E: Flux through Surface

1) Parametrize the surface S: $\vec{r}(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$
2) Compute :

$$\hat{n}\, dS = \left(\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}\right) ds\, dt$$

$a \leq s \leq b$
$c \leq t \leq d$

3) Integral :

$$\boxed{\iint_S \vec{F} \cdot \hat{n}\, dS = \int_a^b \int_c^d \vec{F}\langle x(s,t), y(s,t), z(s,t) \rangle \cdot \left(\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}\right) ds\, dt}$$

Part 2F: Vector Operators

Nabla

input : scalar
output : vector

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$$

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

- nabla operator applied to scalar function gives you the gradient

Divergence : how much vector field is "created" at a point

input : vector
output : scalar

$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_x, F_y, F_z \rangle$$

$$= \frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} F_y + \frac{\partial}{\partial z} F_z$$

Curl : how much "rotation" of vector field at a point

$\begin{cases} \text{mag : speed} \\ \text{dir : axis of rotation} \end{cases}$

input : vector
output : vector

$$\nabla \times \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle F_x, F_y, F_z \rangle$$

Part 2G: Three Major Theorems

Divergence Theorem

Let R be a region in $\mathbb{R}^3$ and $S = \partial R$ be the boundary surface, then:

$$\boxed{\oiint_{\partial R} \vec{F}\cdot\hat{n}\,dS = \iiint_R \nabla\cdot\vec{F}\,dV}$$

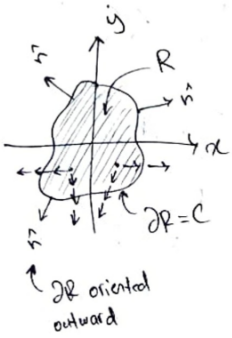
$\leftarrow \partial R = S$ oriented outwards

Green's Theorem (2D Version)

Let R be region in $\mathbb{R}^2$ and $C = \partial R$ be the boundary curve, then:

$$\boxed{\oint_{\partial R} \vec{F}\cdot\hat{n}\,ds = \iint_R \nabla\cdot\vec{F}\,dA}$$

$\leftarrow C = \partial R$ oriented outwards



$$\underbrace{\oint_{\partial R} \vec{F}\cdot\hat{n}\,ds}_{\text{how much vector field leaves the edges}} = \underbrace{\iint_R \nabla\cdot\vec{F}\,dA}_{\text{measures how much vector field is created inside}}$$

Stokes Theorem

Let S be a surface in $\mathbb{R}^3$ (with some orientation) and let $C = \partial S$ be the boundary curve (with induced orientation):

$$\boxed{\iint_S \nabla\times\vec{F}\cdot\hat{n}\,dS = \oint_{C=\partial S} \vec{F}\cdot\hat{T}\,ds}$$

∂S is the part of surface that can give you a papercut

Green's Theorem (2D Version)

Let S be a region in $\mathbb{R}^2$ then:

$$\boxed{\iint_S \text{curl}(\vec{F})\,dA = \oint_{\partial S} \vec{F}\cdot\hat{T}\,ds}$$

∂S has induced orientation

Intuition

$$\underbrace{\iint_S \text{curl}(\vec{F})\,dA}_{\text{total rotation generated in } S} = \underbrace{\oint_{\partial S} \vec{F}\cdot\hat{T}\,ds}_{\text{flow of vector field along } \partial S}$$

$\rightarrow$ (+)ve curl implies CCW rotation

$\therefore$ induced orientation is also CCW

∂S is combination of two boundries

$\leftarrow$ induced orientation CCW on outside and CW on inside