Week 1 : Review of Vectors, Lines, Planes

30.08.2024

Lecture 0 (extracted)

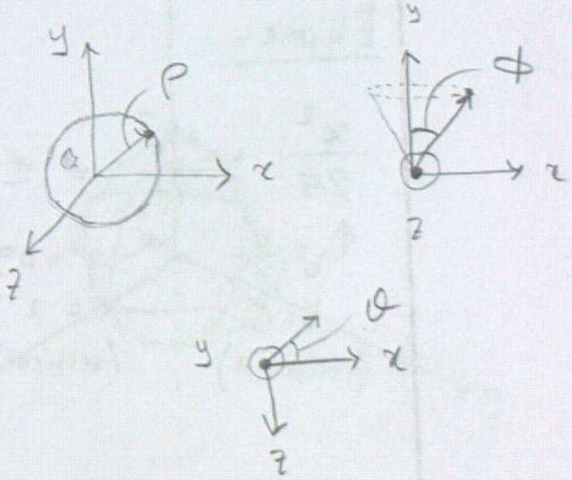
- scalar field: return scalar output from scalar inputs
  eg: temp distr. in a room : $T(x,y) = \cos\left(\frac{x}{2}\right) + \sin\left(\frac{y}{2}\right)$
- Vector field: returns vector w.r.t. position given by scalar values
  eg: $\vec{E}$ given by charge Q : $E(x,y,z) = \frac{Q}{4\pi\varepsilon_0 (x^2+y^2+z^2)^{3/2}} (x\hat{i} + y\hat{j} + z\hat{k})$
- Linear Dependence: Vectors $\vec{v}_1 \cdots \vec{v}_n$ linearly independent if
  $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = 0$ IAOI $\underline{a_1 \cdots a_n = 0}$
- Basis : set of linearly independent vectors spanning entire vector space
- dot product : $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} = \|\vec{u}\| \|\vec{v}\| \cos\theta$
- cross product : $\vec{u} \times \vec{v} = \|\vec{u}\| \|\vec{v}\| \sin\theta = -\vec{v} \times \vec{u}$ (magnitude gives area of parallelogram)
- scalar triple product: $\vec{w} \cdot (\vec{v} \times \vec{u})$ (magnitude gives volume of parallelepiped)
  ↳ cyclical permutation: $w \cdot (v \times u) = v \cdot (u \times w) = u \cdot (w \times v)$
- Vector triple product: $\vec{w} \times (\vec{v} \times \vec{u}) = v(w \cdot u) - u(w \cdot v)$

→ Coordinate Systems:

- Cartesian: $\vec{r} = r_x\hat{x} + r_y\hat{y} + r_z\hat{z}$
- Cylindrical: $\vec{v} = \vec{v}_r\hat{r} + \vec{v}_\theta\hat{\theta} + \vec{v}_z\hat{z}$
- Spherical : $V_\rho\hat{\rho} + V_\phi\hat{\phi} + V_\theta\hat{\theta}$

> “A coordinate system of an n-dimentional space is a system at which a point in space is defined by the intersection point of n objects having (n−1) dimensional form”

Spherical System



13.5 Lines and Planes in Space

- Vector equation of a line:

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle \quad \text{or} \quad r = r_0 + tV$$

- parametric equation: $l \begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$

Eg 1: given $P_0(1,2,4)$ and dir. vec. $V = \langle 5,-3,1 \rangle$

$$\langle x, y, z \rangle = \langle 1,2,4 \rangle + t\langle 5,-3,1 \rangle \quad , \ t \in \mathbb{R}$$

$$l \begin{cases} x = 1 + 5t \\ y = 2 + -3t \\ z = 4 + t \end{cases}$$

- To generate line segment, restrict value of "t"

13.6 Cylinders and Quadric Surfaces

- Cylinders and Traces, Quadric Surfaces
- Know: sphere, ellipsoid, paraboloid, hyperboloid
- cylinder: surface parallel to a line
- trace: set of points (line) where surface intersects coordinate plane

Ellipses

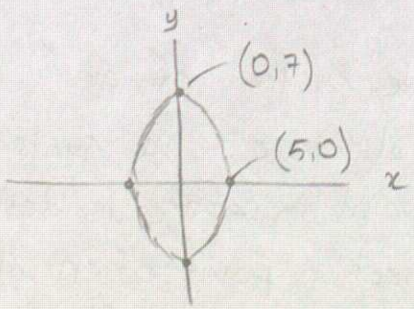
$$\frac{x^2}{25} + \frac{y^2}{49} = 1$$

$b^2 = 25$, $b = 5$ (horizontal)

$a^2 = 49$, $a = 7$ (vertical)

$$\boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1}$$

a is larger number



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Hyperbolas

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

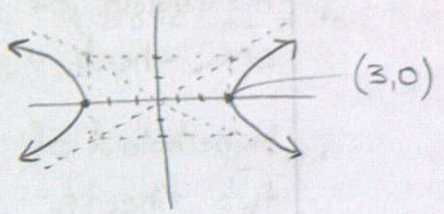
$a^2 = 9$, $a = 3$

$b^2 = 4$, $b = 2$

$$\boxed{\frac{(x-h)^2}{a^2} - \frac{(y-h)^2}{b^2} = 1}$$

a is the first one, or a is the one with the positive, b with negative

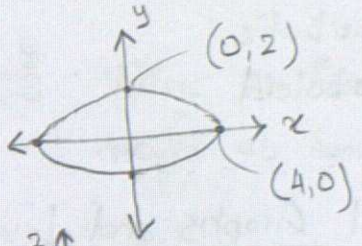
- if x first 
- if y first 



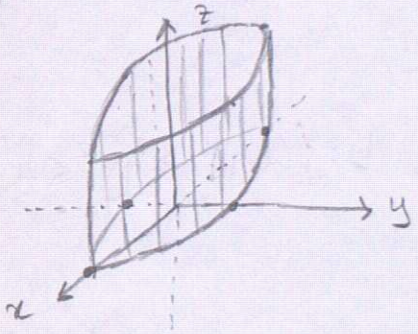
Eg 1: a) $x^2 + 4y^2 = 16$

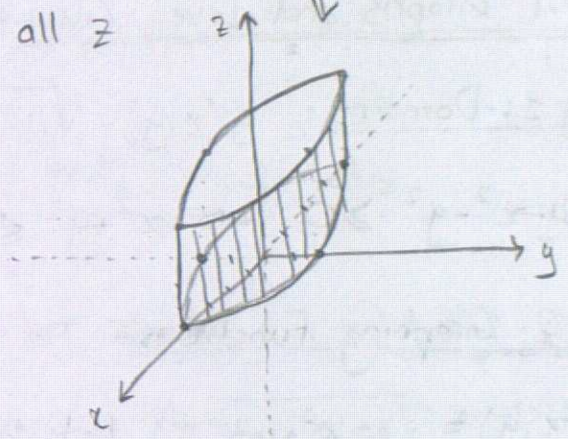
$$x^2 + 4y^2 = 16$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1 \rightarrow \frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$$



- missing variable z, so can extend all z





Eg 2: Ellipsoid

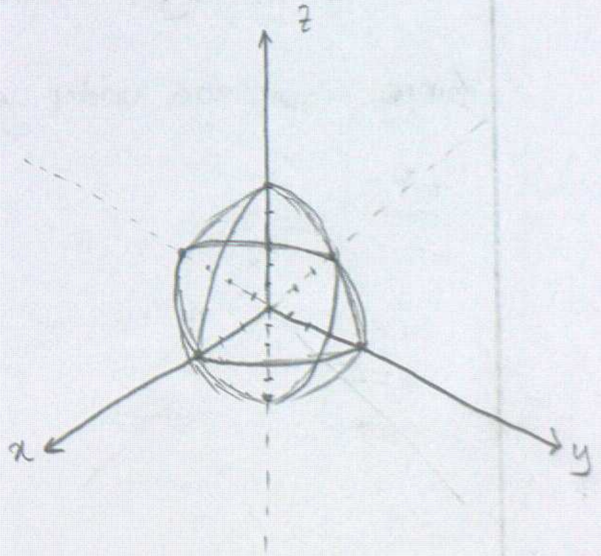
$$\frac{x^2}{3^2} + \frac{y^2}{4^2} + \frac{z^2}{5^2} = 1$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$$

xy trace (no z): $\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$

xz trace (no y): $\frac{x^2}{3^2} + \frac{z^2}{5^2} = 1$

yz trace (no x) = $\frac{y^2}{4^2} + \frac{z^2}{5^2} = 1$



- Eliptic Paraboloid $z = \frac{x^2}{a} + \frac{y^2}{b^2}$
- Hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Hyperboloid of two sheets $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- Hyperbolic paraboloid $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

15.1 Graphs and Level Curves

Eg 1: Domains: $g(x,y) = \sqrt{4-x^2-y^2}$

$$4-x^2-y^2 \geqslant 0 \Rightarrow x^2+y^2 \leq 4, \quad D: \{(x,y): x^2+y^2 \leq 4\}$$

Eg 2: Graphing Functions

$h(x,y) = \sqrt{1+x^2+y^2}$, Let $h(x,y) = z$

hyberboloid of 2 sheets, opens along z-axis

$z = \sqrt{1+x^2+y^2}$ Now square both sides

$$z^2 = 1+x^2-y^2 \Rightarrow \boxed{-x^2-y^2+z^2 = 1}$$

- Since expression under sqrt > 0, we only take the top hyperbola

Sep 4, 2024

Lecture 1

- scalars: you can use numbers to represent things, count things, but these can be different. eg: 4 cats v.s. 4 dollars, these use scalar "4" but they're diff. "units"
  - → operations:
    - addition /subtraction
    - multiplication /division
    - exponential
- Scalar functions

$$\underbrace{\text{input}}_{\text{scalar}} \longrightarrow \boxed{\quad f \quad} \longrightarrow \underbrace{\text{output}}_{\text{scalar}}$$

eg:
- Voltage as function of time
- Height as function of x

- Subsets of Scalars
- used in calculus: open sets and closed sets
- Vectors
- a vector is an ordered pair of scalars

$$\underbrace{(x,y)}_{\text{2d vector}} \qquad \underbrace{(x,y,z)}_{\text{3d vector}} \qquad \underbrace{(a,b,c,d)}_{\text{4d vector}}$$

- eg: position, $\underbrace{\text{velocity, force}}_{\text{direction + magnitude}}$

→ (money, apples)

→ (height, weight)

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Lecture 2

Vector Operations

- multiplication
  - ↳ scalar
  - ↳ dot product
  - ↳ cross product
  - ↳ matrix multiplication

} all used to be just one kind of multiplication when using scalars
Vectors → new types of multiplic.

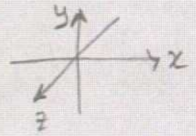
- addition / subtraction

Vector Representations

- $(x, y, z) = x\underbrace{(1,0,0)}_{\hat{x}} + y\underbrace{(0,1,0)}_{\hat{y}} + z\underbrace{(0,0,1)}_{\hat{z}}$

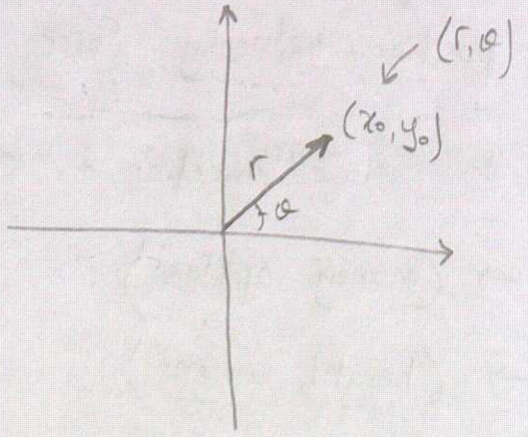
basis of vector spaces

- Graphical



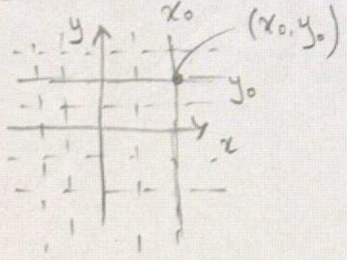
- Alternate Coordinates
  - ↳ polar (2D)
  - ↳ cylindrical (3D)
  - ↳ spherical (3D)

Polar Coord.



- Grid Lines

cartesian



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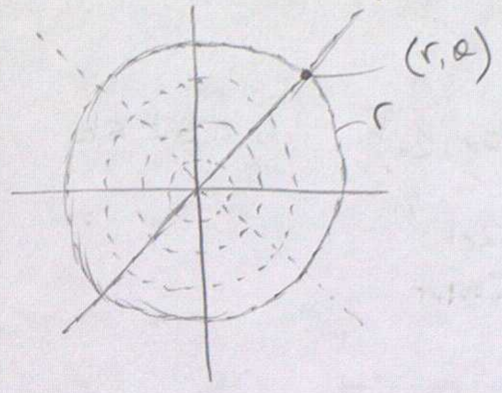
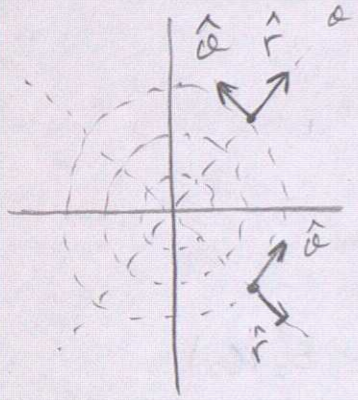
Grid Lines

- polar coordinates



- in 3d, the "grid lines" become "grid surfaces"
  - ↳ spherical coord.
  - ↳ cylindrical coord.

Basis in Other Coordinate Systems



$\hat{r}$ and $\hat{\theta}$ depend on the position

"direction you move when that direction increases"

↳ $\hat{r}$ is direction of movement when r increases

↳ $\hat{\theta}$ is direction of movement when θ increases

Subsets of Vector Spaces

- open and closed sets
- subspaces like lines, planes
- circle (in 2D/3D)
- hollow sphere (in 3D)

Lines

- 1 dimensional line in 2D or 3D

Parametric Repr. : $\underset{\text{direction vector}}{(x_0, y_0, z_0)}\,\underset{\text{parameter}}{t} + \underset{\text{offset}}{(a_0, b_0, c_0)}$ $\quad a \le t \le b$

Equation : $ay + bx + c = 0$ —— 2D

$$\left.\begin{aligned} a_0x + b_0y + c_0z + d_0 &= 0 \\ a_1x + b_1y + c_1z + d_1 &= 0 \end{aligned}\right\} \text{—— 3D}$$

Planes (in 3D)

parameters s, t

Parametric Repres. : $(a_0, b_0, c_0)s + (a_1, b_1, c_2)t + (a_2, b_2, c_2)$

equations : $$\left.\begin{aligned} ax + by + cz + d &= 0 \\ (a,b,c)\cdot(x,y,z) + d &= 0 \end{aligned}\right\} \text{dot product actually}$$

- If "d" = 0, then (a,b,c) normal to plane

Lecture 3 Sep 6, 2024

Multivariable Functions

→ multivariable scalar valued function has multiple inputs with scalar output

$$\underbrace{(x,y)}_{\text{vector/multiple inputs}} \longrightarrow \boxed{\quad f \quad} \longrightarrow \underbrace{f(x,y)}_{\text{scalar}}$$

→ a vector field takes in vector input (multiple inputs) and outputs a vector of the same dimension

$$(x,y) \longmapsto \boxed{\quad f \quad} \longmapsto \underbrace{f(x,y)}_{\text{vector}} = (\underbrace{f_1(x,y)}_{\text{scalar}}, \underbrace{f_2(x,y)}_{\text{scalar}})$$

eg: $(x,y) \mapsto (x+y, x-y)$

Scalar Valued Functions

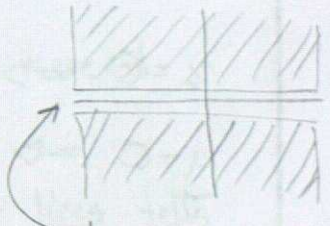
① Eg: $f(x,y) = x + y$ $\quad f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

domain (possible inputs)

codomain (possible outputs)

② Eg: $f(x,y) = \frac{x}{y}$ $\quad f : \mathbb{R}^2 \setminus \{y = 0\} \longrightarrow \mathbb{R}$

minus



domain: everything except y=0

③ Eg: $x^2 + y^2 + z^2 = 1$ ← implicitly defined

$$z = \pm\sqrt{1 - x^2 - y^2}$$

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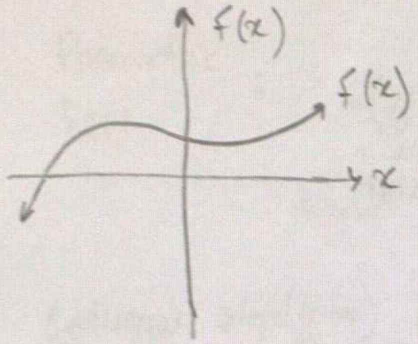
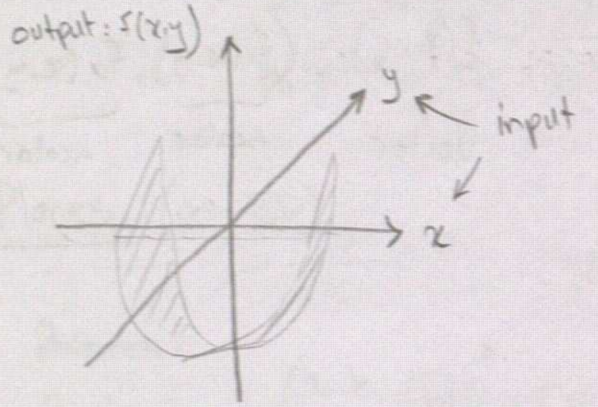
Graphs

- given single variable function $f(x)$



$\Leftarrow$ visual representation of f

- same idea for multivariable $f(x,y)$



$\Leftarrow$ graphs useful representations for 2D functions

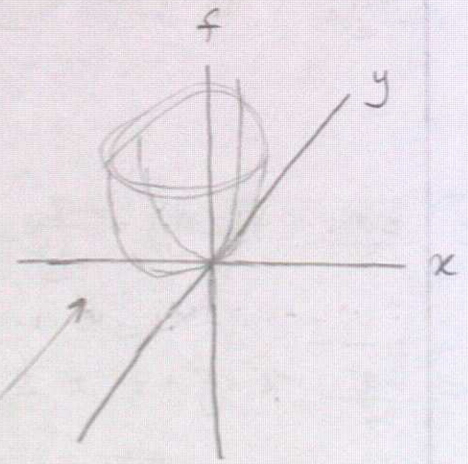
① **Eg:** $f(x,y) = x^2+y^2$

- consider trace by restricting input

$x=0 \Rightarrow f(x,y) = y^2$

$y=0 \Rightarrow f(x,y) = x^2$

"sharp 3-d bowl"



3D is a paraboloid

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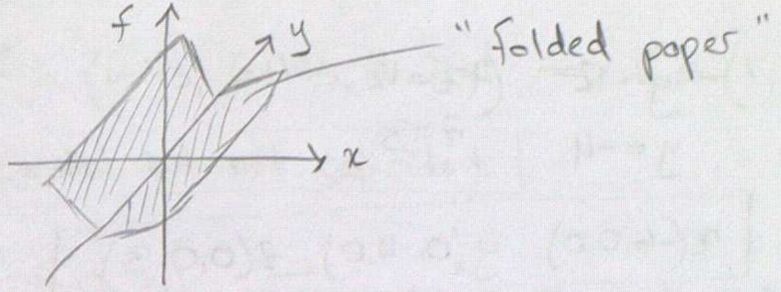
② **Eg:** $f(x,y) = |x|$



Level Curves and Contours

- a level curve of function f is a set of points in its domain s.t. $f(x,y) = C$
  - ↳ level curve depends on choice of "C" — "C-level curve"

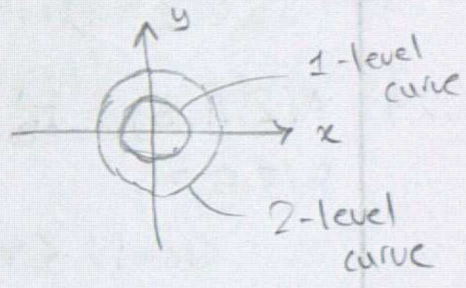
① **Eg:** $f(x,y) = x^2+y^2$

1-level curve

$f(x,y) = 1 = x^2+y^2$

2-level curve

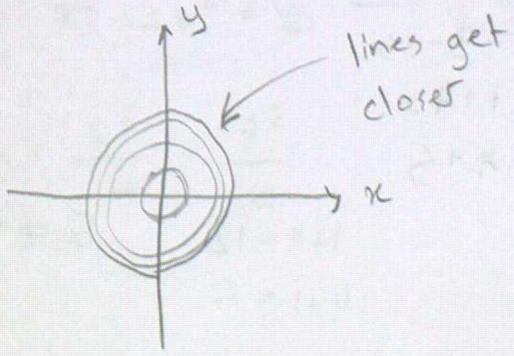
$f(x,y) = 2 = x^2+y^2$



- level curve is lower dimensional than graph
- can use for 3D functions

Contour Plot

- a contour plot of function f is a plot of several equally-spaced level curves.



- spacing of lines tells us steepness/ROC of the graph/function.

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Week 1: Tutorial 1

Section 13.5

13.5.9. $-2x - 3y + 4z = 12$ $\quad -3y = 12 \quad 4z = 12$

$y = -4 \quad z = 3$

$-2x = 12$

$x = -6$

$\boxed{x(-6,0,0) \quad y(0,-4,0) \quad z(0,0,3)}$

13.5.11. $P(0,0,1) \quad \vec{v} = \langle 4, 7, 0 \rangle$

$\langle x, y, z \rangle = (0,0,1) + t\langle 4, 7, 0 \rangle$

$\left.\begin{aligned} x &= 4t \\ y &= 7t \\ z &= 1 \end{aligned}\right\}$ parametric equations

13.5.29 $A(2,4,8)$ $\quad \vec{AB} = \langle 7-2, 5-4, 3-8 \rangle = \langle 5, 1, -5 \rangle$

$B(7,5,3)$

$\langle x, y, z \rangle = (2,4,8) + t\langle 5, 1, -5 \rangle$

$\left.\begin{aligned} x &= 2 + 5t \\ y &= 4 + t \\ z &= 8 - 5t \end{aligned}\right\}$ parametric equations

13.5.33 ① $(4,6,1) + t(0,-1,1)$ $\quad$ ·not parallel

② $(-3,1,4) + s(-7,4,1)$

$\boxed{\text{no intersection}}$

$l_1 \begin{cases} x = 4 \\ y = 6 - t \\ z = 1 + t \end{cases} \qquad l_2 \begin{cases} x = -3 - 7s \\ y = 1 + 4s \\ z = 4 + s \end{cases}$

$4 = -3 - 7s \qquad 6 - t = 1 + 4s \qquad 1 + 11 = 12 \qquad 12 \neq 5$

$s = \frac{7}{-7} = -1 \qquad t = 11 \qquad 4 + 1 = 5$

13.5.37 $\vec{r} = (4+t, -2t, 3t) \Longrightarrow \vec{r} = (4,0,0) + t\langle 1, -2, 3 \rangle$

$\vec{R} = (1-7s, 6+14s, 4-21s) \Longrightarrow \vec{R} = (1,6,4) + s\langle -7, 14, -21 \rangle$

$\boxed{\text{parallel but not coincident}}$

13.5.49 ☆ $P(1,0,3)$ $\quad \vec{QP} = \langle 1-0, 0-4, 3-2 \rangle = \langle 1, -4, 1 \rangle$

$Q(0,4,2)$ $\quad \vec{QR} = \langle 1-0, 1-4, 1-2 \rangle = \langle 1, -3, -1 \rangle$

$R(1,1,1)$

$$\vec{QP} \times \vec{QR} = \begin{vmatrix} i & j & k \\ 1 & -4 & 1 \\ 1 & -3 & -1 \end{vmatrix} = i(-4(-1) - 1(-3)) + -j(1(-1) - 1(1)) + k(1(-3) + 4(1))$$

$4 + 3 = 7$

$-1 - 1 = +2$

$-3 + 4 = +1$

$= 7i + 2j + 1k$

$7x + 2y + z + d = 0 \Longrightarrow d = -(7 + 2 + 1) = -10$

$\boxed{\therefore \pi : 7x + 2y + z - 10 = 0}$

13.6.1 ① $x^2 + 2y^2 = 8 \longrightarrow \parallel z$

② $z^2 + 2y^2 = 8 \longrightarrow \parallel x$

③ $x^2 + 2z^3 = 8 \longrightarrow \parallel y$

☆ 13.6.23 $y = \frac{x^2}{6} + \frac{z^2}{16}$

13.6.35 $z = \frac{x^2}{9} - y^2$ $x(0,0,0)$ $y(0,0,0)$ $z(0,0,0)$

zy-trace: $z = -y^2$

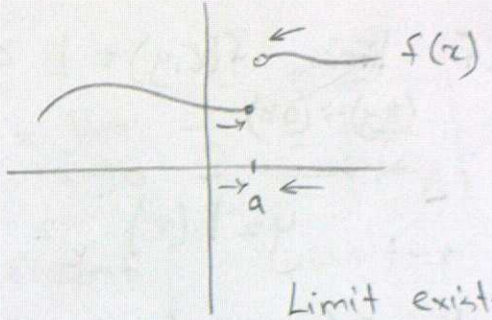
zx-trace: $z = \frac{1}{9}x^2$

xy-trace: $\frac{x^2}{9} - y^2 = 0$

$\lim_{r \to 0} \cos\theta \sin\theta = \cos 0 \sin 0 = 0$

Week 2: lec 1 Sep 9, 2024

Limits



Limit exists if left-hand side limit equals right-hand side limit.

- Single Variable Case:

$$\lim_{x \to a} f(x) = L$$

- 2 possible directions
  - ↳ from left
  - ↳ from right

  } from domain

- Multivariable Case

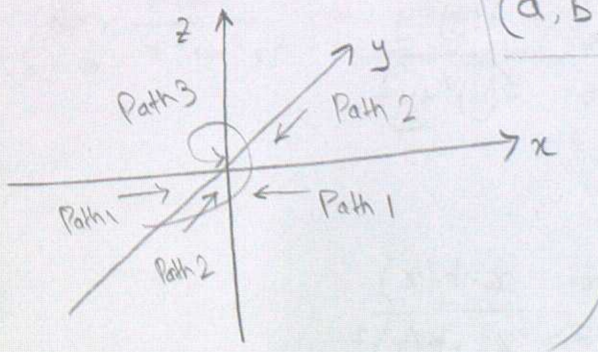
$$\lim_{(x,y) \to (a,b)} f(x,y)$$ ⇐ limit exists if limit is the same along any path in the domain

What is a path?

① $\boxed{y = f(x)}$, f is continuous and $\boxed{f(a) = b}$

② $\boxed{x = g(y)}$, g is continuous and $\boxed{g(b) = a}$

③ $\boxed{(x,y) = (f(t), g(t))}$, both f and g continuous and $\boxed{(a,b) = (f(t_0), g(t_0))}$



$\lim_{(x,y) \to (0,0)} f(x,y)$ must go through (0,0)

Path 1: $y = 0$

Path 2: $x = 0$

Path 3: $(x,y) = (t\cos(t), t\sin(t))$

note: $t = 0 \Rightarrow (x,y) = (0,0)$

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- If $\lim_{(x,y)\to(a,b)} f(x,y) = L$ exists then for any path,

$y = h(x)$ or $x = g(y)$

$\underbrace{\lim_{x\to a} f(x, h(x)) = L}_{\text{single-variable limit}}$ or $\underbrace{\lim_{y\to b} f(g(y), y) = L}_{\text{single-variable limit}}$

⇒ if limit along 2 paths is different, then the multivariable limit DNE.

$(x,y) = (g(t), h(t))$ $\Longrightarrow$ $\lim_{t\to t_0} f(g(t), h(t)) = L$

$(a,b) = (g(t_0), h(t_0))$

Eg: $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ show this DNE

- consider paths:

① $x = 0 = f(y)$ $\longrightarrow$ $\lim_{y\to 0} \frac{f(y)\cdot y}{f(y)^2+y^2}$

$\lim_{y\to 0} \frac{0\cdot y}{0^2+y^2} = \boxed{0}$

② $y = 0 = h(x)$ $\longrightarrow$ $\lim_{x\to 0} \frac{x\cdot h(x)}{x^2+h(x)^2}$

$\lim_{x\to 0} \frac{x\cdot 0}{x^2+0^2} = \boxed{0}$

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③ $y = x = g(x)$

$\lim_{x\to 0} \frac{x\cdot x}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \boxed{\frac{1}{2}}$

∴ multivariable DNE because different along two paths → 0, 0, and 1/2

Eg 2: Does $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ exist?

- consider all linear paths $\boxed{y = mx}$ ← passes thru (0,0) if m=0

$\lim_{x\to 0} \frac{x^2(mx)}{x^4+(mx)^2} = \lim_{x\to 0} \frac{mx^3}{x^4+m^2x^2} = 0$ for m=0

- consider path $\boxed{y = x^2}$

$\lim_{x\to 0} \frac{x^2\cdot x^2}{x^4+(x^2)^2} = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$

- limit exists along all linear paths, but multivariable limit DNE

$\lim_{x\to 0} \frac{mx^3}{x^4+m^2x^2} = \lim_{x\to 0} \frac{mx}{x+m^2} = \lim_{x\to 0} \frac{m\cdot 0}{0+m^2} = 0$ ✓

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Eg: 3 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-y^4}{x^2+y^2}$

$$= \lim_{(x,y)\to(0,0)} \frac{\cancel{(x^2+y^2)}(x^2-y^2)}{\cancel{(x^2+y^2)}}$$

$$= \lim_{(x,y)\to(0,0)} x^2-y^2 = \boxed{0}$$

Eg: 4 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y}{x^2+y^2}$ $\Leftarrow$ polar coordinates

$(x,y) = (r\cos\theta, r\sin\theta)$

$(x,y)\to(0,0) \Leftrightarrow r=0$

$$\lim_{r\to0} \frac{r^3\cos^3\theta \cdot r\sin\theta}{r^2}$$

$$= \lim_{r\to0} r^2\cos^3\theta\sin\theta = \boxed{0}$$

Tips on Nasty Limits

- change to polar: $x^2+y^2=r^2$, $x=r\cos\theta$, $y=r\sin\theta$
- use 2 paths (that pass thru point) to disprove limit
- seeing xy, let a path be $z=xy$
- $\lim\limits_{x\to0} \dfrac{\sin x}{x} = 1$ $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 0$

Week 2: Lecture 2   Sep 11, 2024

I **Eg:** $\lim\limits_{(x,y)\to(4,1)} \dfrac{xy-4y^2}{\sqrt{x}-2\sqrt{y}}$

$$\frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}} \cdot \frac{\sqrt{x}+2\sqrt{y}}{\sqrt{x}+2\sqrt{y}} = \frac{(xy-4y^2)(\sqrt{x}+2\sqrt{y})}{x-4y} = \frac{y\cancel{(x-4y)}(\sqrt{x}+2\sqrt{y})}{\cancel{x-4y}}$$

$$\lim_{(x,y)\to(4,1)} y(\sqrt{x}+2\sqrt{y}) = \boxed{4}$$

Continuity

- A function $f(x,y)$ is continuous at a point (a,b)
  - ① f is defined at (a,b)
  - ② $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists
  - ③ $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$
- continuity means no "weird" jumps in the graph
- there are infinite paths to consider in multivariable, which means continuity in multivariables is more "strict" than single variable.

II **Eg:** $f(x,y) = \begin{cases} \dfrac{x+2y}{x-2y} & (x,y)\neq(0,0) \\ 0 & (x,y)=0 \end{cases}$

- Check the 3 conditions
  - ① $f(0,0)=0$ is defined

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② $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x+2y}{x-2y}$

Check paths $x=0$ and $y=0$

$\lim_{y\to 0} \frac{0+2y}{0-2y} = -1$ $\lim_{x\to 0} \frac{x+2(0)}{x-2(0)} = 1$

$\therefore$ limit DNE, so $f(x,y)$ not cont. at $(0,0)$

Properties of Cont. Functions

① Sums of cont. func. are cont. (also differences)

↳ f and g cont. at (a,b), then $(f \pm g)$ cont. at (a,b)

② Products of cont. func. are cont.

③ Quotients of cont. func. are cont. at points where denominator not zero.

④ Compositions of cont. func. are cont.

↳ $f(x,y) = x+y$ $g(z) = \log(z)$

$g \circ f = \log(x+y)$ ⟸ cont. when $x+y>0$

⑤ Projection func. are cont.

$f_x(x,y) = x$ or $f_y(x,y) = y$

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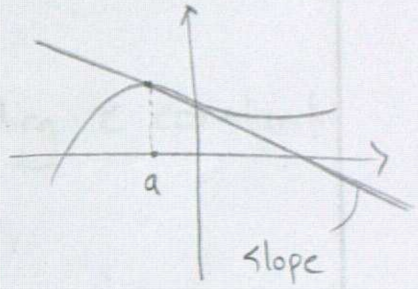
III Eg: $f(x,y) = \frac{x^3\sin(y) + e^{\sin(y)}y^2}{(x^2\sin(x))^2+1+y^2}$ ⟵ Is this continuous?

- $f(x,y)$ is sum/diff./prod./quotient of a projection func.
- $f(x,y)$ is continuous bc it's made up of cont. functions and denominator is never zero

Partial Derivatives



- single variable derivative

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

- slope of tangent
- the change f with the change in the input (variable x)

- Generalize to multivarables:

↳ multiple inputs (x,y)

↳ start by considering change in one input at a time

- Given $f(x,y)$

$$\frac{\partial f}{\partial x}(a,b) = \lim_{t\to 0} \frac{f(a+t,b)-f(a,b)}{t}$$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{t\to 0} \frac{f(a,b+t)-f(a,b)}{t}$$

- alternate notation: $f_x(a,b) = \frac{\partial f}{\partial x}(a,b)$
- the partial is change in f with change in one variable while other variable is held constant.
- to compute: $\frac{\partial f}{\partial x}$, take single variable derivative w.r.t x while treating y as a constant.

Derivatives of Multivariable Functions

- **Partial Derivative**: the idea of treating one dependent variable, such as "x" as constant and thereby ensuring that the tangent line is in the direction of the other variable (in this case, in direction of y-axis)
- **Notation**: for $f(x,y) = z$

$\frac{\partial f}{\partial x}$: tangent line along x-direction, holding y constant

$\frac{\partial f}{\partial y}$: tangent line along y-direction, holding x constant

$\frac{\partial f}{\partial x}, \frac{\partial z}{\partial x}, f_x, z_x$ or $\frac{\partial f}{\partial y}, \frac{\partial z}{\partial y}, f_y, z_y$

I **Ex**: $f(x,y) = 2x^2y^3 - 3x^2y + 2x^2 + 3y^2 + 1$

$$\frac{\partial f}{\partial x} = 4xy^3 - 6xy + 4x$$

$$\frac{\partial f}{\partial y} = 6x^2y^2 - 3x^2 + 6y$$

II **Ex:** $f(x,y) = e^{x}\cos y + e^{y}\sin x$

$f_x = e^{x}\cos y + e^{y}\cos x$

$f_y = -e^{x}\sin y + e^{y}\sin x$

III **Ex:** $z = xe^{xy^2}$

$$z_x = \frac{\partial}{\partial x}[x]\cdot e^{xy^2} + x\cdot\frac{\partial}{\partial x}\left[e^{xy^2}\right]$$

$$= e^{xy^2} + x\cdot y^2 e^{xy^2}$$

$$z_x = e^{xy^2}(1+xy^2)$$

$$z_y = 2yxe^{xy^2}$$

IV **Ex:** $z = y^{x}$

$\frac{\partial z}{\partial x} = y^{x}\ln y$ $\qquad$ $\frac{\partial z}{\partial y} = xy^{x-1}$

V **Ex:** $g(x,y) = x^2\cosh\left(\frac{x}{y}\right)$

$$g_x = 2x\cosh\left(\frac{x}{y}\right) + x^2\sinh\left(\frac{x}{y}\right)\cdot\frac{1}{y}$$

$$g_y = x^2\sinh\left(\frac{x}{y}\right)\cdot\left(-\frac{x}{y^2}\right)$$

VI **Ex:** $w = 2x^3 + 3xy + 2yz - z^2$

$\frac{\partial w}{\partial x} = 6x^2 + 3y$ $\qquad$ $\frac{\partial w}{\partial z} = 2y - 2z$

$\frac{\partial w}{\partial y} = 3x + 2z$

VII **Ex:** $f(x,y,z,w) = \frac{xw^2}{y+\sin(zw)}$

$$f_x = \frac{w^2}{y+\sin(zw)}$$

$$f_w = \frac{\frac{\partial}{\partial w}[xw^2]\cdot(y+\sin(zw)) - xw^2\cdot\frac{\partial}{\partial w}[y+\sin(zw)]}{[y+\sin(zw)]^2}$$

$$= \frac{2xw(y+\sin(zw)) - xw^2(z\cos(zw))}{(y+\sin(zw))^2}$$

Implicit Derivatives

z is implicitly defined

$- z = f(x,y)$

$$x^2y + xz + yz^2 = 8$$

$$\frac{\partial z}{\partial x} : \quad \frac{\partial}{\partial x}\left[x^2y + xz + yz^2\right] = \frac{\partial}{\partial x}[8]$$

$$2xy + z + x\frac{\partial z}{\partial x} + y \cdot 2z \cdot \frac{\partial z}{\partial x} = 0$$

$$x\frac{\partial z}{\partial x} + 2yz\frac{\partial z}{\partial x} = -2xy - z$$

$$\frac{\partial z}{\partial x} = \frac{-2xy - z}{x + 2yz}$$

$$\frac{\partial z}{\partial y} : \quad \frac{\partial}{\partial y}\left[x^2y + xz + yz^2\right] = \frac{\partial}{\partial y}[8]$$

$$x^2 + x\frac{\partial z}{\partial y} + z^2 + y \cdot 2z\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-x^2 - z^2}{x + 2yz}$$

Higher Derivatives

$$\begin{tikzcd}
 & & \frac{\partial^2 f}{\partial x^2}, f_{xx}, z_{xx} \\
 & \frac{\partial f}{\partial x} \arrow[ru, dash] \arrow[r, dash] & \frac{\partial^2 f}{\partial y \partial x}, f_{xy}, z_{xy} \\
z = f(x,y) \arrow[ru, dash] \arrow[rdd, dash] & & \\
 & & \frac{\partial^2 f}{\partial y^2}, f_{yy}, z_{yy} \\
 & \frac{\partial f}{\partial y} \arrow[ru, dash] \arrow[r, dash] & \frac{\partial^2 f}{\partial x \partial y}, f_{yx}, z_{yx}
\end{tikzcd}$$

VIII Ex: $g(x,y) = x^3y^2 + xy^3 - 2x + 3y + 1$

$$\frac{\partial g}{\partial x} = 3x^2y^2 + y^3 - 2 \qquad \frac{\partial g}{\partial y} = 2x^3y + 3xy^2 + 3$$

$$\frac{\partial^2 g}{\partial x^2} = 6xy^2$$

$$\frac{\partial^2 g}{\partial y^2} = 2x^3 + 6xy$$

$$\frac{\partial^2 g}{\partial y \partial x} = 6x^2y + 3y^2$$

$$\frac{\partial^2 g}{\partial x \partial y} = 6x^2y + 3y^2$$

IX **Eg:** $f(x,y) = x\sin^2 y + y^2\cos x$ f_{xy}, f_{yx}

$$\frac{\partial f}{\partial x} = \sin^2 y - y^2 \sin x$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2\sin y\cos y - 2y\sin x$$

- Mixed Partial Derivatives: for any function that is cont. on a region, mixed partial derivatives are equal

$$f_{xy} = f_{yx}$$

$$g_{xyz} = g_{zyx} = g_{xzy} \neq g_{xyx}$$

Laplace Equations

$$h(x,y) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(yx^{-1}) = \tan^{-1}\left(\frac{1}{x}\cdot y\right)$$

$$\frac{\partial h}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot -\frac{y}{x^2} = -\frac{y}{x^2+y^2}$$

$$\frac{\partial^2 h}{\partial x^2} = \frac{y}{(x^2+y^2)^2} \cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

We know $\frac{\partial^2 h}{\partial y^2} = -\frac{2xy}{(x^2+y)^2}$

If $h_{xx} + h_{yy} = 0$, h is called a Laplace Equation

Week 2 - Lecture 3   Sep 13, 2024

- the partial is the single-variable derivative of the trace.

$$\frac{\partial f}{\partial x}(a,b) = \frac{d}{dx} f_{y=b} \ (x=a)$$

$$\frac{\partial f}{\partial y}(a,b) = \frac{d}{dy} f_{x=a} \ (y=b)$$

I **Ex:** $f(x,y) = x^2 + xy$

$$\frac{\partial f}{\partial x} = 2x + y \qquad \frac{\partial f}{\partial y} = x$$

II **Ex:** $f(x,y,z) = e^{xy} + xyz + z^2$

$$\frac{\partial f}{\partial x} = ye^{xy} + yz \qquad \frac{\partial f}{\partial y} = xe^{xy} + xz \qquad \frac{\partial f}{\partial z} = xy + 2z$$

III **Ex:** compute partial at (0,0)

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(0+t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0) - 0}{t}$$

$$= \lim_{t\to 0} \frac{\frac{t\cdot y}{t^2+0^2}}{t} = 0 \quad \overset{}{\Longrightarrow} \quad \frac{\partial f}{\partial y}(0,0) = 0$$

by symmetry

Higher Order Partials

- apply partial multiple times (similar to second derivative)

$$\frac{\partial f}{\partial x} \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial f}{\partial y} \Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

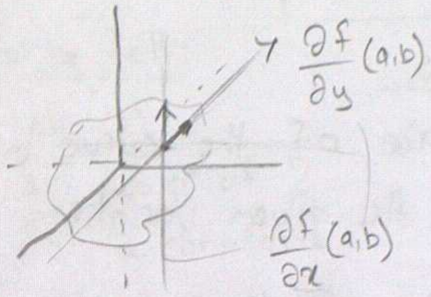
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

Clariot's Thm:

If f is cont. and 2nd order partials of f are cont. in a neighborhood of (a,b) then order of mixed partial doesn't matter

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b)$$

Tangent Plane → span of 2 tangent lines (vectors)



$\langle 0, 1, \frac{\partial f}{\partial y} \rangle \leftarrow$ y-dir, T_y

$\langle 1, 0, \frac{\partial f}{\partial x} \rangle \leftarrow$ x-dir, T_x

- for the vector in x-direction, you hold y constant. You pick $x = 1$ to make $z = \frac{\partial f}{\partial x} \cdot x = \frac{\partial f}{\partial x} \longrightarrow \langle 1, 0, \frac{\partial f}{\partial x} \rangle$ and vice versa for y-dir vector.
- normal vector = cross product of $T_y \times T_x$

$$\vec{n} = T_y \times T_x = \left\langle \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) - 1 \right\rangle$$

- the plane, given $\vec{n}$ and offset to the point $(a, b, f(a,b))$

$$0 = \vec{n} \cdot \left((x,y,z) - (a,b,f(a,b))\right)$$

$$\boxed{z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)}$$

↑

tangent plane to the graph of f

Cylinders and Surfaces in 3D

Cylinders

1) Equations have only <u>2 variables</u>
   - These equations give a trace of the curve on the coord. plane denoted by the given variables
2) Curve is directed along the axis of the <u>missing variable</u>
3) Curve <u>does not change</u> along the directional axis.

General Surfaces Intro

- Have all 3 variables
- Traces occur on coordinate planes and/or on planes parallel to coordinate planes
- Still directed along an axis, but the trace changes along that axis

Steps

1. Determine the type of surface
2. Determine the direction axis
3. Find trace on coord. plane
4. Find at least <u>two</u> other traces along direction axis

Ellipsoids:

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$$

How to tell:

1) All (+)
2) All power of 2
3) Has a constant

Notes:

- Std form helps with traces

Intercepts

$x = \pm a$

$y = \pm b$

$z = \pm c$

1-Sheet Hyperboloids:

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1}$$

How to tell:

1) Has <u>one</u> (-)
2) All power of 2
3) Has a constant

Notes:

- Always directed along the axis with the negative variable
- Set (-) variable = 0 and equal to $\pm\sqrt{\text{denominator}}$ to get three traces (circles or ellipses)

2-Sheet Hyperboloid:

$$\boxed{-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$$

How to tell

1) Has <u>one</u> (+)
2) All power of 2
3) Has a constant

Notes:

- Always directed along the axis with the positive variable
- Set both (-) variables = 0 to get axis intercept
- Set (+) variable = to numbers divisible by its denominator

Cones: $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0}$

How to tell:

1) Has one (-)
2) All power of 2
3) No constant

Notes:

- Always directed along negative variable
- Plug in values for the negative variable that are divisible by its denominator to get circle/ellipse traces

Paraboloids: $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz}$

How to tell:

1) 3 variables w/ 2 squares
2) Var. w/ squares are (+)

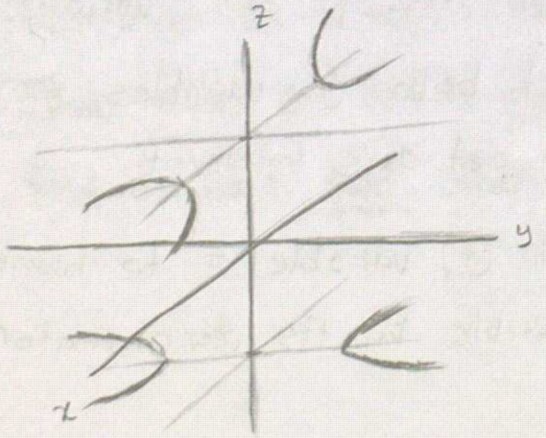
Notes:

- Opens along axis of var. with degree 1 and its coefficient sign → dir. of opening.

Hyperbolic Paraboloids: $\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = cz}$

How to tell:

1) 3 variables w/ 2 squares
2) One square has (-)



Notes

- Opens along axis of var. with degree 1
- Plug in (+) and (-) for degree 1 variable to get 2 sets of hyperbola traces (inverted to each other)

15.4 The Chain Rule

15.09.24

One Independent Var.

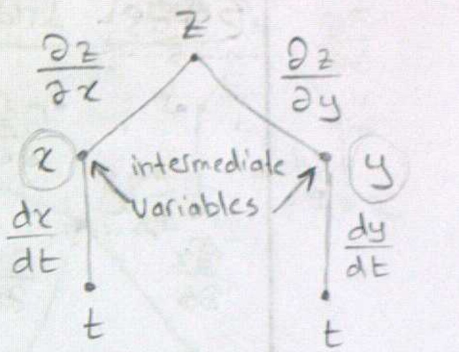
- If $y(u)$ and $u(t)$, $\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}$
- Given $z(x,y)$ and $x(t)$ and $y(t)$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

- ordinary derivative because dz/dt ultimately depends on t only

partial derivative because z is depending on both x and y

ordinary derivatives because x and y depend only on t



$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Ex: $z = x^2 - 3y^2 + 20$, $x = 2\cos t$, $y = 2\sin t$, $\frac{dz}{dt} = ?$, $\frac{dz}{dt}\Big|_{t=\pi/4}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2x)(-2\sin t) + (-6y)(2\cos t)$$

$$= -4x\sin t - 12y\cos t$$

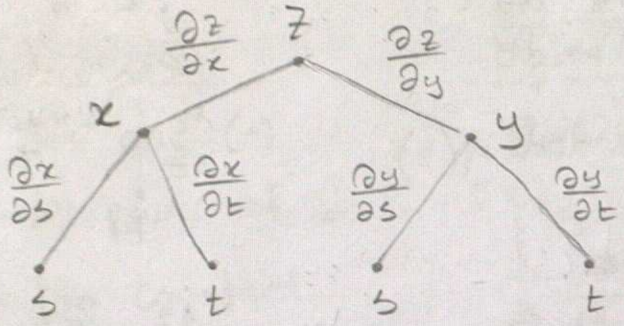
$$= -4(2\cos t)\sin t - 12(2\sin t)\cos t$$

$$= -8\cos t\sin t - 24\cos t\sin t$$

$$\frac{dz}{dt} = -16\sin 2t$$

$$\frac{dz}{dt}\Big|_{t=\frac{\pi}{4}} = -16 \leftarrow$$ at $t = \frac{\pi}{4}$, the z value or "height" is decresing at a rate of 16.

Several Independent Var.



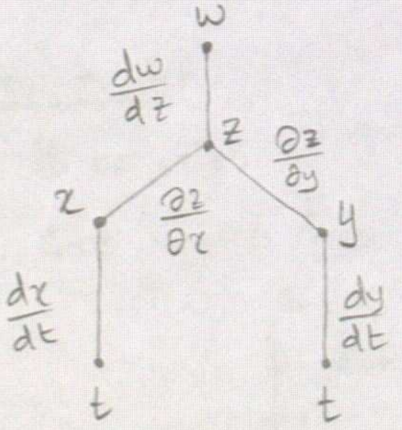
- z is a function of x and y, and x and y are functions of s and t. To find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$:

$$\boxed{\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\end{aligned}}$$

} we have partials along all branches since z, x, and y depend on multiple parameters

Deeper Trees

- Let w be function of z, and z is function of x and y, both of which are functions of t.

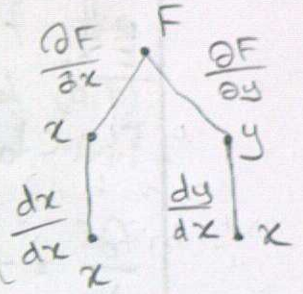


- To find $\frac{dz}{dt}$, we have 2 paths
  - $w \to z \to x \to t$
  - $w \to z \to y \to t$

$$\therefore \frac{dw}{dt} = \frac{dw}{dz}\cdot\frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{dw}{dz}\cdot\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{dw}{dt} = \frac{dw}{dz}\left(\frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\right)$$

Implicit Differentiation



$$\sin xy + \pi y^2 = x \Longrightarrow \underbrace{\sin xy + \pi y^2 - x}_{F(x,y)} = 0$$

- $F(x,y) = 0$

$F(x, y(x)) = 0$

↳ To find $\frac{dx}{dy}$: $\frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0$

$$F_x + F_y\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

Week 3: Lecture 1

Sep 16, 2024

Ex: $f(x,y) = 4-2x^2-y^2$ at $(2,2,-8)$

Find tangent plane : $(a,b,f(a,b)) = (2,2,-8)$

$\frac{\partial f}{\partial x} = -4x \qquad \frac{\partial f}{\partial x}(2,2) = -4(2) = -8$

$\frac{\partial f}{\partial y} = -2y \qquad \frac{\partial f}{\partial y}(2,2) = -2(2) = -4$

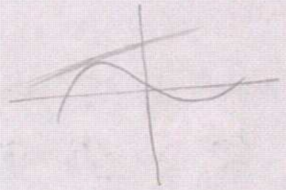
$\boxed{z = -8 - 8(x-2) - 4(y-2)}$ ← tangent plane

$z = 16 - 8x - 4y$

Differentiability

Recal: single variable derivative is the slope of the tangent line

$f'(a) = \lim_{h\to 0} \frac{f(x+h) - f(a)}{h}$



- f is differentiable if the tangent line is a good approximation for the function/graph

$$0 = \lim_{h\to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} \qquad h = x-a$$

$$0 = \lim_{x\to a} \frac{f(x) - \overbrace{(f(a) + f'(a)(x-h))}^{\text{tangent line}}}{x-a}$$

- We say $f(x,y)$ is differentiable at $f(a,b)$ if the tangent plane is a good approx. of f around (a,b):

  ① $\frac{\partial f}{\partial x}(a,b)$ and $\frac{\partial f}{\partial y}(a,b)$ exist

  ② $\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L_{(a,b)}(x,y)}{\|(x,y)-(a,b)\|} = 0$

  $$L_{(a,b)}(x,y) = f(a,b) + \frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b)$$

- If f is differentiable at (a,b), then f is continuous at (a,b)
  - ↳ partial derivatives don't imply continuity
- If the partials of f exist in a neighborhood of (a,b) and are cont. on that neighborhood, then f is differentiable at (a,b)

Ex: $f(x,y) = |xy|$ ← differentiable at $(0,0)$

$\frac{\partial f}{\partial x}(0,0) = (0) = 0 \qquad \frac{\partial f}{\partial y}(0,0) = 0$

- note: theorem doesn't apply since partials don't exist in neighborhood

  ex: $\frac{\partial f}{\partial x}(0,0.1) = \frac{d}{dx} 0.1|x|$ ← derivative DNE

  $\frac{\partial f}{\partial x}(0,a) = \frac{d}{dx} |a||x|$ ← DNE for any $a \neq 0$

- Use definition

$$L_{(0,0)}(x,y) = 0 + 0(x-0) + 0(y-0)$$

$$= 0 \leftarrow \text{tangent plane}$$

$$\lim_{(x,y)\to(0,0)} \frac{|xy| - 0}{\sqrt{(x-0)^2+(y-b)^2}} \boxed{=0} \leftarrow \text{polar coordinates}$$

∴ f is differentiable at (0,0)

Tutorial Week 3

Differentiabity:

1. if f_x and f_y exist at specific point of $O \to (a,b)$
2. f_x and f_y are cont. at (a,b)

↳ then f is differentiable at (a,b)

Ex: $$f(x,y) = \begin{cases} \frac{2xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

a) cont. or not: $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} \rightarrow$ path: $x = y^2$

$$\lim_{y\to 0} \frac{2y^4}{2y^4} = 1 \leftarrow \text{not cont.}$$

15.5 Directional Derivatives and the Gradient

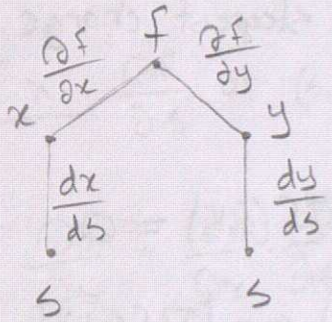
- Fix a direction $\vec{u} = \langle u_1, u_2 \rangle$, where $|\vec{u}| = 1$
- We start at (x_0, y_0) and move s units along $\vec{u}$

$$x(s) = x_0 + su_1 \qquad y(s) = y_0 + su_2$$

Definition: Directional Derivative

$$D_{\vec{u}} f(x_0,y_0) = \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s}$$

- the directional derivative of f at (x_0,y_0) w.r.t. $\vec{u}$
- we see this is a composition of functions → chain rule



$$f \rightarrow f(x(s), y(s)) \qquad \left.\frac{df}{ds}\right|_{s=0}$$

since limit def. means $s \to 0$

$$\frac{d}{ds} f = \frac{\partial f}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial f}{\partial y} \cdot \frac{dy}{ds}$$

$$\frac{d}{ds}[x_0+su_1] = u_1 \qquad \frac{d}{ds}[y_0+su_2] = u_2$$

$$\therefore D_{\vec{u}} f(x_0,y_0) = \left(\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} u_1 \right) + \left(\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} u_2 \right)$$

- Let $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ be "gradient of f"

$$\boxed{D_{\vec{u}} f(x_0,y_0) = \nabla f|_{(x_0,y_0)} \cdot \vec{u}} \leftarrow$$

a dot product b/n the gradient vector and unit vector $\vec{u}$

- **Directional Derivative:** the directional derivative of f at (a,b) in direction of $\vec{u} = \langle u_1, u_2 \rangle$ is

$$\boxed{D_{\vec{u}} f(a,b) = \left\langle \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right\rangle \cdot \langle u_1, u_2 \rangle}$$

- **Gradient:** the gradient of f at (x,y) is the vector-valued function

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle$$

$$\nabla f(x,y) = f_x(x,y)\,\hat{i} + f_y(x,y)\,\hat{j}$$

the gradient vector points in the direction of steepest change

- **Directions of change:** f diff. @ (a,b), $\nabla f(a,b) \neq 0$
  1) f has max inc. at (a,b) in dir. $\nabla f(a,b) \leftarrow |\nabla f(a,b)|$
  2) f has max dec. at (a,b) in dir. $-\nabla f(a,b) \leftarrow -|\nabla f(a,b)|$
  3) dir. derivative zero along any dir. orthogonal to $\nabla f(a,b)$

- the line tangent to a level curve of f at (a,b) is orthogonal to the gradient $\nabla f(a,b)$

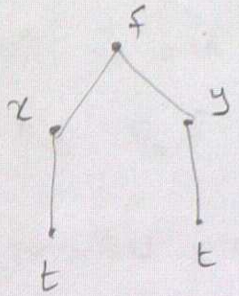
the triangle is called "nabla" operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

Week 3: Lecture 2

Sep 19, 2024

Ex: Chain Rule ~ $f(x,y) = \sin x \cos 2y$, $x = t^2$ $y = t^3$



$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{\partial f}{\partial x} = \cos x \cos 2y \qquad \frac{dx}{dt} = 2t$$

$$\frac{\partial f}{\partial y} = -2\sin x \sin 2y \qquad \frac{dy}{dt} = 3t^2$$

$$\frac{df}{dt} = 2t\cos x \cos 2y - 6t^2 \sin x \sin 2y$$

$$= 2t\cos t^2 \cos 2t^3 - 6t^2 \sin t^2 \sin 2t^3$$

Ex: $\frac{\partial f}{\partial s}$ at $(s,t) = (0,0)$ for $f(x,y) = e^{x+y}$ $x = e^s t$ $y = \cos(s+t)$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial f}{\partial y} \cdot \frac{dy}{ds}$$

$$= e^{x+y} e^s t - e^{x+y} \sin(s+t)$$

$s = 0$, $x = 0$
$s = 0$, $y = 1$

$$\frac{\partial f}{\partial s}(0,0) = e^{0+1} e^0 \cdot 0 - e^{0+1} \sin(0+0) = 0$$

Directional Derivative

- rate of change as we move along arbitrary dir. $\hat{u} = \langle u_1, u_2 \rangle$

Week 3: Lecture 3

Sep 20, 2024

Ex $f(x,y) \begin{cases} \frac{xy}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$ $u = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$$D_{\vec{u}}(0,0) = \lim_{t \to 0} \frac{f(0+\frac{t}{\sqrt{2}}, 0+\frac{t}{\sqrt{2}}) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{f(t/\sqrt{2}, t/\sqrt{2}) - 0}{t}$$

$$= \lim_{t \to 0} \frac{\frac{(t/\sqrt{2})(t/\sqrt{2})}{(t/\sqrt{2})^2 + (t/\sqrt{2})^2}}{t}$$

$$= \lim_{t \to 0} \frac{\frac{t^2/2}{t^2/2 + \frac{t^2}{2}}}{t} = \lim_{t \to 0} \frac{\frac{\frac{t^2}{2}}{\frac{2t^2}{2}}}{t}$$

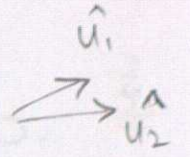
$$= \lim_{t \to 0} \frac{\frac{t^2}{2}}{t^3} = \lim_{t \to 0} \frac{1}{2} \cdot \frac{t^2}{t^3} = \lim_{t \to 0} \frac{1}{2} \frac{1}{t} \text{ DNE}$$

∴ limit DNE

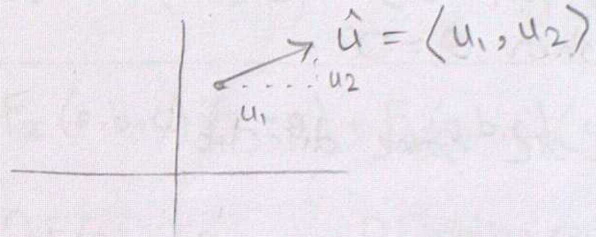
so $D_u f(0,0)$ DNE

→ recall that partials do exist, but derivative doesn't exist in this direction

Relationships B/n directional derivatives



- if $\hat{u}_1$ and $\hat{u}_2$ are similar directional vectors, then the $D_{u_1}f$ and $D_{u_2}f$ might be similar
- another motivation



is the change in f with a step in $\hat{u}$ the same as u_1 step in x and u_2 step in y?

$$D_u f = u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y} \leftarrow$$

is this true? (only true for differentiable functions)

- differentiable functions are nice in that directional derivative is predictable
- not true in general

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$ but $D_{\vec{u}}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ DNE

Theorem

If f is differentiable at (a,b) then

$$D_{\hat{u}} f(a,b) = u_1 \frac{\partial f}{\partial x}(a,b) + u_2 \frac{\partial f}{\partial y}(a,b) = \langle \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \rangle \cdot \hat{u}$$

All this is true if f is differentiable and $\nabla f \neq 0$. If $\nabla f = 0$, then it is a critical point (max/min or saddle)

What is gradient?

$D_{\hat{u}} f = \nabla f \cdot \hat{u}$ $\|\hat{u}\| = 1$ unit vector

$= \|\nabla f\| \|\hat{u}\| \cos\theta$

$= \|\nabla f\| \cos\theta$ ← θ is angle b/n $\hat{u}$ and ∇f

① $\cos\theta = 1$ maximum when $\theta = 0°$

→ $\theta = 0$ ∇f and $\hat{u}$ are same direction

→ $D_{\hat{u}} f$ is maximum possible

∴ gradient is direction of steepest ascent (maximal increase)

② $\cos\theta = -1$ at $\theta = 180°$

∴ gradient is opposite direction of steepest descent (maximal decrease)

③ $\cos\theta = 0$ at $\theta = 90°$ and $\theta = 270°$

→ $D_u f = 0$ when ∇f and $\hat{u}$ are orthogonal

→ no change in f (f is constant)

∴ gradient is orthogonal to the level curve

15.6 Tangent Planes and Linear Approximations

Tangent Planes for $F(x,y,z) = 0$

Let F be differentiable at $P_0(a,b,c)$ with $\nabla F(x,y,z) \neq 0$.

The plane tangent to the surface $F(x,y,z) = 0$ at P_0, is the plane passing through P_0 orthogonal to $\nabla F(a,b,c)$.

↳ $\nabla f(a,b,c) \cdot \langle x-a, y-b, z-c \rangle = 0$

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$$

$$\frac{\partial F(a,b,c)}{\partial x}[x-a] + \frac{\partial F}{\partial y}(a,b,c)[y-b] + \frac{\partial F}{\partial z}(a,b,c)[z-c] = 0$$

Tangent Planes for $z = f(x,y)$

Let f be differentiable at the point (a,b).

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

Linear Approximations

Given $z = f(x,y)$ or $w = f(x,y,z)$ and the point(s) $(a,b,f(a,b))$ or $(a,b,c,f(a,b,c))$, tangent planes are lin. approx.

$$L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

$$L(x,y,z) = f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) + f(a,b,c)$$

The differential "dz"

$$\boxed{\Delta z \approx dz = f_x(a,b)\,dx + f_y(a,b)\,dy}$$

the change in z when we move dx and dy from original point (a,b)

Relative or Percentage Change

Given $B(w,h) = \frac{w}{h^2}$ ← weight, ← height

BMI ↗

$$dB = B_w(a,b)\,dw + B_h(a,b)\,dh$$ ← absolute change

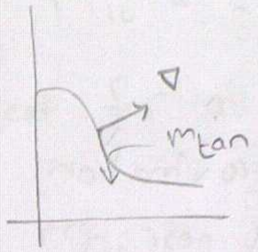
how to get relative changes $\frac{dB}{B}$, $\frac{dw}{w}$, $\frac{dh}{h}$?

↳ divide both sides by $B(w,h) = \frac{w}{h^2} = wh^{-2}$

$$\Longrightarrow \frac{dB}{B} = B_w \frac{dw}{wh^{-2}} + B_h \frac{dh}{wh^{-2}}$$

Week 4: Lecture 1 Sep 23

Ex: $\nabla f(1,2) = \langle 3,4 \rangle$, what is slope of tangent to level curve of f at (1,2).



$\nabla f(1,2) \longrightarrow \frac{4}{3}$

slope of tangent ⟶ −reciprocal : $-\frac{3}{4}$

$$\boxed{\therefore m_{tan} = -\frac{3}{4}}$$

Tangent Plane

- we talked about tangent plane of a graph.
- we can also consider tangent plane of implicitly defined surfaces

$f(x,y,z) = \text{constant}$ ex: $x^2+y^2+z^2 = 1$ (sphere)

$x^2+y^2 = 1$ (cylinder)

- tangent plane: all points orthogonal to gradient

Tangent Plane: $\nabla f(a,b,c) \cdot (x-a,\ y-b,\ z-c) = 0$

of $f(x,y,z) = C$ at point (a,b,c)

Ex: $x^2+y^2-z^2=0$, $(3,4,5)$, $F(x,y,z)=0$

$$0 = F_x(x-3) + F_y(y-4) + F_z(z-5)$$

$$\boxed{0 = 6(x-3) + 8(y-4) - 10(z-5)}$$

Linear Approximation

$y = mx + b$

- Recall first order Taylor series

$$\underbrace{P_1(x) = f(a) + f'(a)(x-a)}_{\text{linear}} \approx \underbrace{f(x)}_{\text{good approximation for } f(x) \text{ near } a}$$

- in multivariable, given $f: \mathbb{R}^n \to \mathbb{R}^m$, try to find linear approximation around $\vec{a} = \langle a_1, a_2, \cdots, a_n \rangle$

$$\vec{L}(\vec{x}) = M\vec{x} + \vec{b}$$

same as $y = mx+b$ except that M is $m \times n$ matrix

- linear approximation for $f(\vec{x})$ around $\vec{a}$

$$⊛ \quad f(x) \approx L(\vec{x}) = f(\vec{a}) + \underbrace{M}_{m \times n \text{ matrix}}(\vec{x} - \vec{a})$$

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \\ M_{m1} & & M_{mn} \end{bmatrix}$$

Now look at equ ⊛ component by component

$$f_1(x) \approx f_1(a) + M_{11}(x_1 - a_1) + \cdots + M_{1n}(x_n - a_n)$$ — row 1 of output

- Take the $\frac{\partial}{\partial x_i}$ and evaluate at $\vec{x} = \vec{a}$

$$\frac{\partial}{\partial x_i} f_1(x) = 0 + 0 + \cdots + M_{1i} + \cdots + 0$$

$$M_{1i} = \frac{\partial}{\partial x_i} f_1(\vec{a})$$

- repeat for all components:

$$m_{ji} = \frac{\partial f_j}{\partial x_i}(\vec{a})$$

- therefore matrix $\vec{M}$ is given by:

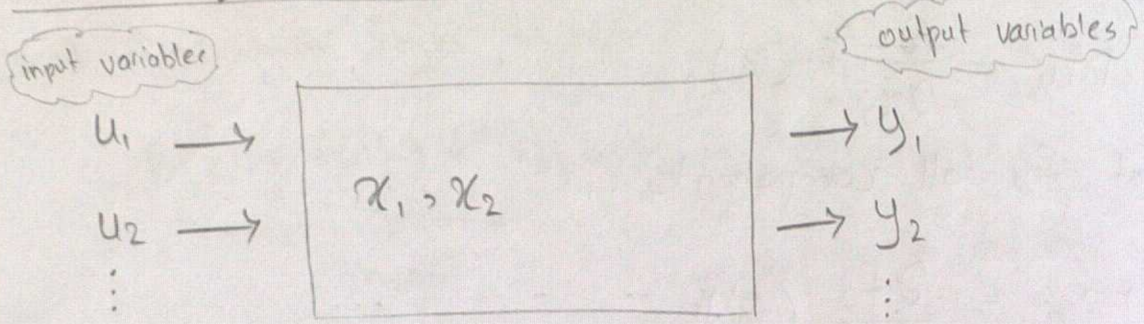
$$J_f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{a}) \\ \vdots & \ddots & \\ \frac{\partial f_m}{\partial x_1}(\vec{a}) & & \frac{\partial f_m}{\partial x_n}(\vec{a}) \end{bmatrix} \longleftarrow \text{called "Jacobian" of } f \text{ (at } \vec{a})$$

- if f is differentiable, then the Jacobian is the counterpart to the single-variable derivative

- Linear approx:

$$\boxed{f(\vec{x}) \approx f(\vec{a}) + J_f(\vec{a})(\vec{x} - \vec{a})}$$

State Equations and Small Signal Modelling



System

→ x_i variables give the current state of system

ex:// ① Electric circuit : voltages at nodes, current through wires etc.

② Population model : population, birth rate, death rate etc.

Input Variables

→ user-defined inputs

ex// ① input voltage, potentiometer

② food levels

Output Variables

→ measured quatities

→ variables you care about

Week 4: Lecture 2

Sep 25, 2024

1) assume the evolution of x_i is a first order ODE

$$\frac{dx_i}{dt} = f(x_1, x_2, \cdots x_n, u_1, \cdots, u_m)$$

2) assume output depends on input & state variables

$$y_i = g_i(x_1, \cdots, x_n, u_1, \cdots u_m)$$

n vs. m

# of state var's don't have to match # of input variable

Goal: Approximate output variables and evolution with linear functions

→ linear approx. accurate around $\vec{a}$

→ assume $\boxed{\underbrace{u_i(t)}_{\text{input}} = \underbrace{U_i}_{\text{equilibrium point}} + \underbrace{\hat{u}_i(t)}_{\text{small perturbation}}}$

⇒ output and state variables are also perturbations around equilibrium

$\boxed{y_i(t) = Y_i + \hat{y}_i(t)}$ $\boxed{x_i(t) = X_i + \hat{x}_i(t)}$

→ use linear approx :

$$\frac{d\vec{x}}{dt} = F(x_1, \cdots, x_n, u_1, \cdots, u_m) \longleftarrow$$ approximate around $\vec{a} = (x_1, \cdots, x_n, u_1, \cdots, u_m)$

$$\frac{d\vec{x}}{dt} = \vec{F}(x_1, \cdots, x_n, u_1, \cdots u_m)$$

$$\approx \vec{F}(x_1, \cdots, x_n, u_1, \cdots, u_m) \leftarrow \frac{d\vec{x}}{dt} = 0 \text{ at equilibrium}$$

$$+ \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & & & & \\ \frac{\partial f_k}{\partial x_1} & & \cdots & & & \frac{\partial f_k}{\partial u_m} \end{pmatrix}}_{\text{partials evaluated at equilibrium}} \begin{pmatrix} x_1 - X_1 \\ \vdots \\ x_n - X_n \\ u_1 - U_1 \\ \vdots \\ u_m - U_m \end{pmatrix}$$

$$\downarrow$$

$$\frac{d\vec{x}}{dt} = \frac{d\hat{x}}{dt} = 0 + \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \partial f \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_n \\ \hat{u}_1 \\ \vdots \\ \hat{u}_m \end{pmatrix}$$

$$\frac{d\hat{x}}{dt} = \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & & \\ & & \frac{\partial f_n}{\partial x_n} \end{pmatrix}}_{\text{call this } J_f} \begin{pmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_n \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ & & \\ & & \frac{\partial f_n}{\partial u_m} \end{pmatrix}}_{\text{Call this } B_f} \begin{pmatrix} \hat{u}_1 \\ \vdots \\ \hat{u}_m \end{pmatrix}$$

$$\boxed{\frac{d\hat{x}}{dt} = J_f \hat{x} + B_f \hat{u}} \qquad \boxed{y = J_g \hat{x} + B_g \hat{u}}$$

Ex: $\frac{dx_1}{dt} = \underbrace{x_1 + x_2 u_1}_{f_1(x_1, x_2, u_1, u_2)}$ $\qquad \frac{dx_2}{dt} = \underbrace{x_1 - x_2 u_2}_{f_2(\cdots)}$

$U_1 = 2$

$y_1 = \underbrace{-x_1 + u_1}_{g_1(\cdots)}$ $\qquad y_2 = \underbrace{-x_2 u_2}_{g_2(\cdots)}$

$U_2 = 1$

$u_1 = U_1 + \hat{u}_1$

$u_2 = U_2 + \hat{u}_2$

① Equilibrium is when $u_i = U_i$ and no change $\frac{dx_i}{dt} = 0$

$$\Longrightarrow \left.\begin{aligned} 0 &= \frac{dx_1}{dt} = x_1 + x_2 U_1 = x_1 + 2x_2 \\ 0 &= \frac{dx_2}{dt} = x_1 - x_2 U_2 = x_1 - x_2 \end{aligned}\right\} \text{solve to get solution}$$

$$\boxed{X_1 = 0 \quad X_2 = 0}$$

→ equilibrium for y_i

$$Y_1 = g_1(X_1, X_2, U_1, U_2) \qquad Y_2 = g_2(X_1, X_2, U_1, U_2)$$

$$= -X_1 + U_1 \qquad = \cdots$$

$$= -0 + 2 = 2 \qquad = -(0)(1)$$

$$= 0$$

② $\frac{d\hat{x}}{dt} = J_f \hat{x} + B_f \hat{u}$

$$f_1 = x_1 + x_2 u_1$$

$$f_2 = x_1 - x_2 u_2$$

$$J_f = \begin{pmatrix} 1 & u_1|_{u_1=U_1} \\ 1 & -u_2|_{u_2=U_2} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

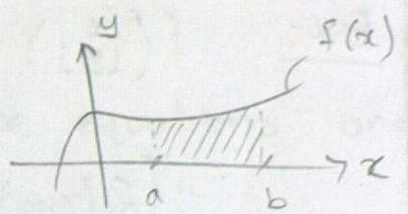
$$B_f = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

⇒ the small signal state equations

$$\frac{d\hat{x}}{dt} = \begin{pmatrix} \frac{d\hat{x}_1}{dt} \\ \frac{d\hat{x}_2}{dt} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{x}_1 + 2\hat{x}_2 \\ \hat{x}_1 - \hat{x}_2 \end{pmatrix}$$

Double Integrals: Rectangles



area $= \int_a^b f(x)\,dx$

Recall: 1D Integral

→ area under curve

→ generalizes discrete sums to continuous variables

$$\sum_{i \in I} m_i \sim \int_I m(i)$$

Multivariable Integral

→ signed volume under graph

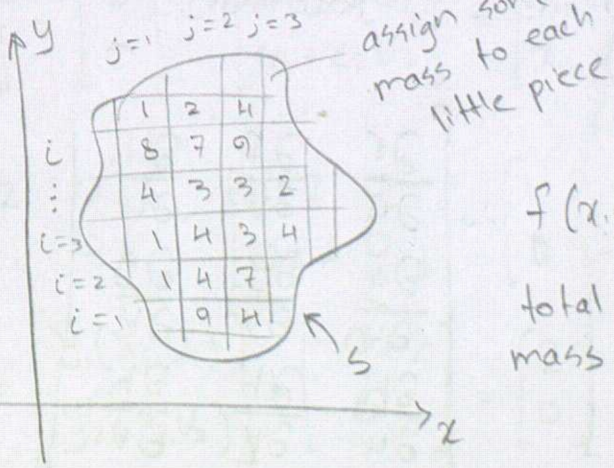
$$V = \iint_R f(x,y)\,dR$$

R is the area of the rectangle over which we integrate

→ generalizes discrete sums with multiple variables

$$\boxed{\sum_{i,j} x_{ij} \sim \iint x(i,j)}$$

total mass $= \sum_{ij} m_{ij}$



f(x,y) = density at each point

total mass $= \iint_S f(x,y)\,dA$ ☆

Jacobian Matrix

Ex: $f\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x+\sin y\\ y+\sin x\end{bmatrix}_{(-2,1)}$ $\quad x = f_1(x,y)$, $y = f_2(x,y)$

$$\begin{bmatrix}f_1(x,y)\\ f_2(x,y)\end{bmatrix} = \begin{bmatrix}x+\sin y\\ y+\sin x\end{bmatrix}$$

Jacobian : $\begin{bmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{bmatrix} = \begin{bmatrix}1 & \cos y\\ \cos x & 1\end{bmatrix}$

Ex: $x = f(u,v)$, $y = g(u,v) \longrightarrow \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}f(u,v)\\ g(u,v)\end{bmatrix}$

Jacobian : $\begin{bmatrix}\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v}\\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v}\end{bmatrix}$

Ex: $x = f(u,v,w)$
$y = g(u,v,w)$
$z = h(u,v,w)$
$\downarrow$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}f(u,v,w)\\ g(u,v,w)\\ h(u,v,w)\end{bmatrix}$$

Jacobian: $\begin{bmatrix}\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w}\\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial v}\\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial v}\end{bmatrix}$

Small Signal Model Example

Consider the following state space equations and output equations:

$$\frac{dx_1}{dt} = x_1 + x_2 u_1 \qquad \frac{dx_2}{dt} = x_1 - x_2 u_2$$

$$y_1 = -x_1 + u_1 \qquad y_2 = -x_2 u_2$$

Assume the input states are small perturbations around an equilibrium : $u_1 = U_1 + \hat{u}_1$ and $u_2 = U_2 + \hat{u}_2$ with the equilibrium points $U_1 = 2$ and $U_2 = 1$

1) What are the equilibrium points for x_1, x_2 and y_1, y_2?

At the equilibrium point, our $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}X_1\\X_2\end{bmatrix}$ and $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ and $\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}g(X_1,U_1)\\ g_2(X_2,U_2)\end{bmatrix} = \begin{bmatrix}Y_1\\Y_2\end{bmatrix}$ and we want to find X_1, X_2, Y_1, and Y_2

∴ equilibrium points are $X_1 = 0$ $Y_1 = 2$
$X_2 = 0$ $Y_2 = 0$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}x_1+x_2U_1\\ x_1-x_2U_2\end{bmatrix} = \begin{bmatrix}x_1+2x_2\\ x_1-x_2\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 0\\ 1 & -1 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 0\\ 0 & -3 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \qquad \begin{aligned}x_1 &= 0 = X_1\\ x_2 &= 0 = X_2\end{aligned}$$

$$\begin{bmatrix}Y_1\\Y_2\end{bmatrix} = \begin{bmatrix}y_1(X_1,U_1)\\ y_2(X_2,U_2)\end{bmatrix} = \begin{bmatrix}-0+2\\ -0(1)\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} \qquad \begin{aligned}Y_1 &= 2\\ Y_2 &= 0\end{aligned}$$

2.) What are the state space equations in the small signal model?

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}x_1 + x_2u_1\\ x_1 - x_2u_2\end{bmatrix} = \begin{bmatrix}f_1(x_1,x_2,u_1,u_2)\\ f_2(x_1,x_2,u_1,u_2)\end{bmatrix}$$

$$\text{Jacobian} = \begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2}\end{bmatrix}_{(X_1,X_2,U_1,U_2)}$$

$$= \begin{bmatrix}1 & u_1 & x_2 & 0\\ 1 & -u_2 & 0 & -x_2\end{bmatrix}_{(0,0,2,1)} = \begin{matrix}\begin{bmatrix}1 & 2 & 0 & 0\\ 1 & -1 & 0 & 0\end{bmatrix}\\ \underbrace{\hspace{2em}}_{x_1x_2}\;\underbrace{\hspace{2em}}_{u_1u_2}\end{matrix}$$

$$\boxed{\therefore \vec{\dot{x}} = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{u}_1\\ \hat{u}_2\end{bmatrix}}$$

3) What are the output equations in small signal?

$$\bar{y} = \begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}-x_1+u_1\\ -x_2u_2\end{bmatrix} = \begin{bmatrix}g_1(x_1,x_2,u_1,u_2)\\ g_2(x_1,x_2,u_1,u_2)\end{bmatrix}$$

$$\text{Jacobian} = \begin{bmatrix}\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2}\\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2}\end{bmatrix}_{(X_1,X_2,U_1,U_2)}$$

$$= \begin{bmatrix}-1 & 0 & 1 & 0\\ 0 & -u_2 & 0 & -x_2\end{bmatrix}_{(0,0,2,1)} = \begin{matrix}\begin{bmatrix}-1 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\end{bmatrix}\\ \underbrace{\hspace{2em}}_{x_1x_2}\;\underbrace{\hspace{2em}}_{u_1u_2}\end{matrix}$$

$$\boxed{\therefore \vec{y} = \begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{u}_1\\ \hat{u}_2\end{bmatrix}}$$

Week 4: Lecture 3

Sep 27, 2024

- domain of integration can be any set for multivariables
- in single variables, we only consider $[a,b]$
- notation:

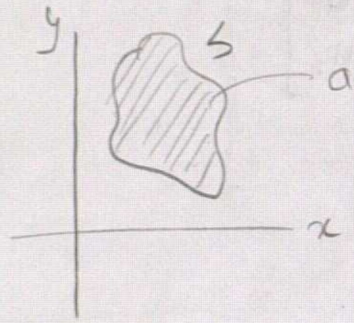
$$\iint \underbrace{f(x,y)}_{\text{density}} \underbrace{dA}_{\text{Infinitesimal area element}}$$

sum

$\underbrace{\text{density} \quad \text{Infinitesimal area element}}$

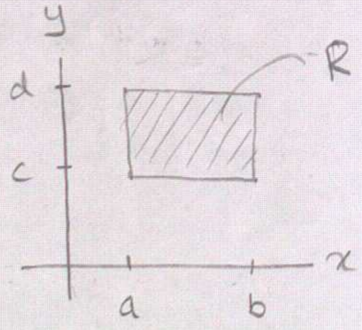
density · area = mass, charge, etc.

- if $f(x,y) = 1$ then $\iint_S 1 \, dA = \text{area}(S)$



area of $S = \iint_S 1 \, dA$

Integration Over Rectangles



- consider domain of integration as $R = [a,b] \times [c,d]$
- Fubini's for Rectangles

If f is continuous, then

$$\boxed{\iint_R f(x,y) \, dA = \int_a^b \int_c^d f(x,y) \, dy \, dx = \int_c^d \int_a^b f(x,y) \, dx \, dy}$$

nested single variable integrals

Ex: $\iint_R \underbrace{t^2 e^{st}}_{f(s,t)} dA$ where $R = \underbrace{[0,1]}_{s} \times \underbrace{[0,1]}_{t}$

$$= \int_{t=0}^{t=1} \int_{s=0}^{s=1} t^2 e^{st} \, ds \, dt = \int_{t=0}^{t=1} \Big(\underbrace{\int_{s=0}^{s=1} t^2 e^{st} \, ds}\Big) dt$$

$$\int_0^1 t^2 e^{st} \, ds = t^2 \int_0^1 e^{st} \, ds = t^2 \left[\frac{1}{t} e^{st} \right]_0^1 = te^t - t$$

$$\Rightarrow \int_{t=0}^{t=1} (te^t - t) \, dt = \int_0^1 te^t \, dt - \int_0^1 t \, dt$$

choose order wisely!

$$= \left[te^t - e^t - \frac{1}{2} t^2 \right]_0^1 = \boxed{\frac{1}{2}}$$

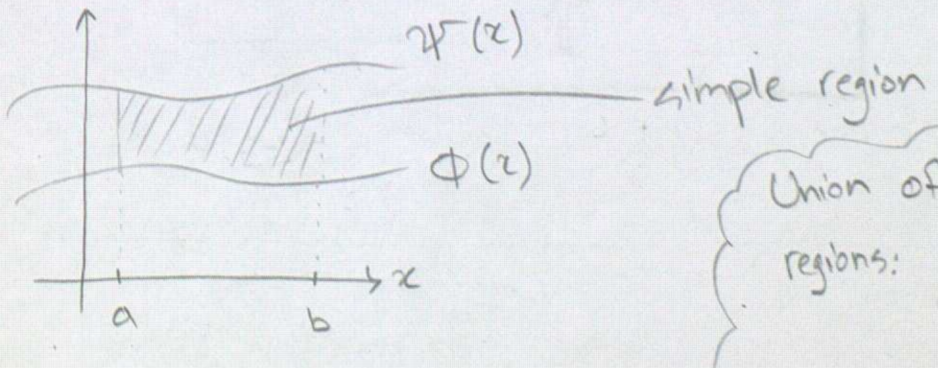
Double Integrals

- A region $S \subseteq \mathbb{R}^2$ is a simple region if it's of the form:

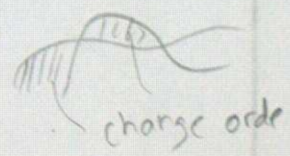
$$S = \{ (x,y) : a \leq x \leq b, \ \Phi(x) \leq y \leq \Psi(x) \}$$

$$S = \{ (x,y) : a \leq y \leq b, \ \Phi(y) \leq x \leq \Psi(y) \}$$

where Φ, Ψ are cont. functions



Union of simple regions:



Fubini's for Simple Regions

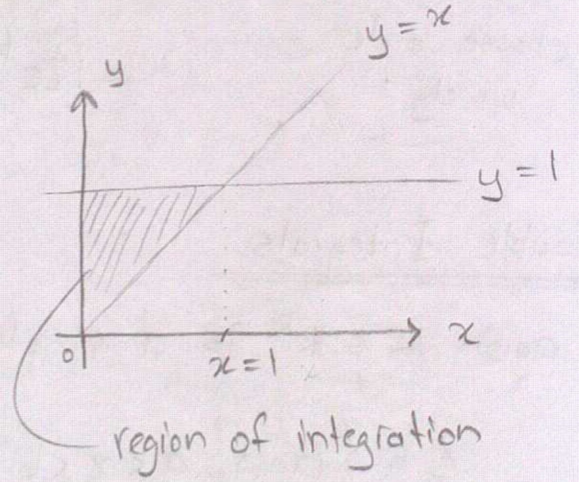
- If f is continuous and $S = \{(x,y) : a \le x \le b, \Phi(x) \le y \le \Psi(x)\}$

then
$$\boxed{\iint_S f(x,y)\,dA = \int_a^b \int_{\Phi(x)}^{\Psi(x)} f(x,y)\,dy\,dx}$$

- Order of integrals can't be easily solved.

$$\int_a^b \int_{\Phi(x)}^{\Psi(x)} f(x,y)\,dy\,dx \neq \int_{\Phi(x)}^{\Psi(x)} \int_a^b f(x,y)\,dx\,dy$$

Ex: $\int_0^1 \int_x^1 6y\,dy\,dx$



$$= \int_{x=0}^{x=1} \int_{y=x}^{y=1} 6y\,dy\,dx$$

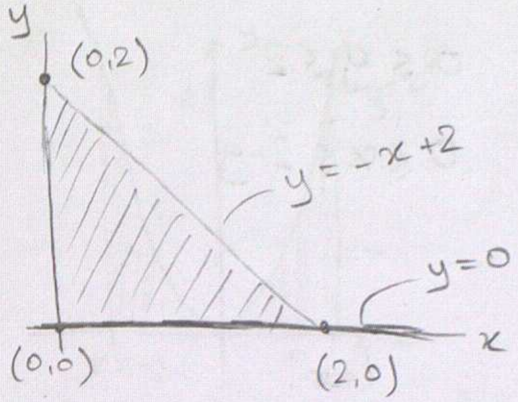
$$= \int_0^1 \left[3y^2\right]_{y=x}^{y=1} dx$$

↓ y-variable is gone, only left with x

$$= \int_0^1 (3 - 3x^2)\,dx$$

$$= 2$$

Ex: $\iint_R (x+y)\,dA$ where R is the rectangle with vertices (0,0), (2,0), (0,2)



→ as simple region

$0 \le x \le 2$

$0 \le y \le -x + 2$

↓

$0 \le y \le 2 - x$

$$\iint_R (x+y)\,dA = \int_{x=0}^{x=2} \int_{y=0}^{y=2-x} (x+y)\,dy\,dx$$

$$= \int_{x=0}^{x=2} \left[xy + \frac{1}{2}y^2\right]_{y=0}^{y=2-x} dx$$

$$= \int_{x=0}^{x=2} \left(x(2-x) + \frac{1}{2}(2-x)^2\right) dx$$

$$= 5 - \frac{8}{6}$$

Average Value of Func. over Plane Region

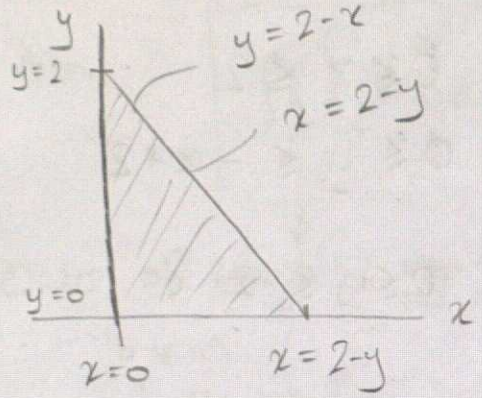
average value $\boxed{\bar{f} = \frac{1}{\text{area of R}} \iint_R f(x,y)\,dA}$

Week 5: Lecture 1 Sep 30, 2024

Double Integrals: Swapping Bounds

- in previous ex $0 \le y \le 2$

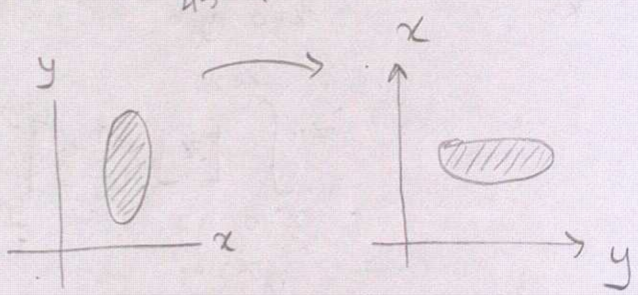
$0 \le x \le 2-y$



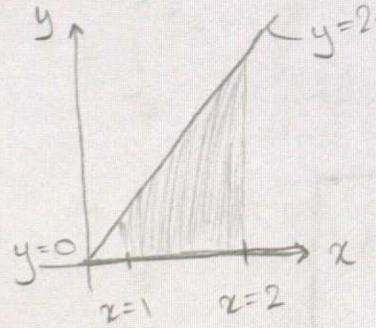
General Strategy

1) Draw the region
2) Swap the axes
3) Compute new bounds

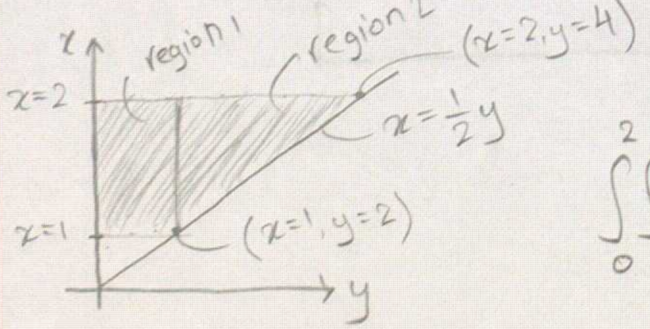
reflection over 45° line



Ex: Swap $\int_1^2 \int_0^{2x} f(x,y)\,dy\,dx$



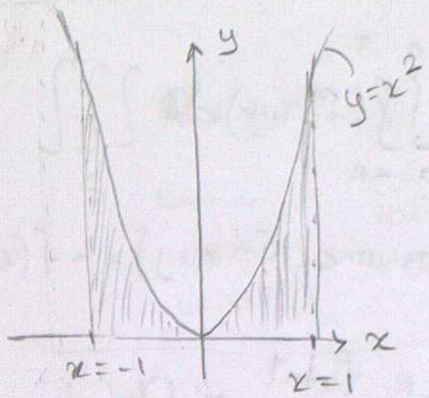
region 1	region 2
$0 \le y \le 2$	$2 \le y \le 4$
$1 \le x \le 2$	$\frac{1}{2}y \le x \le 2$



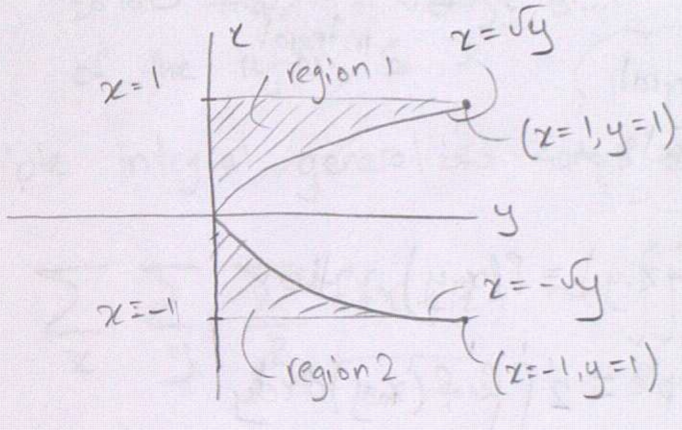
$$\int_1^2 \int_0^{2x} f(x,y)dydx =$$

$$\int_0^2 \int_1^2 f(x,y)dxdy + \int_2^4 \int_{\frac{1}{2}y}^2 f(x,y)dxdy$$

Ex: Swap $\int_{-1}^1 \int_0^{x^2} f(x,y)\,dy\,\boxed{dx}$ ← the one outside should be constants, so "y" must be our new constanst



region 1	region 2
$0 \le y \le 1$	$0 \le y \le 1$
$\sqrt{y} \le x \le 1$	$-1 \le x \le -\sqrt{y}$



$$\int_{-1}^1 \int_0^{x^2} f(x,y)dydx =$$

$$\int_0^1 \int_{\sqrt{y}}^1 f(x,y)dxdy +$$

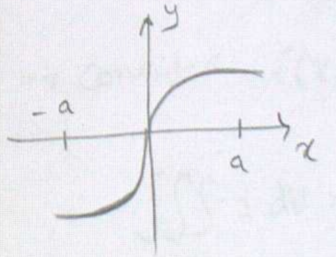
$$\int_0^1 \int_{-1}^{-\sqrt{y}} f(x,y)dxdy$$

Double Integrals: Symmetry

Recall: from single-variable integral

- odd function

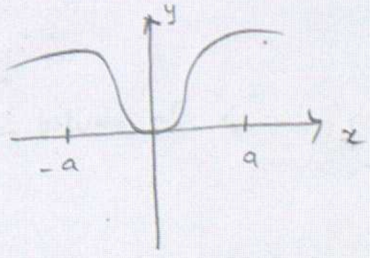
$f(-x) = -f(x)$



$$\int_{-a}^a f(x)dx = 0$$

- even function

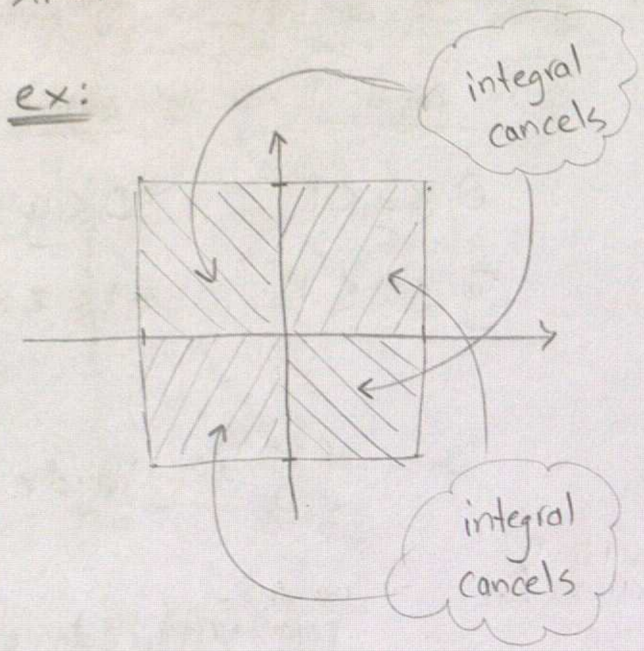
$f(-x) = f(x)$



$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$$

Similar idea in multivariables:

ex:



$$\int_{-a}^{a}\int_{-a}^{a} f(x,y)\,dA$$

assume $f(-x,-y) = -f(x,y)$

$\Longrightarrow$ total integral $= 0$

- for prev example, if $f(-x,y) = f(x,y)$, then

  integral from prev. example $= 2\int_0^1\int_{\sqrt{y}}^1 f(x,y)\,dx\,dy$

Triple Integrals

$$\iiint_S \underbrace{f(x,y,z)}_{\substack{\text{density}\\ \text{units}/m^3}}\ \underbrace{dV}_{\substack{\text{infinitesimal}\\ \text{Volume}}}$$

→ S is a region in 3d

total "mass" or "charge" or... of the region S

- triple integral generalizes sums over 3 indexes

$$\sum_x \sum_y \sum_z \underbrace{f(x,y,z)}_{\text{ex: mass}} \sim \iiint \underbrace{f(x,y,z)}_{\text{density}}\,dV$$

Fubini's for Rectangles

$R = [a,b] \times [c,d] \times [e,f]$

$$\iiint_R f(x,y,z)\,dV = \underbrace{\int_a^b\int_c^d\int_e^f f(x,y,z)\,dz\,dy\,dx}_{\text{other orders valid:}}$$

if f is cont.

→ consider $f(x,y,z) = 1$

$$\iiint_S 1\,dV = \mathrm{vol}(S)$$

volume of "S"

Tutorial 4 Oct 1, 2024

Small Signal Analysis

- state space equation: explain dynamics

$$\rightarrow \begin{cases} \frac{dx_1}{dt} = f_1(x_1, x_2, u_1(t), u_2(t)) \\ \frac{dx_2}{dt} = f_2(x_1, x_2, u_1(t), u_2(t)) \end{cases}$$

- output equation:

$$\rightarrow \begin{cases} y_1 = g_1(x_1, x_2, u_1(t), u_2(t)) \\ y_2 = g_2(x_1, x_2, u_1(t), u_2(t)) \end{cases}$$

- if $u_1(t) = U_1 + \hat{u}_1(t)$ $\quad \dot{U}_1 \longrightarrow U_1 + \hat{u}_1(t)$

$$\left.\begin{aligned} u_1(t) &= U_1 + \hat{u}_1(t) \\ x_1(t) &= X_1 + \hat{x}_1(t) \\ y_1(t) &= Y_1 + \hat{y}_1(t) \end{aligned}\right\} \text{same for } u_2(t),\ x_1(t), \text{ and } y_1(t)$$

Taylor Expantion

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = J_f \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + B_f \begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \end{bmatrix} + \begin{bmatrix} f_1(X_1\ X_2\ U_1\ U_2) \\ f_2(X_1\ X_2\ U_1\ U_2) \end{bmatrix}$$

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = J_g \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + B_g \begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \end{bmatrix} + \begin{bmatrix} g_1(X_1\ X_2\ U_1\ U_2) \\ g_2(X_1\ X_2\ U_1\ U_2) \end{bmatrix}$$

- equilibrium point → DC condition, so set the

$$\begin{bmatrix} f_1(X_1\ X_2\ U_1\ U_2) \\ f_2(X_1\ X_2\ U_1\ U_2) \end{bmatrix} \text{ and } \begin{bmatrix} g_1(X_1\ X_2\ U_1\ U_2) \\ g_2(X_1\ X_2\ U_1\ U_2) \end{bmatrix} \text{ equal to } \underline{\underline{0}}$$

Quiz 3, 2023

EQ. Point

$U_1 = 0$

$U_2 = 1$

$$① \begin{cases} \frac{dx_1}{dt} = x_1 \sin(x_2) - u_1 - u_2 \\ \frac{dx_2}{dt} = x_1 \cos(x_2) + u_1^2 - u_2^2 \end{cases}$$

$$\begin{cases} y_1 = x_1 u_1 \\ y_2 = x_2^2 \end{cases} \qquad \begin{cases} u_1(t) = U_1 + \hat{u}_1(t) \\ u_2(t) = U_2 + \hat{u}_2(t) \end{cases}$$

a) X_1, X_2, Y_1, Y_2 ← find these

$$① \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} X_1 \sin X_2 - U_1 - U_2 \\ X_1 \cos X_2 + U_1^2 - U_2^2 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \sin X_2 - 0 - 1 = 0 \\ X_1 \cos X_2 + 0 - 1 = 0 \end{bmatrix} \Rightarrow \left[\begin{array}{c|c} X_1 \sin X_2 & 1 \\ X_1 \cos X_2 & 1 \end{array}\right]$$

$$X_1 \sin X_2 = 1 = X_1 \cos X_2$$

$$\sin X_2 = \cos X_2$$

$$\therefore X_2 = \frac{\pi}{4} \Rightarrow X_1 = \frac{1}{\sin \frac{\pi}{4}} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

$$\boxed{\begin{aligned} X_2 &= \frac{\pi}{4} \\ X_1 &= \sqrt{2} \end{aligned}}$$

$Y_1 = X_1 U_1 = \sqrt{2} \cdot 0 = \boxed{0 = Y_1}$

$Y_2 = X_2^2 = \left(\frac{\pi}{4}\right)^2 = \boxed{\frac{\pi^2}{16} = Y_2}$

b) State space equation for $\hat{x}_1$ $\hat{x}_2$ $\hat{y}_1$ $\hat{y}_2$

Jacobian: $\begin{bmatrix} \sin(x_2) & x_1\cos(x_2) & -1 & -1 \\ \cos(x_2) & -x_1\sin(x_2) & 2u_1 & -2u_2 \end{bmatrix}_{(X_1 X_2 U_1 U_2)}$

$$\begin{bmatrix} \sin\frac{\pi}{4} & \sqrt{2}\cos\frac{\pi}{4} & -1 & -1 \\ \cos\frac{\pi}{4} & -\sqrt{2}\sin\frac{\pi}{4} & 0 & -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & -1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}$$

Jacobian: $\begin{bmatrix} u_1 & 0 & x_1 & 0 \\ 0 & 2x_2 & 0 & 0 \end{bmatrix}_{(X_1 X_2 Y_1 Y_2)}$

$$\begin{bmatrix} 0 & 0 & \sqrt{2} & 0 \\ 0 & \frac{\pi}{2} & 0 & 0 \end{bmatrix} \quad \therefore \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\pi}{2} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}$$

Week 5: Lecture 2 Oct. 2, 2024

- A simple region in $\mathbb{R}^3$ is a set of the form:

$$S = \{(x,y,z) : a \le x \le b,\ g(x) \le y \le h(x),\ \Phi(x,y) \le z \le \Psi(x,y)\}$$

where g, h, Φ, and Ψ are cont. functions

- other orders possible as well

ex: $a \le y \le b$, $g(y) \le z \le h(y)$, $\Phi(y,z) \le x \le \Psi(y,z)$

Fubini's for Simple Regions

If f is cont. and S is a simple region, then

$$\boxed{\iiint_S f(x,y,z)\,dV = \int_o^b \int_{g(x)}^{h(x)} \int_{\Phi(x,y)}^{\Psi(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

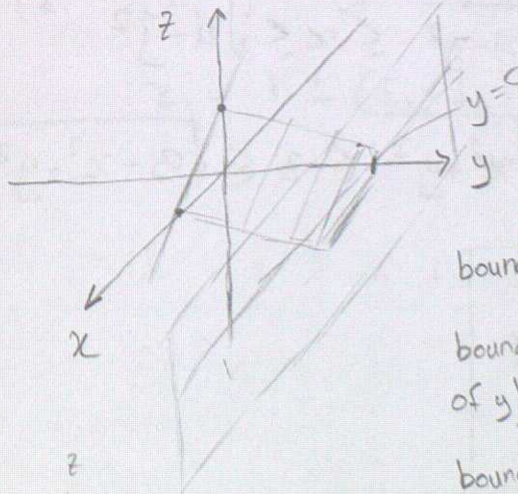
Ex: Volume of prism, bounded by $\underbrace{z = 2-4x}_{\text{exists on } zx \text{ plane}}$ and $y=8$ $\underbrace{(1^{st} \text{ octant})}_{x,y,z \ge 0}$

Volume $= \iiint_S 1\,dV$

z int: $z = 2$

x int: $x = \frac{1}{2}$

"triangular prism"

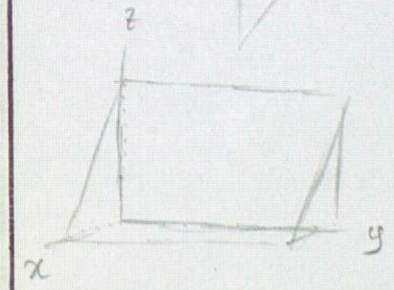


· pick order dzdxdy

bounds y (constants): $0 \le y \le 8$

bounds x (function of y) look at xy plane : $0 \le x \le \frac{1}{2}$ ← x-int.

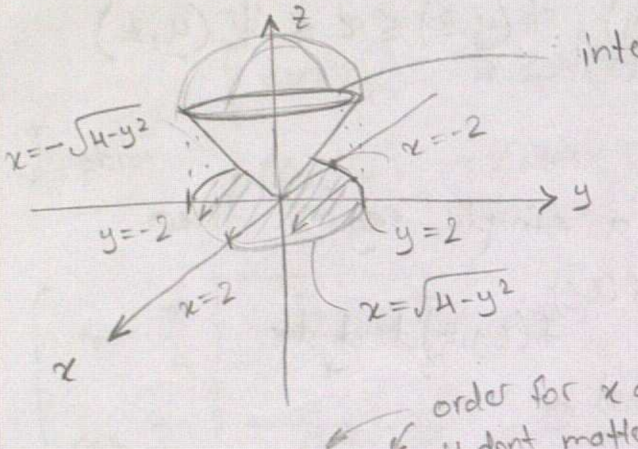
bounds z (function of x and y) look at whole shape : $0 \le z \le 2-4x$



$V = \int_0^8 \int_0^{1/2} \int_0^{2-4x} 1\,dz\,dx\,dy =$

Ex: Volume of solid bounded below by cone $z=\sqrt{x^2+y^2}$ and above by $x^2+y^2+z^2=8$

- cone : $z=\sqrt{x^2+y^2} = r \leftarrow$ polar, 45° cone
- sphere: $x^2+y^2+z^2=8$ with radius $\sqrt{8}$



intersection: $x^2+y^2+z^2=8$ and

$z=\sqrt{x^2+y^2}$

$$x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=8$$

$$\boxed{x^2+y^2=4} \text{ at } \boxed{z=2}$$

$$z=\sqrt{4}=2$$

- pick order $dz\,dx\,dy$ ← order for x and y dont matter since symmetric

  easiest to do inside ↗

bound for y (constant bounds) : $-2 \le y \le 2$

bound for x (function of y) : $-\sqrt{4-y^2} \le x \le \sqrt{4-y^2}$

bound for z (function of x,y): $\sqrt{x^2+y^2} \le z \le \sqrt{8-x^2-y^2}$

$$\boxed{\therefore \text{Volume} = \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} 1 \, dz\,dx\,dy}$$

Quiz 3 Small Signal Examples Oct 2, 2024

Example 2

$$\frac{dx_1}{dt} = x_1^2 - x_2 i_1 - i_2^2 + 1 \qquad y_1 = \sin(x_2) i_1 - x_1^2 i_2^2$$

$$\frac{dx_2}{dt} = -2x_1 + x_2 i_1 - 2 i_1 i_2 - i_1^2 \qquad y_2 = x_2 i_1 - x_1^2 i_2^2$$

operating point $i_1 = I_1$ and $i_2 = I_2$

a) At operating point, $\vec{\dot{x}} = 0$

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix} = \begin{bmatrix}x_1^2 - x_2 I_1 - I_2^2 + 1 \\ -2x_1 + x_2 I_1 - 2I_1 I_2 - I_1^2\end{bmatrix}$$

$$x_1^2 - \cancel{x_2 I_1} - I_2^2 + 1 - 2x_1 + \cancel{x_2 I_1} - 2I_1 I_2 - I_1^2 = 0$$

$$x_1^2 - 2x_1 - I_2^2 + 1 - 2I_1 I_2 - I_1^2 = 0$$

$$x_1^2 - 2x_1 + 1 = I_2^2 + 2I_1 I_2 + I_1^2$$

$$(x_1 - 1)(x_1 - 1) = (I_1 + I_2)^2$$

$$x_1 = 1 \pm (I_1 + I_2)$$

After plugging x_1 into second equation, we get

$$x_2 = \frac{I_1^2 + 2I_1 I_2 \pm 2(I_1 + I_2) + 2}{I_1}$$

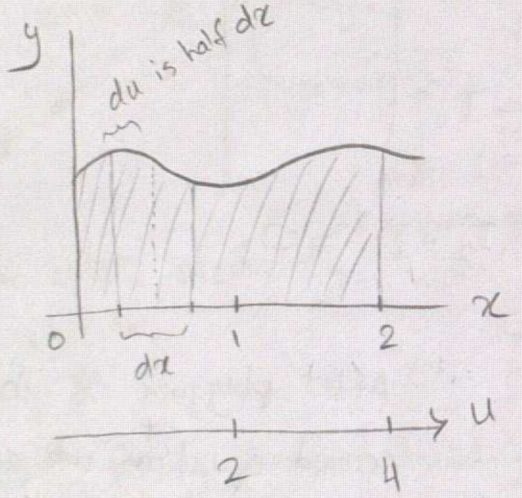
Week 5: Lecture 3 Oct 4, 2024

Change of Variables

Recall: Integration by substitution

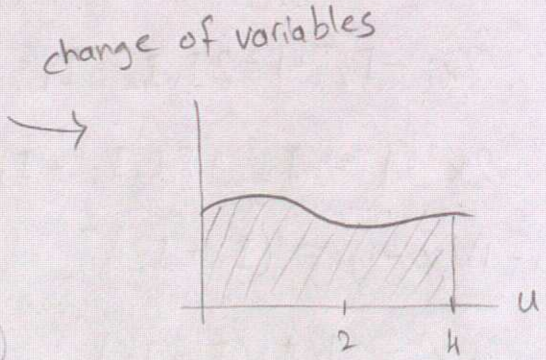
$$\int_{x=a}^{x=b} F(x)\,dx = \int_{u=g^{-1}(a)}^{u=g^{-1}(b)} F(g(u))\,g'u\,du$$

$$x = g(u) \Longrightarrow dx = \underbrace{g'(u)}_{\text{scaling factor}}du$$



$$\int_0^2 F(x)\,dx = \text{area under function}$$

Let $u = 2x \Longrightarrow du = 2dx$



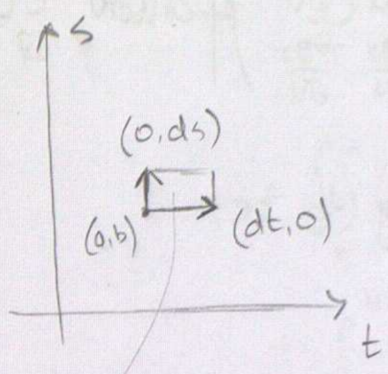
in general, scaling factor given by $g'(x)$ whee $u=g(x)$

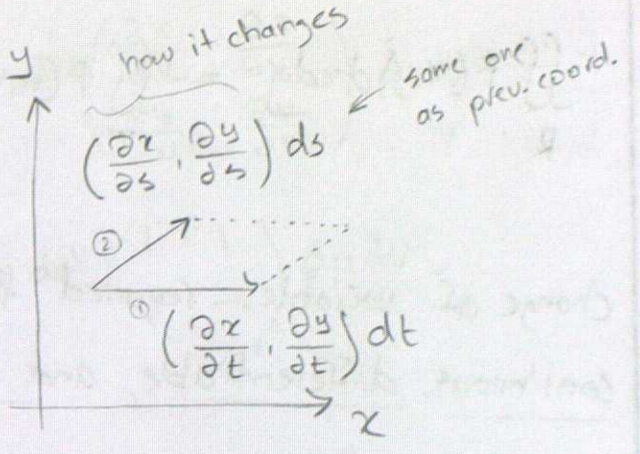
- to compensate, we must divide the integrand by scaling factor
- In 1D, we do change of variables to make integral easier.

ex: $\underset{\uparrow \text{ hard}}{\int 2t\sin(t^2)\,dt} = \underset{\uparrow \text{ easy}}{\int \sin u\,du}$

In 2D

$(s,t) \to (x,y)$





area = dsdt

$$\text{area} = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}\right)dt \times \left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}\right)ds$$

- the area of parallelogram defined by ① and ② vectors is the determinant of the vectors in matrix form, which gives scaling factor

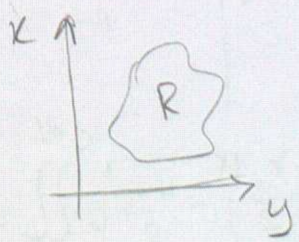
$$= \det\begin{bmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial s} \end{bmatrix} dtds$$

$$\det\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \leftarrow \text{determinant of Jacobian matrix}$$

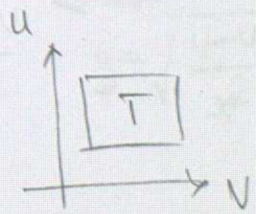
$$= \underbrace{\det|J|}_{\text{scaling factor is Jacobian}}\ \underbrace{dtds}_{\text{area of first coordinate system}}$$

Ex: R is region on xy plane, and let

$(x,y) = (g_1(u,v), g_2(u,v)) = g(u,v)$ and g maps region T in uv to R:



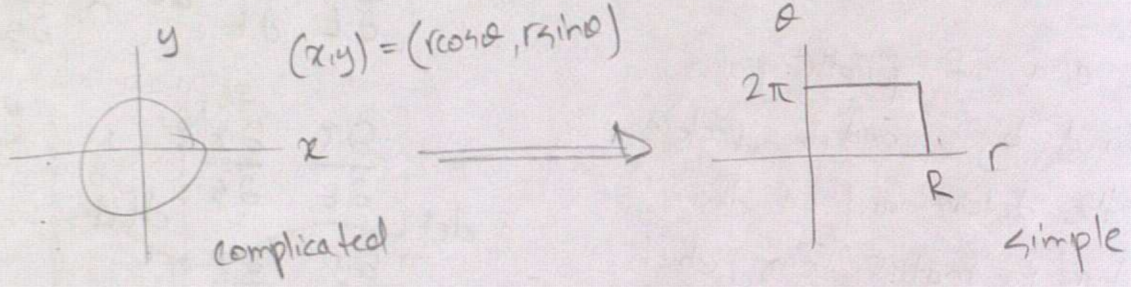
$$\begin{tikzcd} (x,y) \arrow[r, bend left, "g^{-1}"] & (u,v) \arrow[l, bend left, "g"] \end{tikzcd}$$



$$\iint_R F(x,y)\,dxdy = \iint_T F(u,v)\left|\det\begin{pmatrix}\frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v}\\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v}\end{pmatrix}\right| dudv$$

- change of variables required the map g to be continuous, differentiable, and bijective.
- In 2D, we do change of variables to make region simpler

$(x,y) = (r\cos\theta, r\sin\theta)$



- Sometimes, changing the variables may make integrand more complicated

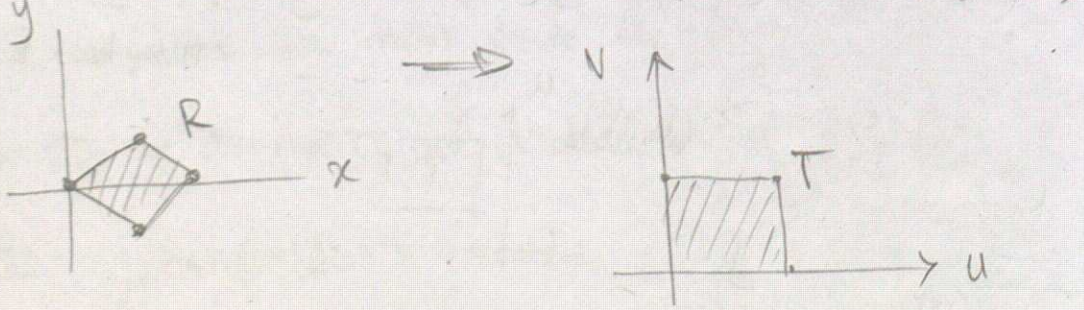
Ex: $\iint_R xy\,dA$ R formed by $(0,0)$ $(1,1)$ $(2,0)$ $(1,-1)$

and $x = u+v$ $y = u-v$

to transfer, we plug into our equations

$\Rightarrow u = \frac{1}{2}(x+y)$ $v = \frac{1}{2}(x-y)$

$(0,0)$ $(1,0)$ $(1,1)$ $(0,1)$



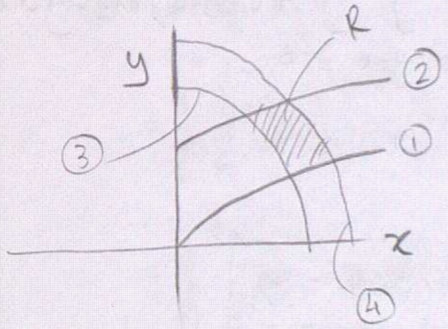
$$\iint_R xy\,dxdy = \iint_T (u+v)(u-v)\cdot\det\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix} dudv$$

$$= \iint_T (u+v)(u-v)\cdot\det\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} dudv$$

$$= \int_{v=0}^{v=1}\int$$

Ex 2: $\iint_R \sqrt{y^2-x^2}\,dA$, 1st quadrant bounded by parabolas

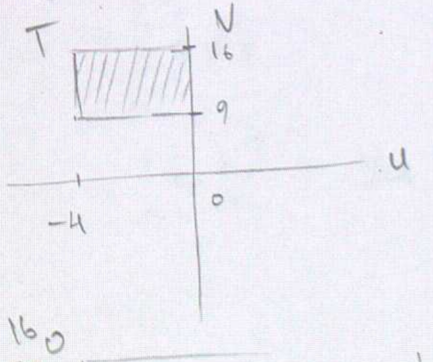
$x=y^2$ $x=y^2-4$ $x=9-y^2$ $x=16-y^2$



$$\begin{cases} x-y^2=0\\ \underbrace{x-y^2}_{u}=-4\end{cases}\quad\begin{cases} x+y^2=9\\ \underbrace{x+y^2}_{v}=16\end{cases}$$

$$\hookrightarrow\begin{cases} u=x-y^2 : -4\leq u\leq 0\\ v=x+y^2 : 9\leq v\leq 16\end{cases}$$

- R in region xy is the rectangular in region uv



$$\iint_R \sqrt{y^2-x^2}\,dxdy = \iint_T \sqrt{y^2-x^2}\cdot\det\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} dudv$$

$$x = \frac{u+v}{2}\quad y=\sqrt{\frac{v-u}{2}},\quad \det = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2\sqrt{2(v-u)}} & \frac{1}{2\sqrt{2(v-u)}}\end{pmatrix}$$

$$\det = \frac{1}{2\sqrt{2(v-u)}}$$

$$= \iint_{9\ \ -4}^{16\ \ 0} \sqrt{\frac{v-u}{2} - \left(\frac{u+v}{2}\right)^2}\cdot\frac{1}{2\sqrt{2(v-u)}}\,dudv$$

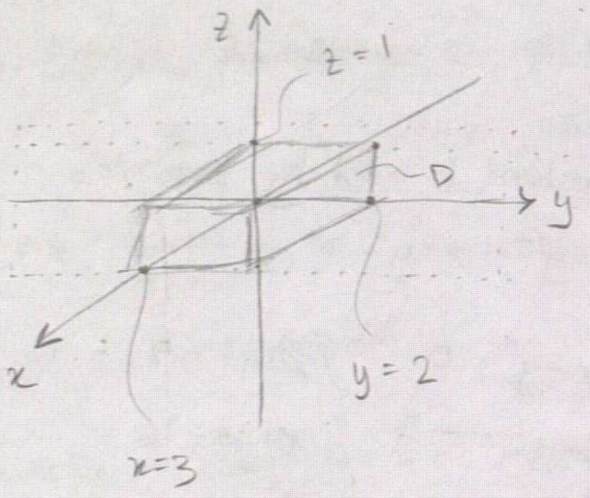
Triple Integrals Reading Assigment

Example 1

bound by planes $x=0,\ x=3,\ y=0,\ y=2,\ z=0,\ z=1$.

density given by $f(x,y,z) = 2-z$. Find mass

$$\text{mass} = \iiint_D f(x,y,z)\,dV$$



$0 \leq x \leq 3$

$0 \leq y \leq 2$

$0 \leq z \leq 1$

$$D: \int_{x=0}^{x=3}\int_{y=0}^{y=2}\int_{z=0}^{z=1} f(x,y,z)\,dz\,dy\,dx$$

$$M = \int_0^3\int_0^2\int_0^1 2-z\,dz\,dy\,dx$$

$$= \int_0^3\int_0^2 \left[2z - \frac{z^2}{2}\right]_0^1 dy\,dx$$

$$= \int_0^3\int_0^2 \frac{3}{2}\,dy\,dx$$

$$= \int_0^3 \left[\frac{3}{2}y\right]_0^2 dx$$

$$= \int_0^3 3\,dx$$

$$= [3x]_0^3 = 9$$

$$\boxed{\therefore \text{mass} = 9}$$

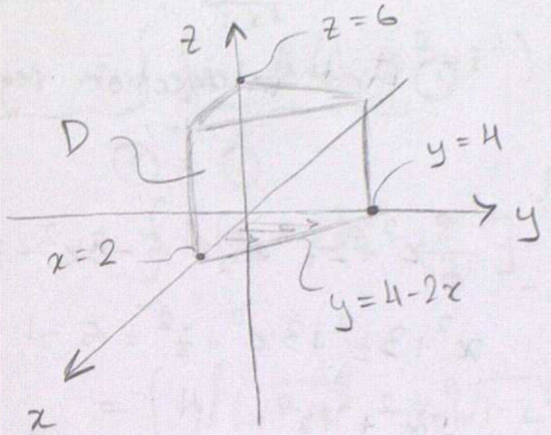
Example 2: prism D in 1st octant, bounded by

$y = 4-2x$ and $z = 6$

$0 \leq x \leq 2$

$0 \leq y \leq 4-2x$

$0 \leq z \leq 6$



$$V_D = \int_0^6\int_0^2\int_0^{4-2x} 1\,dy\,dx\,dz$$

$$= \int_0^6\int_0^2 [y]_0^{4-2x}\,dx\,dz$$

$$= \int_0^6\int_0^2 4-2x\,dx\,dz$$

$$= \int_0^6 \left[4x - \frac{2x^2}{2}\right]_0^2 dz$$

$$= \int_0^6 4\,dz$$

$$= [4z]_0^6 = 24$$

① slicing along $z \to z$ is outer

② for individual triangles, slicing along x, so x is next outer

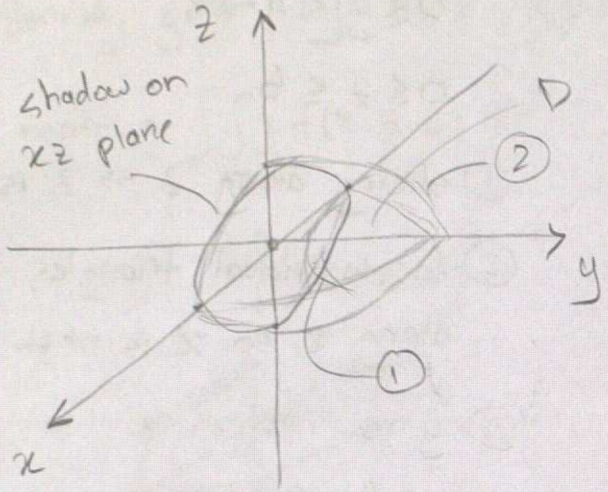
③ y is inner

$$V_D = \int_{z=0}^{z=6}\int_{x=0}^{x=2}\int_{y=0}^{y=4-2x} 1\,dy\,dx\,dz$$

$$\boxed{\therefore \text{Volume} = 24}$$

Example 3 Volume bound by paraboloids

$y = x^2 + 3z^2 + 1$ ① and $y = 5 - 3x^2 - z^2$ ② $\Rightarrow -y = 3x^2 + z^2 - 5$



① Find intersection region

① = ②

$$x^2 + 3z^2 + 1 = 5 - 3x^2 - z^2$$

$$x^2 + 3z^2 + 3x^2 + z^2 = 5 - 1$$

$$4x^2 + 4z^2 = 4$$

$$\boxed{x^2 + z^2 = 1}$$

circle on xz plane with radius 1

$$z = \pm\sqrt{1-x^2}$$

$$x^2 + 3z^2 + 1 \le y \le 5 - 3x^2 - z^2$$

$$-1 \le x \le 1$$

$$-\sqrt{1-x^2} \le z \le \sqrt{1-x^2}$$

① y is f(x, z) — inside,

② z is f(x) — next

③ x is bounded by constants so it is the outer

$$V_D = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+3z^2+1}^{5-3x^2-z^2} 1\, dy\, dz\, dx$$

$$V_D = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [y]_{x^2+3z^2+1}^{5-3x^2-z^2}\, dz\, dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 5 - 3x^2 - z^2 - x^2 - 3z^2 - 1\, dz\, dx$$

$$V_D = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 4 - 4x^2 - 4z^2\, dz\, dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 4(1 - x^2 - z^2)\, dz\, dx$$

$$= \int_{-1}^{1} 4\left[z - x^2 z - \frac{z^3}{3}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} 4\left[\left(\sqrt{1-x^2} - x^2\sqrt{1-x^2} - \frac{(\sqrt{1-x^2})^3}{3}\right) - \left((-\sqrt{1-x^2}) - x^2(-\sqrt{1-x^2}) - \frac{(-\sqrt{1-x^2})^3}{3}\right) dx\right.$$

$$= \int_{-1}^{1} 4\left(\sqrt{1-x^2} - x^2\underset{\uparrow}{\sqrt{1-x^2}} - \overset{\downarrow}{\frac{(1-x^2)^{3/2}}{3}} + \sqrt{1-x^2} - x^2\underset{\uparrow}{\sqrt{1-x^2}} - \overset{\downarrow}{\frac{(1-x^2)^{3/2}}{3}}\right) dx$$

$$= \int_{-1}^{1} 4\left(2\sqrt{1-x^2} - 2x^2\sqrt{1-x^2} - \frac{2}{3}(1-x^2)^{3/2}\right) dx$$

$$= 8\int_{-1}^{1} \sqrt{1-x^2} - x^2\sqrt{1-x^2} - \frac{1}{3}(1-x^2)^{3/2}\, dx$$

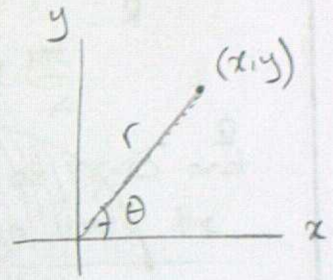
Week 6: Lecture 1

Oct. 7, 2024

$\iint_R xy\,dA$ R bounded by $y=0$, $x-2y=0$, $x-y=1$

Polar Coordinates

- special case of change of variables



$$(x,y) = (r\cos\theta, r\sin\theta)$$

$$(r,\theta) = \left(\sqrt{x^2+y^2}, \pm\arctan\left(\frac{y}{x}\right)\right)$$

$$\iint_R f(x,y)\,dx\,dy = \iint_T f(r\cos\theta, r\sin\theta) \left| \underbrace{\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}}_{} \right| dr\,d\theta$$

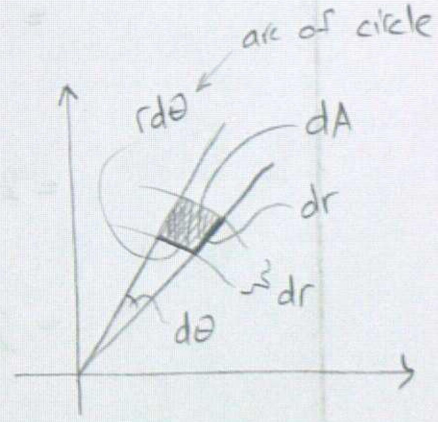
$$= \det\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$\boxed{\therefore \iint_R f(x,y)\,dx\,dy = \iint_T f(r\cos\theta, r\sin\theta)\, r \cdot dr\,d\theta}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

- we want new variables in "denominator" for the partials → w.r.t those variables

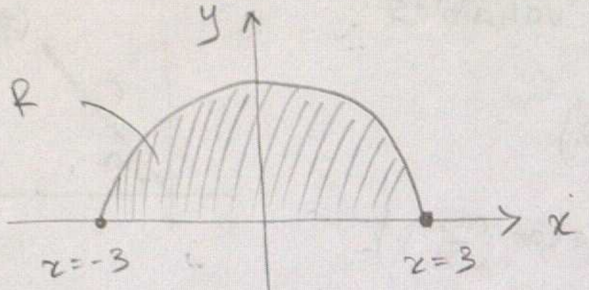
$$\boxed{\iint_R f(x,y)\,dx\,dy = \iint_{\tilde{R}} f(r,\theta)\,\underbrace{r\,dr\,d\theta}_{dA}}$$



$\therefore dA = r\,dr\,d\theta$

- $\tilde{R}$ is R in polar coordinates
- $f(r,\theta)$ is f in terms of polar
  - ↳ $f(r\cos\theta, r\sin\theta)$

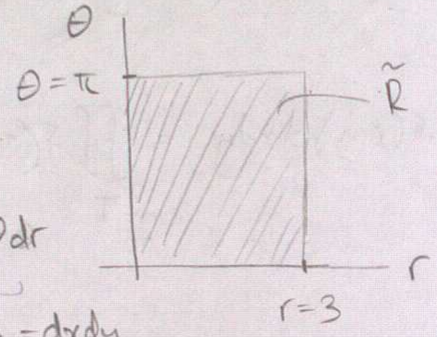
Ex: $\iint\limits_R 2xy\,dxdy \quad R=\{(x,y): x^2+y^2 \leq 9,\ y>0\}$



bounds of integration

$0 \leq r \leq 3$

$0 \leq \theta \leq \pi$



$$\iint\limits_R 2xy\,dxy = \int_0^3\int_0^\pi 2(r\cos\theta)(r\sin\theta)\underbrace{r\,d\theta\,dr}_{dA\,=\,dxdy}$$

$$= \int_0^3\int_0^\pi 2r^2\cos\theta\sin\theta\, r\,d\theta\,dr$$

$$= 2\int_0^3\int_0^\pi r^3\sin 2\theta\,d\theta\,dr$$

$$= 2\int_0^3 r^3\left[\frac{1}{2}\cdot -\cos 2\theta\right]_0^\pi dr$$

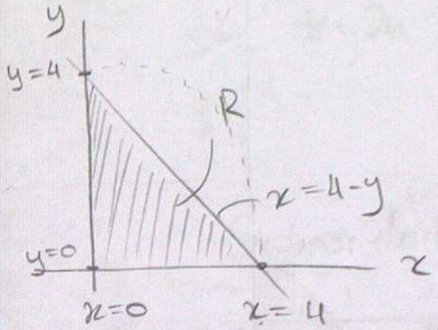
$$= 2\int_0^3 r^3\left[-\frac{1}{2}\overset{1}{\cancel{\cos 2\pi}} + \frac{1}{2}\overset{1}{\cancel{\cos 0}}\right] dr$$

$$= 2\int_0^3 r^3\left(-\frac{1}{2}+\frac{1}{2}\right) dr$$

$$= 2\int_0^3 0\,dr$$

$$= 0$$

Ex: $\int_0^4\int_0^{4-y} \frac{1}{16-x^2-y^2}\,dxdy$ ... Express in polar coord.



→ bounds θ: $0 \leq \theta \leq \frac{\pi}{2}$

→ bounds r: you start at r=0 and you only go until the line $x=4-y$

$\downarrow$

$r\cos\theta = 4 - r\sin\theta$

$r\cos\theta + r\sin\theta = 4$

$r(\cos\theta + \sin\theta) = 4$

$r = \frac{4}{\cos\theta+\sin\theta}$ ← upper bound

r=0 lower bound

bounds for new variables

θ: $0 \leq \theta \leq \frac{\pi}{2}$

r: $0 \leq r \leq \frac{4}{\cos\theta+\sin\theta}$

$$\therefore \int_0^4\int_0^{4-y}\frac{1}{16-x^2-y^2}\,dxdy = \int_0^{\pi/2}\int_0^{\frac{4}{\cos\theta+\sin\theta}} \frac{1}{16-r^2}\,r\,dr\,d\theta \quad //$$

Changing Variables

$$\iint_R f(x,y)\,dA = \iint_S f(g(u,v),h(u,v))\,|J(u,v)|$$

Steps

1) sketch the region in xy plane (initial region)

2) Find the limits of integration for the new integral based on the given transformation formula for u and v, then sketch the new region

3) Calculate the absolute value of Jacobian

4) Change variables and evaluate the new integral

To find a substitution formula

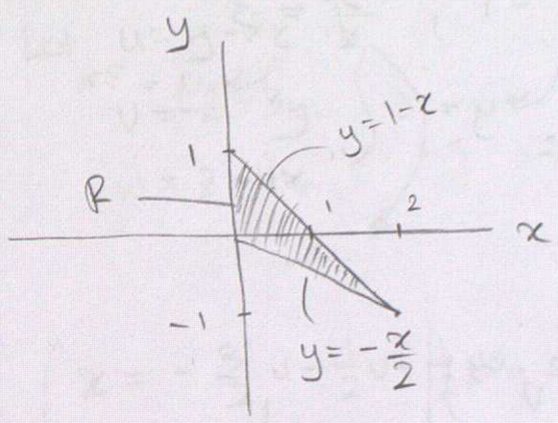
Look at the bounded points/lines/planes

⇒ if they are the same and only shifted w.r.t. each other, then use them as transformation formula

⇒ if the points are representing parallelogram, then derive the equations and use them.

29. $\iint_R x^2\sqrt{x+2y}\,dA$ where R: $0 \le x \le 2$, $-\frac{x}{2} \le y \le 1-x$

use $x=2u$ $y=v-u$

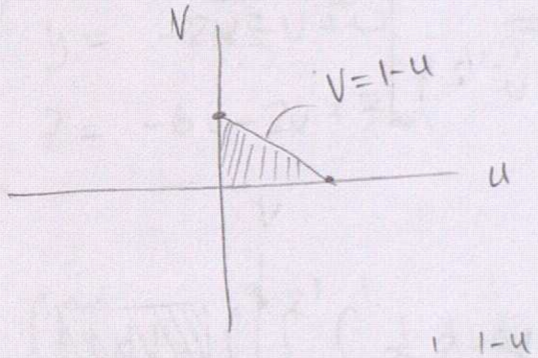


$x=2u \Rightarrow u=\frac{x}{2} \Rightarrow \boxed{0 \le u \le 1}$

$y=v-u \Rightarrow -\frac{x}{2} \le y \le 1-x$

$-u \le v-u \le 1-2u$

$\boxed{0 \le v \le 1-u}$



$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = -2 \Rightarrow |J| = 2$$

$$\iint_A x^2\sqrt{x+2y}\,dA = \int_0^1\int_0^{1-u} 4u^2\sqrt{2v} \times 2\,dv\,du =$$

34. $\iint_R e^{xy}\,dA$, R is in 1st quadrant bounded by

the hyperbolas $xy=1$; $xy=4$

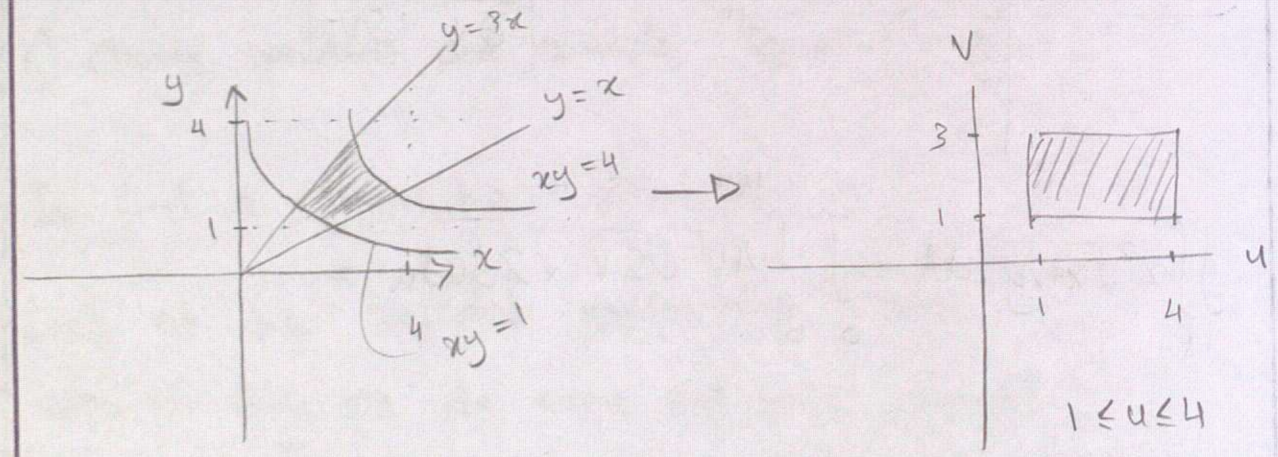
and the lines $\frac{y}{x}=1$, $\frac{y}{x}=3$

$\hookrightarrow y=x$ $\quad\hookrightarrow y=3x$

Let $u=xy$ and $v=\frac{y}{x}$

$\Rightarrow x=\sqrt{\frac{u}{v}}$, $y=\sqrt{uv}$

$$J=\begin{vmatrix}\frac{1}{2}(u^{-1/2}v^{-1/2}) & -\frac{1}{2}(u^{-1/2}v^{-3/2}) \\ \frac{1}{2}(u^{-1/2}v^{1/2}) & \frac{1}{2}(u^{1/2}v^{-1/2})\end{vmatrix}=\frac{1}{2v}$$



$1\le u\le 4$

$1\le v\le 3$

$$\iint_R e^{xy}\,dA=\int_1^4\int_1^3 e^4\frac{1}{2v}\,dudu=\frac{\ln 3}{2}(e^4-c)$$

$u=xy \qquad v=\frac{y}{x}$

$x=\frac{u}{y} \qquad y=vx \qquad y=v\cdot\frac{u}{y}\Rightarrow y^2=vu$

$x=\frac{u}{vx} \qquad y=\sqrt{vu}$

$x^2=\frac{u}{v}\Rightarrow x=\sqrt{\frac{u}{v}}$

42. $\iiint_D dv$ when D bounded by

$y-2x=0 \qquad z-3y=0$

$y-2x=1 \qquad z-3y=1$

$z-4x=0$

$z-4x=3$

Let $u=y-2x$

$v=z-3y$

$w=z-4x$

$0\le u\le 1$

$0\le v\le 1$

$0\le w\le 3$

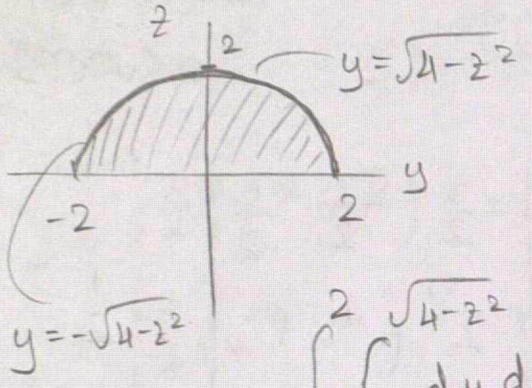
$$\begin{cases} x=-\frac{3}{2}u-\frac{1}{2}v+\frac{1}{2}w \\ y=-2u-v+w \\ z=-6u-2v+3w \end{cases}\Rightarrow J\begin{vmatrix}-\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -2 & -1 & 1 \\ -6 & -2 & 3\end{vmatrix}=\frac{1}{2}$$

$$\iiint_D dv=\int_0^3\int_0^1\int_0^1\frac{1}{2}\,du\,dv\,dw=\frac{3}{2}$$

Changing Order of Integration

48: $\int_0^1\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} dz\,dy\,dx \Longrightarrow dy\,dz\,dx$

↓



$0 \le z \le \sqrt{4-y^2}$ →

$z=\sqrt{4-y^2}$

$y = \pm\sqrt{4-z^2}$

$z^2 = 4-y^2$

$z^2+y^2=4$

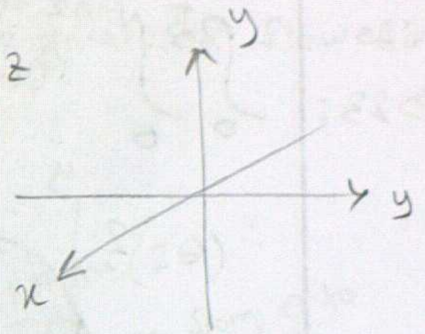
only take top half since $z \geq 0$

$\int_0^2\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dy\,dz$

↓

$\int_0^1\int_0^2\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dy\,dz\,dx$

50. $\int_0^4\int_0^{\sqrt{16-x^2}}\int_0^{\sqrt{16-x^2-z^2}} dy\,dz\,dx \Longrightarrow dx\,dy\,dz$



$0 \le y \le \sqrt{16-x^2-z^2}$

$y = \sqrt{16-x^2-z^2}$

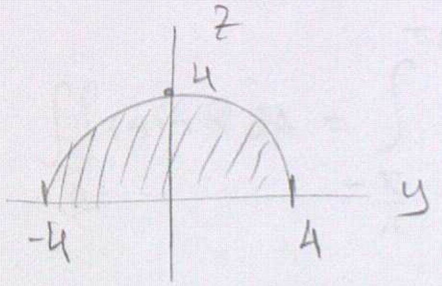
$y^2 = 16-x^2-z^2$

$x^2+y^2+z^2 = 16$ ← sphere of radius 16

so for inner "dy" → same as "dx", so only thing that changes is $y=\sqrt{16-x^2-z^2} \longrightarrow x=\sqrt{16-y^2-z^2}$

so new upper bound in dx is $x=\sqrt{16-y^2-z^2}$

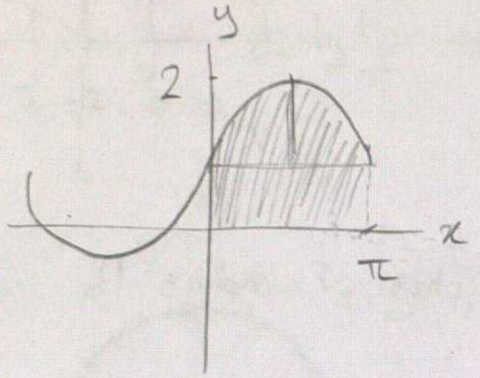
→ for dy, we project to yz plane (since we already defined dx)



$0 \le y \le \sqrt{16-z^2}$

$\int_0^4\int_0^{\sqrt{16-z^2}}\int_0^{\sqrt{16-y^2-z^2}} dx\,dy\,dz$

Quiz 2023: $\int_0^{\pi}\int_0^{\sin x+1} f(x,y)\,dy\,dx \longrightarrow dx\,dy$



$y=\sin x+1$

$$x=\begin{cases}\arcsin(y-1) & 0\le x\le \frac{\pi}{2}\\ -\arcsin(y-1)+\pi & \frac{\pi}{2}\le x\le \pi\end{cases}$$

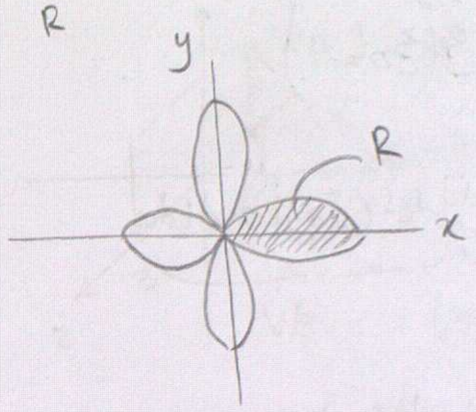
$$\int_1^2\int_{+\arcsin(y-1)}^{-\arcsin(y-1)+\pi} f(x,y)\,dx\,dy + \int_0^1\int_0^{\pi} f(x,y)\,dx\,dy$$

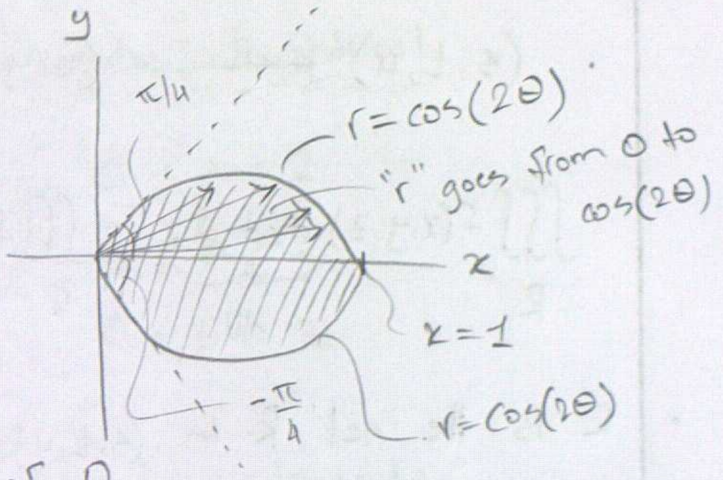
Week 6: Lecture 2

Oct 9, 2024

Ex: flower bounded by $r=\cos(2\theta)$, right petal of flower

$\iint_R f\,dA$, $f(r,\theta)=r\sin\theta$





- bound θ since r is function of θ.
- Let's do both ways:
- polar bounds, order $dr\,d\theta$

symmetry argument

→ bounds for θ: $-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$

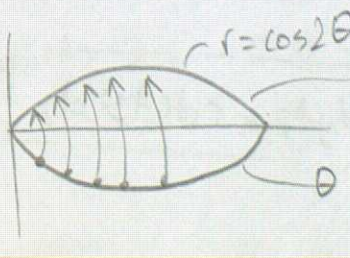
→ bounds for r: $0\le r\le\cos(2\theta)$

$$\iint_R r\sin\theta\,dA=\int_{-\frac{\pi}{4}}^{\pi/4}\int_0^{\cos 2\theta} r\sin\theta\, \underbrace{r\,dr\,d\theta}_{dA}$$

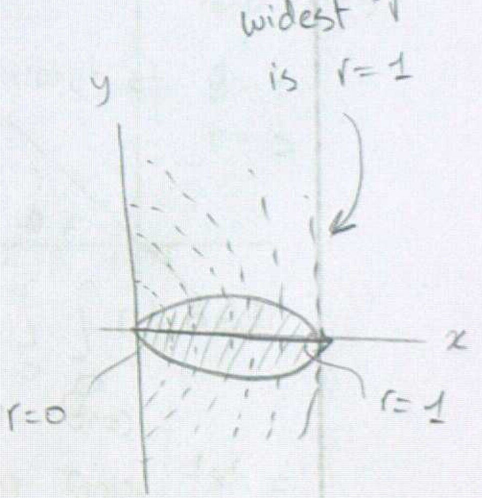
- reverse order: $d\theta\,dr$

→ bounds for r: $0\le r\le 1$

→ bounds for θ: $-\frac{1}{2}\arccos(r)\le r\le\frac{1}{2}\arccos(r)$



solve θ as func. of r



Change of Variables in 3D

- change of variable in 3D

g is bijective, differentiable mapping

$$(s,t,u) \xmapsto{g} (x,y,z) = g(s,t,u)$$

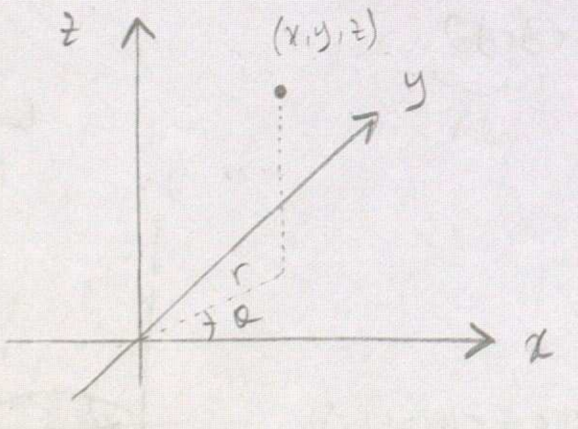
$$\iiint_R f(x,y,z) \underbrace{dxdydz}_{dV} = \iiint_{\tilde{R}} \underbrace{f(s,t,u)}_{f\circ g(s,t,u)} \underbrace{|\det(J_g)|\, dsdtdu}_{dV}$$

- $\tilde{R}$ is the set R in (s,t,u) coordinates
- J_g is jacobian matrix

Cylindrical Coordinates

$$(r,\theta,z) \longmapsto (x,y,z) = (r\cos\theta, r\sin\theta, z)$$

$$(x,y,z) \longmapsto (r,\theta,z) = \left(\sqrt{x^2+y^2}, \pm\arctan\left(\frac{y}{x}\right), z\right)$$



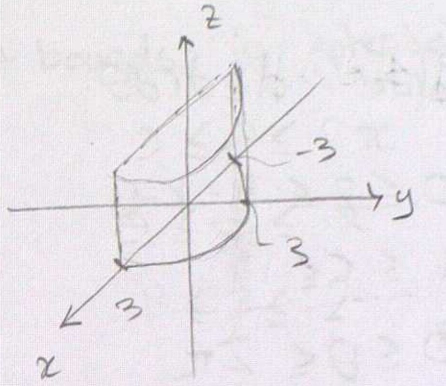
change of variables

$$\det(J_g) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \det \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\boxed{\therefore dV = dxdydz = r\,dr\,d\theta\,dz}$$

Ex: $\iiint_R \frac{1}{1+x^2+y^2} dU$, R bounded by $x^2+y^2=9$ ← cylinder and planes $z=0$, $z=2$, $y \geqslant 0$



→ bounds in cylindrical

$0 \leq z \leq 2$

$0 \leq r \leq 3$

$0 \leq \theta \leq \pi$

$$\iiint_R \frac{1}{1+x^2+y^2} dU = \int_0^2\int_0^3\int_0^\pi \frac{1}{1+r^2} r\,d\theta\,dr\,dz$$

Ex: $\displaystyle\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^{9-3\sqrt{x^2+y^2}} 1 \; dzdxdy$ $\Big\{$

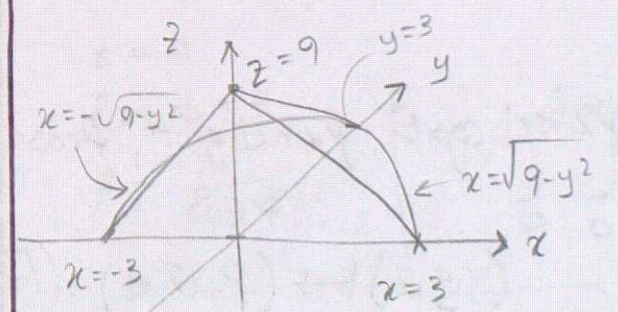
y: $y=0$ to $y=3$

x: $x=-\sqrt{9-y^2}$ to $x=\sqrt{9-y^2}$

z: $z=0$ to $z=9-3\sqrt{x^2+y^2}$

$z = 9-3r$ ← in cylindrical coordinates

↑ intercept at $z=9$, $r=3$



cylindrical coordinates $dzd\theta dr$

bounds r: $0 \leq r \leq 3$

bounds θ: $0 \leq \theta \leq \pi$

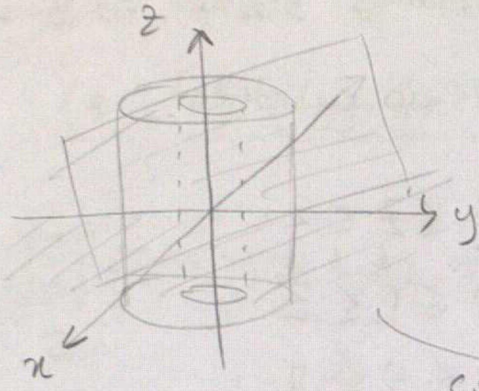
bounds z: $0 \leq z \leq 9-3r$

$$\iiint_R dV = \int_0^2\int_0^\pi\int_0^{9-3r} r\,dz\,d\theta\,dr$$

Week 6: Lecture 3

Oct 11, 2024

Cylinder Ex: bounded by $x^2+y^2=1$, $x^2+y^2=4$, planes $z=4-x-y$, $z=0$



choose order $dz\,dr\,d\theta$

$0 \le z \le 4-x-y$

$1 \le r \le 2$

$0 \le \theta \le 2\pi$

full 2π rotation

Let $r^2 = x^2+y^2$

$\therefore \; 1 \le r^2 \le 4$

$1 \le r \le 2$

$\Rightarrow z = 4-x-y$

$z = 4 - r\cos\theta - r\sin\theta$

$$\boxed{\therefore V = \int_0^{2\pi}\int_1^{2}\int_0^{4-r(\cos\theta+\sin\theta)} r(1)\,dz\,dr\,d\theta}$$

Spherical Coordinates

$$(\rho,\theta,\phi) \longmapsto (x,y,z) = (\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)$$

ρ: distance to point

θ: angle from (+)ve x

ϕ: angle from (+)ve z

$$(x,y,z) \longmapsto (\rho,\theta,\phi) = (\sqrt{x^2+y^2+z^2}\ldots$$

$$J_g = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix}$$

$\det(J_g) = \rho^2\sin\phi$

$$\boxed{dV = \rho^2 \sin\phi \, d\phi\, d\theta\, d\rho}$$

Ex: $\displaystyle\iiint_D \frac{dV}{(x^2+y^2+z^2)^{3/2}}$ where d is solid b/n sphere of radius 1 and sphere of radius 2

→ bounds in spherical

$0 \le \theta \le 2\pi$

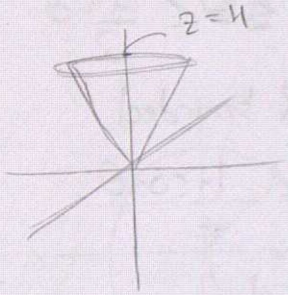
$0 \le \phi \le \pi$

$1 \le \rho \le 2$

← rectangle in spherical

$$\boxed{\int_0^{2\pi}\int_0^{\pi}\int_1^{2} \frac{1}{(\rho^2)^{3/2}}\rho^2 \sin\phi \, dr\,d\phi\,d\theta}$$

Ex: Volume of solid bounded by cone $\phi = \frac{\pi}{3}$ and plane $z=4$



reason to use sherical ↗

choose: $d\rho\, d\phi\, d\theta$

bounds: $0 \le \theta \le 2\pi$

$0 \le \phi \le \frac{\pi}{3}$

$0 \le \rho \le \frac{4}{\cos\phi}$

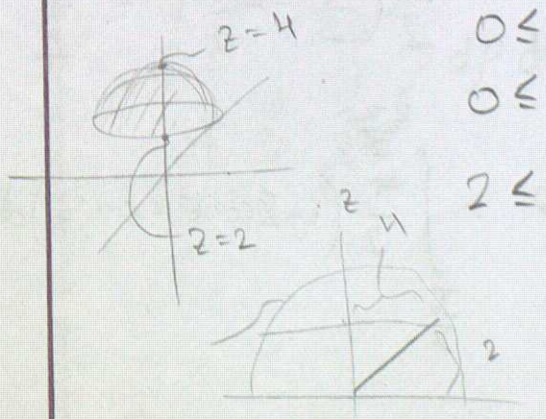
$z = 4$

$\rho\cos\phi = 4$

$\rho = \frac{4}{\cos\phi}$

$$\boxed{V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4/\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}$$

Ex: Volume of shere radius $r=4$ above $z=2$



$0 \le \theta \le 2\pi$

$0 \le \phi \le \frac{\pi}{3}$

$2 \le \rho \le 4$

$x^2+y^2+z^2 = 4^2$

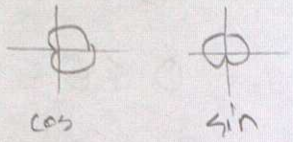
$\rho^2 = 16$

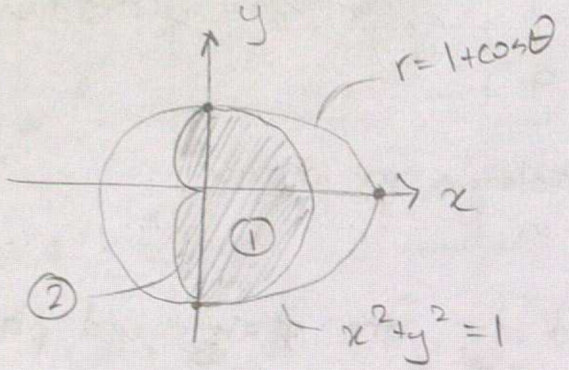
$\rho = 4$

Tutorial 5 Oct 15, 2024

① Compute the area of the region bounded by the cardioid $r = 1 + \cos\theta$ and the circle $x^2 + y^2 = 1$

Hint: $\int \cos^2 t\, dt = \frac{1}{2}(t + \sin t \cos t)$





for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

our r is bounded by 0 and 1

for $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$

our r is bounded by 0 and $1+\cos\theta$

$$①: \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{1} r\,dr\,d\theta$$

$$②: \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{0}^{1+\cos\theta} r\,dr\,d\theta$$

$$\iint_R dA = \iint_{\tilde{R}} r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{1} r\,dr\,d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{0}^{1+\cos\theta} r\,dr\,d\theta$$

$$① \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_0^1 d\theta = \frac{1}{2}\left[\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{2}\left(\frac{\pi}{2} - -\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$② \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[\frac{r^2}{2}\right]_0^{1+\cos\theta} d\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2}(1+\cos\theta)^2\, d\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2}\left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= \frac{1}{2}\left[\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\cos\theta\, d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^2\theta\, d\theta\right]$$

$$= \frac{1}{2}\left[\theta + 2\sin\theta + \frac{1}{2}(\theta + \sin\theta\cos\theta)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} =$$

$$\therefore ① + ② = \frac{\pi}{2} + \quad = \boxed{\frac{5}{4}\pi - 2}$$

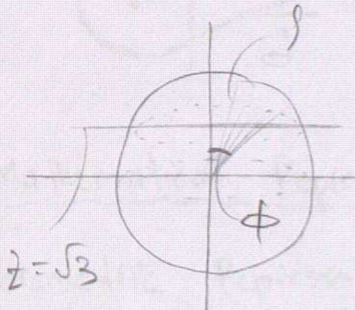
② Change the following integral from cylindrical to spherical

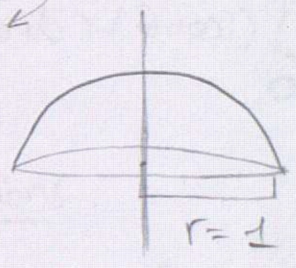
$$\int_0^1 \int_0^{2\pi} \int_{\sqrt{3}}^{\sqrt{4-r^2}} \underbrace{dz\,d\theta\,dr}_{\frac{dV}{r}}$$

$r = \rho\sin\phi$

$0 \leq r \leq 1$
$0 \leq \theta \leq 2\pi$
$\sqrt{3} \leq z \leq \sqrt{4-r^2}$

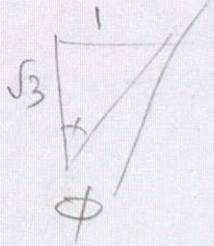
$z = \sqrt{4-r^2}$
$z = \sqrt{4-(x^2+y^2)}$
$z^2 + x^2 + y^2 = 4$
$\therefore \rho = 2$





confirmed

$\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$



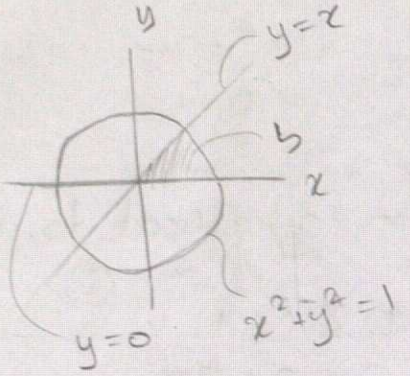
$\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

$$\iiint \frac{dV}{\rho\sin\phi} = \iiint \rho\, d\rho\, d\phi\, d\theta$$

$$\Rightarrow \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_{\frac{\sqrt{3}}{\cos\phi}}^{2} \rho\, d\rho\, d\phi\, d\theta$$

$z = \sqrt{3}$
$\rho\cos\phi = \sqrt{3}$
$\rho = \frac{\sqrt{3}}{\cos\phi}$

③ Compute $\iint_S x\,dx\,dy$, S is the region in the first quadrant bounded by the curve $x^2+y^2=1$, the line $y=0$ and the line $y=x$



transform to polar system

$$0 \le \theta \le \frac{\pi}{4}$$

$$0 \le r \le 1$$

$$x = r\cos\theta$$

$$\iint_S x\,dx\,dy = \int_0^{\frac{\pi}{4}}\int_0^1 (r\cos\theta)\,r\,dr\,d\theta = \frac{\sqrt{2}}{6}$$

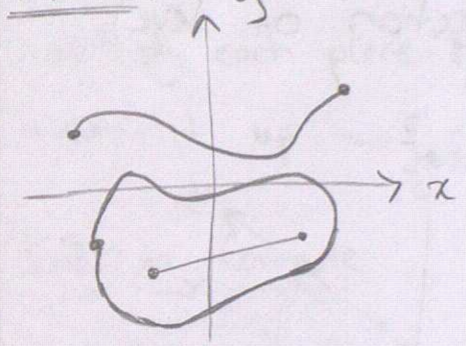
Week 7: Lecture 1 Oct 16, 2024

Curves in 2D and 3D

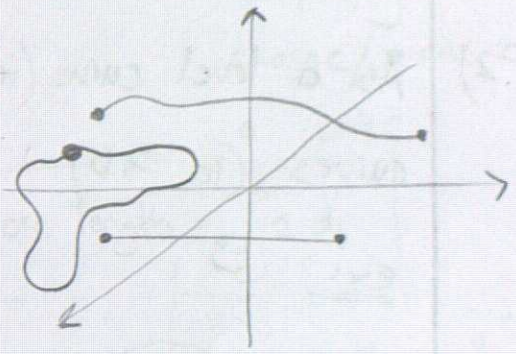
A curve can be thought of as:

1) a path taken by an object in $\mathbb{R}^2$ or $\mathbb{R}^3$ space
2) a 1D object (string) embedded in higher dimensional space

2D ex:



3D ex:



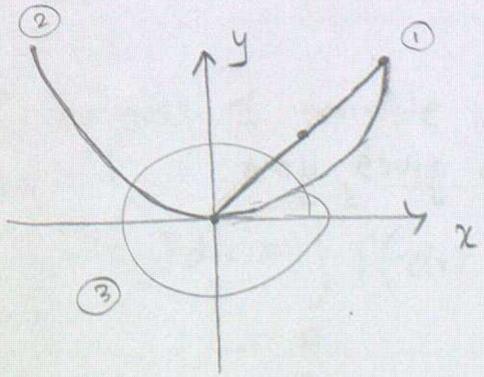
Mathematical Representations:

1) Parametric Representation: specify a curve with a func^{t}

$$\vec{r} : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$$

the image of $\vec{r}$ is the curve

ex: $\vec{r} : t \longrightarrow \langle x(t), y(t) \rangle$



① $\vec{r}(t) = \langle t, t \rangle \quad t \in [0,1]$

② $\vec{r}(t) = \langle t, t^2 \rangle \quad t \in [-1,1]$

$x(t) = t$, $y(t) = t^2$

$\therefore y = x^2$

$t \in [0, 2\pi] \longrightarrow$ ③ $\vec{r}(t) = \langle \cos t, \sin t \rangle$

$x^2+y^2 = 1$

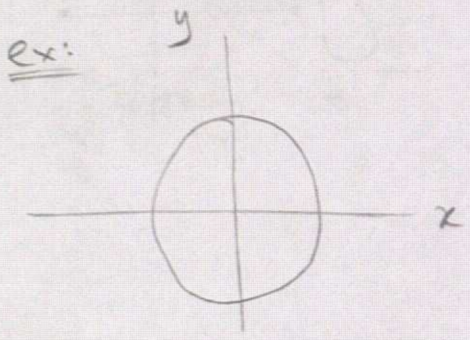
→ the image $\vec{r}$ is the curve

→ if curve represents a path, t can be thought of as time

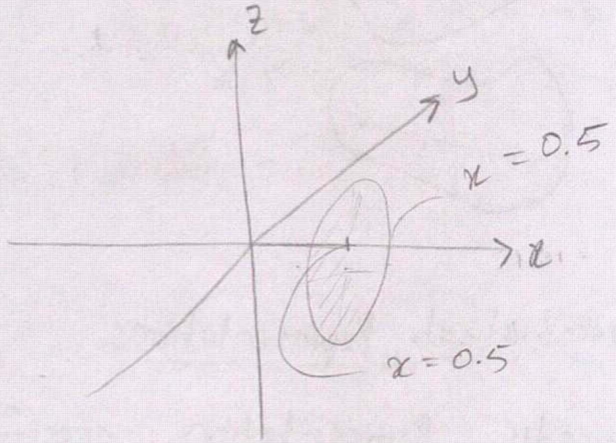
→ if curve represents a string, then t gives you a coordinate system on a curve

2) As a level curve (in 2D) or intersection of level curves (in 3D)

ex:



$x^2+y^2=1$

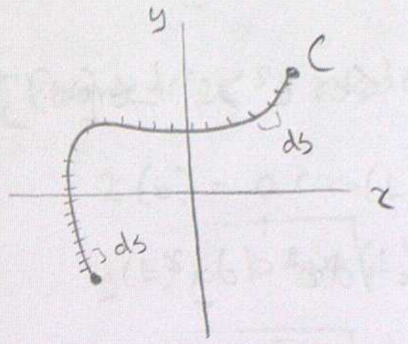


$x^2+y^2+z^2=1$, $x=0.5$

↑ sphere intersected with plane $x=0.5$

- intersection of two 2-d objects gives you 1-d object
- intersection of two 3-d objects gives you a 2-d object (level curve).

Integral of Scalar function over a curve



$$\int_C \underbrace{f(x,y)}_{\substack{\text{linear}\\ \text{density}\\ (\cdots/m)}} \underbrace{ds}_{\text{infinitesimal length element}}$$

add it up

uses

- computing total charge/mass of a wire

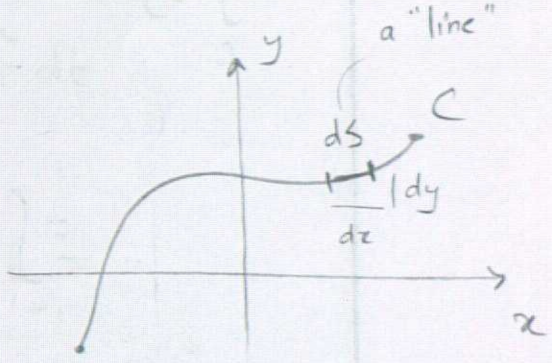
$$\boxed{\text{arc length} = \int_C 1\,ds}$$

- split C into small pieces size ds
- multiply each piece by linear density
- sum it up $= \int_C f(x,y)\,ds$

How to Compute

1) Parametrize the curve

$\vec{r}(t) = \langle x(t), y(t) \rangle \quad t\in[a,b]$



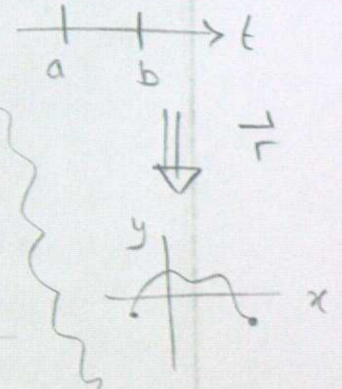
2) Compute ds (pythagorean theorem)

$ds = \sqrt{dx^2+dy^2}$ $\quad dx = x'(t)\,dt$

$dy = y'(t)\,dt$

$$\boxed{ds = \sqrt{(x'(t))^2+(y'(t))^2}\;dt}$$

can be thought of as a change of variables



3) Compute 1 variable integral over t

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\;dt}$$

↑ a,b comes from $t\in[a,b]$ up top

Ex: $\int_C xy\,ds$, where C is $C: r(t) = \langle t, t^2 \rangle$ $t \in [-1,1]$

1) parametrize : already done $\vec{r}(t) = \langle t, t^2 \rangle$, $t \in [-1,1]$

2) Compute ds

$$x = t \Rightarrow dx = 1dt \Rightarrow ds = \sqrt{dx^2 + dy^2}$$

$$y = t^2 \qquad dy = 2t\,dt \qquad = \sqrt{1^2 + (2t)^2}$$

$$ds = \sqrt{1 + 4t^2}\;dt$$

3) Compute integral in t

$$\int_C xy\,ds = \int_{-1}^{1} \underset{\overset{\uparrow}{x}}{(t)}\underset{\overset{\uparrow}{y}}{(t^2)} \sqrt{1+4t^2}\;dt$$

$$= \int_{-1}^{1} t^3 \sqrt{1+4t^2}\;dt \quad \cdots$$

Let $u = \sqrt{1+4t^2} = (4t^2+1)^{1/2}$

$du = \frac{1}{2}(4t^2+1)^{-1/2} \cdot 8t$

$= \frac{4t}{\sqrt{4t^2+1}}$

$du =$

Ex: $\int_C 3x^2 + y^2\,ds$ C is ellipse $3x^2 + y^2 = 9$

1) Parametrize : modified polar coordinates

$$\left.\begin{aligned} x(t) &= a\cos(t) \\ y(t) &= b\sin(t) \end{aligned}\right\} \text{determine } a, b$$

$3x^2 + y^2 = 9$

$\underbrace{3a^2}_{=9}\cos^2 t + \underbrace{b^2}_{=9}\sin^2 t = 9$

$\therefore$ $a^2 = \pm\sqrt{3}$ $b^2 = \pm 3$

$$\vec{r}(t) = \langle \sqrt{3}\cos(t), 3\sin(t) \rangle \quad t \in [0, 2\pi]$$

2) compute ds

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{3\sin^2 t + 9\cos^2 t}\;dt$$

3)

$$\int_C 3x^2 + y^2\,ds = \boxed{\int_0^{2\pi} 9\sqrt{3\sin^2 t + 9\cos^2 t}\;dt} = \cdots$$

Week 7: Lecture 2 Oct 17, 2024

Vector Fields

- A vector field is a function $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$

↳ it assigns a direction vector to a point in space

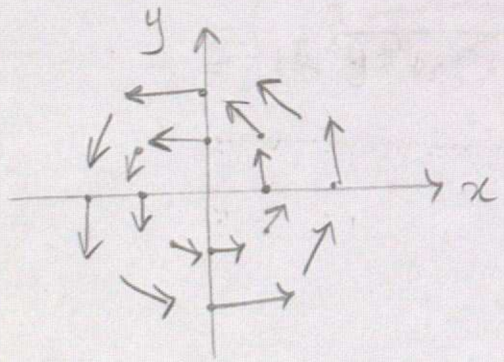
(this is conceptually different from change of variables)

Ex: ① Electric/magnetic field

② Velocity fields (fluid dynamics)

- Vector fields can be visualized by drawing arrows at each point (arrows given by vector fields)

Ex: $F = \langle -y, x \rangle$



consider a few pnts

$(0,1) \mapsto \langle -1, 0 \rangle$

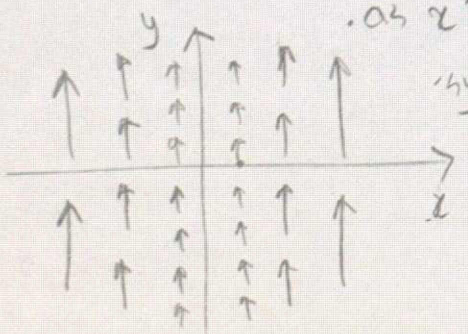
$(1,0) \mapsto \langle 0, 1 \rangle$

$(1,1) \mapsto \langle -1, 1 \rangle$

$F = \langle 0, x^2 \rangle$

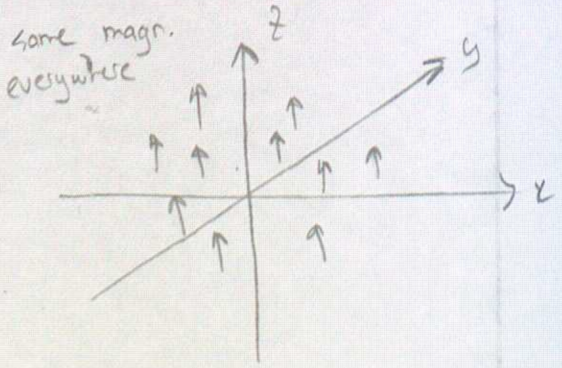
- always point ↑
- no y-dependence
- as x ↑, vectors ↑

symmetry around y-axis



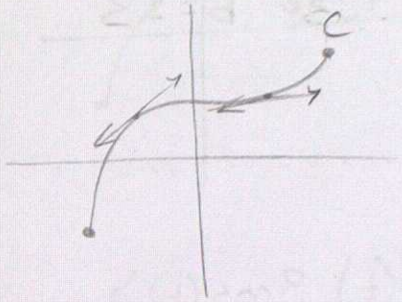
$F = \langle 0, 0, 1 \rangle$

same magn. everywhere



Tangent and Normal Vectors

→ given curve C



- The unit tangent vector is a vector parallel to the curve at a point (of unit lenght)

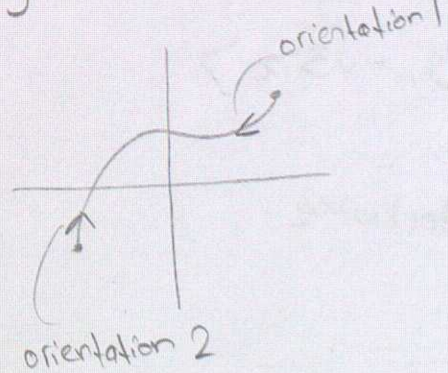
→ in general, there are 2 unit tangents to a curve at each pnt.

- if curve C represents a path, then the unit tangent is the direction of the instantaneous velocity
- given parametrization $\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\boxed{\hat{T} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}}$$

← instananeous Velocity

- given curve C



- The orientation of C is defined as a choice of direction of movement (generally there are 2 orientations
- given orientation, there is a natural choice of unit tangent vector (the one that points in the same direction)

- parametrization of a curve have orientation depending on the vector obtained from formula *

↳ direction as t increases

Ex: $3x^2+y^2=9$

recall from earlier that: $\vec{r}(t)=\langle a\cos(t), b\sin(t)\rangle$

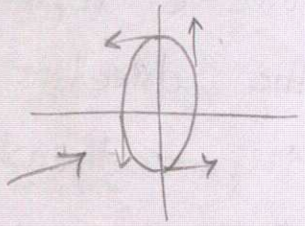
$a=\pm\sqrt{3} \quad b=\pm 3$

Case 1

$\vec{r}(t)=\langle\sqrt{3}\cos(t), 3\sin(t)\rangle$

$$\hat{T}=\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}=\frac{1}{\cdots}\langle-\sqrt{3}\sin(t), 3\cos(t)\rangle$$

$$=\frac{1}{\cdots}\left\langle-\frac{1}{\sqrt{3}}y, \sqrt{3}x\right\rangle$$



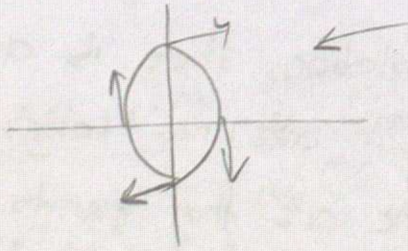
(oriented counterclockwise)

Case 2

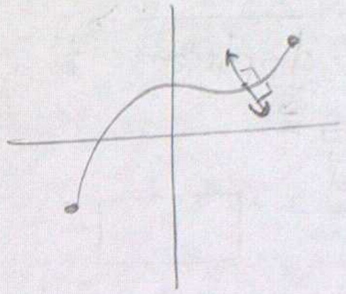
$\vec{r}(t)=\langle-\sqrt{3}\cos(t), 3\sin(t)\rangle$

$$\hat{T}=\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}=\frac{1}{\cdots}\left\langle\frac{1}{\sqrt{3}}y, -\sqrt{3}x\right\rangle$$

oriented clockwise



→ given a curve C



- The unit normal is the vector perpendicular to the curve at a point

(there are 2 normal vectors for curves in 2D)

(in 3D, there are infinte)

- In 2D, given tangent vector $\hat{T}$

$$\boxed{\hat{T}=\langle a,b\rangle \Longrightarrow \hat{n}=\pm\langle -b,a\rangle}$$

Week 7: Lecture 3

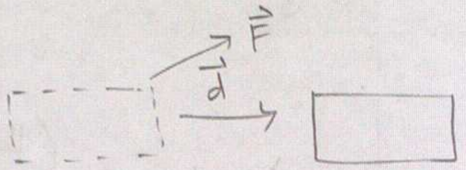
Oct 18, 2024

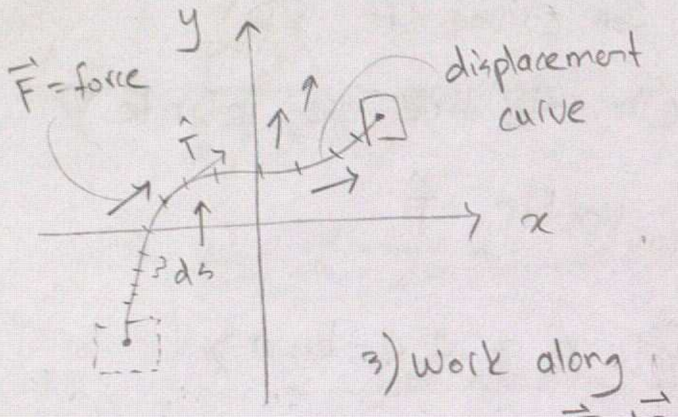
Circulation

→ Motivation: from physics $w = (\overrightarrow{\text{Force}}) \cdot \overrightarrow{\text{displacement}}$

what if $\vec{F}$ not constant?

what if $\vec{d}$ not linear?





1) break curve C into small pieces "ds"

2) displacement along piece

$$\underbrace{\boxed{d\vec{r} = \hat{T}\, ds}}_{\text{displacement}}$$

3) work along piece = $\vec{F} \cdot d\vec{r}$

$$\boxed{\therefore \text{work} = \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \hat{T}\, ds}$$

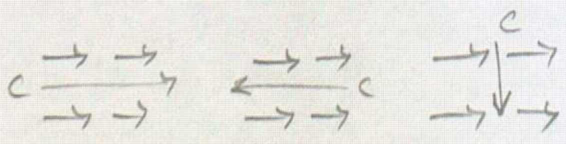
→ The circulation of vector field $\vec{F}$ along curve C is defined as integral

Orientation of C matters (flip orientation, you get minus)

$$\boxed{\int_C \vec{F} \cdot \hat{T}\, ds = \underbrace{\int_C \vec{F} \cdot d\vec{r}}_{\text{alternate notation}} = \underbrace{\int_C F_x dx + F_y dy + F_z dz}}$$

$$\left.\begin{aligned} \vec{F} &= \langle F_x, F_y, F_z \rangle \\ d\vec{r} &= \langle dx, dy, dz \rangle \end{aligned}\right\} \vec{F} \cdot d\vec{r}$$

the circulation is a measure of how vector field $\vec{F}$ "flows" along C



circulation: (+)ve (-)ve zero

$$\int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r}$$

How to Compute

1) Parametrize curve C (should have correct orientation)

2) Compute $\hat{T} ds = d\vec{r}$

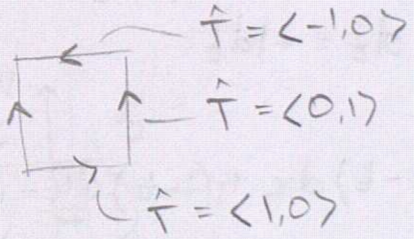
$$\hat{T} = \frac{r'(t)}{\|r'(t)\|}, \quad \underbrace{ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt = \|r'(t)\| dt}$$

$$\boxed{\hat{T} ds = r'(t)\, dt}$$

3) Compute integral

sometimes, it's easier to determine $\hat{T}$, ds by inspection

ex



Ex: $F = \langle -x, y \rangle$ over $C: r(t) = \langle \overset{x}{t}, \overset{y}{t^2} \rangle$, $-1 \le t \le 1$

$$\int_C \vec{F} \cdot dr$$

1) already given

2) $d\vec{r} = r'(t)dt = \langle 1, 2t \rangle dt$

3) $\int_{-1}^{1} \vec{F} \cdot d\vec{r} = \int_{-1}^{1} \langle -x, y \rangle \cdot \langle 1, 2t \rangle dt$

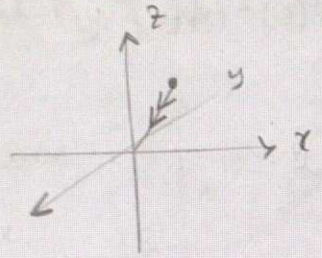
$$= \int_{-1}^{1} \langle -t, t^2 \rangle \cdot \langle 1, 2t \rangle dt = \int_{-1}^{1} (-t + 2t^3)\, dt = \dots$$

Ex: $F = \langle 2y, x, -z\rangle$, C is line segment from $(1,1,1) \rightarrow (0,0,0)$

orientation

$$\int_C 2y\,dx + x\,dy - z\,dz$$

1)



$\vec{r}(t) =$ initial + t(finial - initial)

$= \langle 1,1,1\rangle + t\langle 0-1, 0-1, 0-1\rangle$

$= \langle 1,1,1\rangle + t\langle -1,-1,-1\rangle$

correct orientation $\longrightarrow = \langle \underbrace{1-t}_{x}, \underbrace{1-t}_{y}, \underbrace{1-t}_{z}\rangle \quad t \in [0,1]$

2) Compute dx, dy, dz

$dx = -1dt \quad dy = -1dt \quad dz = -1dt$

3) $\int_C 2y\,dx + x\,dy - z\,dz = \int_0^1 2(1-t)\,dx + (1-t)\,dy - (1-t)\,dz$

$= \int_0^1 2(1-t)(-dt) + (1-t)(-dt) - (1-t)(-dt)$

Surfaces

A surface is a 2d object embedded in 3d

ex: 1) a plane in 3D: $ax + by + cz = d$

2) hollow sphere: $x^2 + y^2 + z^2 = r^2$

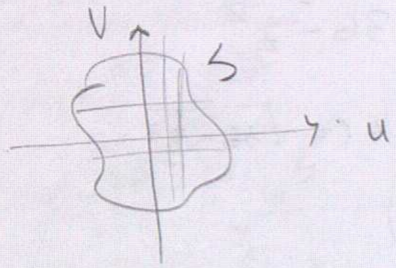
3) a graph of $z = f(x,y)$

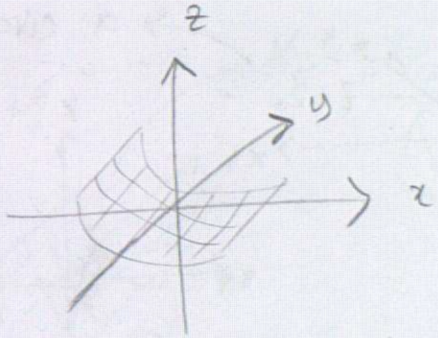
Mathematical Representation

1) parametrization: a function

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$$

where $(u,v) \in S \subseteq \mathbb{R}^2$





(u,v) give coordinates on the surface

2) Level curve: $f(x,y,z) = C$

ex: $x^2 + y^2 + z^2 = 1$

Week 8: Lecture 1 Oct 21, 2024

Surfaces: parametrizing a surface

ex: parametrize the sphere $x^2+y^2+z^2 = 36 = 6^2$

① spherical coordinates

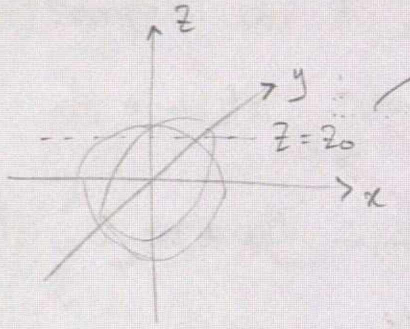
we know $\rho = 6$

$$\vec{r}(\theta,\phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$$

$$\vec{r}(\theta,\phi) = \langle 6\sin\phi\cos\theta, 6\sin\phi\sin\theta, 6\cos\phi\rangle$$

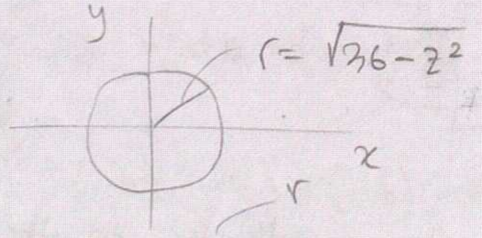
$$0 \le \phi \le \pi$$
$$0 \le \theta \le 2\pi$$

② Consider the z-slices (sort of cylindrical coordinates)



→ a circle

$$x^2+y^2 = 36 - z^2$$



→ parametrize circle: $x = \sqrt{36-z^2}\cos\theta$

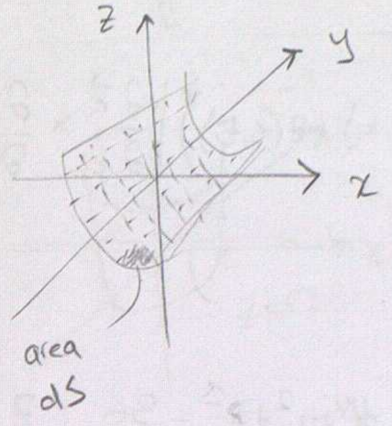
$$y = \underbrace{\sqrt{36-z^2}}_{r}\sin\theta$$

→ parametrize sphere $\vec{r}(\theta, z)$

$$\vec{r}(\theta,z) = \langle\sqrt{36-z^2}\cos\theta, \sqrt{36-z^2}\sin\theta, z\rangle$$

$$0 \le \theta \le 2\pi$$
$$-6 \le z \le 6$$

Surface Integrals



→ break surface into small chunks of area "dS"

→ multiply each chunk by some density $f(x,y,z)$ (kg/m^2, C/m^2 …)

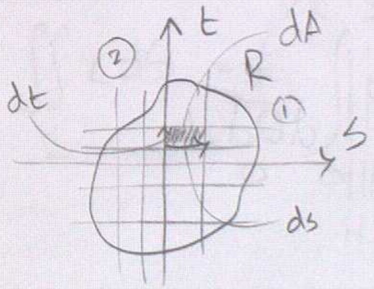
→ add up the pieces

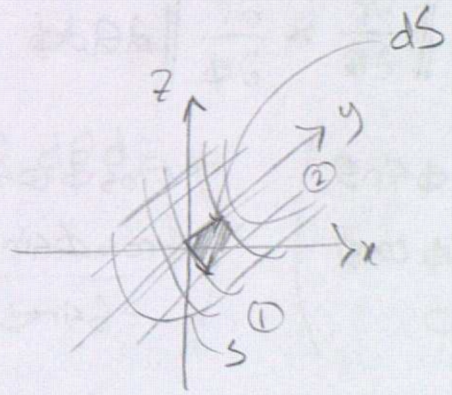
$$\boxed{\iint_S f(x,y,z)\,dS}$$

Uses

- computing mass of 2d sheet given mass density
- compute charge on planar sheet given charge density

How to Compute





- given parametrization: $\vec{r}(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$ $(s,t)\in R$
- compute dS in terms of $dA = ds\,dt$
- dS is a parallelogram with vertices

$$\frac{\partial\vec{r}}{\partial t}dt \quad \frac{\partial\vec{r}}{\partial s}ds \xRightarrow[\text{area of paralelogram}]{} \left|\frac{\partial\vec{r}}{\partial t}\times\frac{\partial\vec{r}}{\partial s}\right|dt\,ds$$

Computing Surface Integrals Cont'd

$$\Longrightarrow \boxed{\iint_S f(x,y,z)\,dS = \iint_R f(x(s,t),y(s,t),z(s,t))\left|\frac{\partial \vec{r}}{\partial t}\times\frac{\partial \vec{r}}{\partial s}\right|dt\,ds}$$

Ex: $\iint_R (x^2+y^2)\,dS$, R is hemisphere $x^2+y^2+z^2=36$, $z \geq 0$

→ spherical coordinates

$$\vec{r}(\theta,\phi) = \langle 6\sin\phi\cos\theta,\ 6\sin\phi\sin\theta,\ 6\cos\phi\rangle \qquad \begin{array}{l} 0\leq\theta\leq 2\pi \\ 0\leq\phi\leq\frac{\pi}{2}\end{array}$$

→ compute $dS = \left\|\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial\phi}\right\|d\theta\,d\phi$

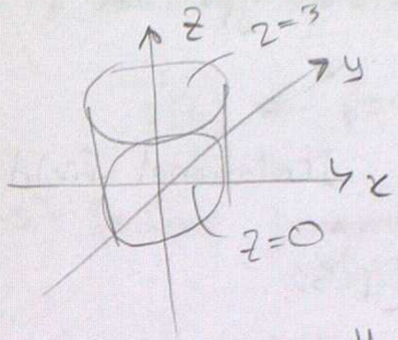
$$dS = \left\|\begin{pmatrix}-6\sin\phi\sin\theta\\ 6\sin\phi\cos\theta\\ 0\end{pmatrix}\times\begin{pmatrix}6\cos\phi\cos\theta\\ 6\cos\phi\sin\theta\\ -6\sin\phi\end{pmatrix}\right\|d\theta\,d\phi$$

$\underline{dS = 36\sin\phi\,d\theta\,d\phi}$ ← in general $dS = R^2\sin\phi\,d\theta\,d\phi$ on a sphere of radius R

→ evaluate in θ and ϕ

$$\boxed{\iint_R (x^2+y^2)\,dS = \int_0^{2\pi}\int_0^{\pi/2}\left[(6\sin\phi\cos\theta)^2+(6\sin\phi\cos\theta)^2\right]36\sin\phi\,d\theta\,d\phi}$$

Ex: $\iint_R x\,dS$, R is cylindrical surface $x^2+y^2=1$, $0\leq z\leq 3$



① Parametrize (cylindrical coord.)

$$\begin{array}{c}\uparrow \quad (x\ ,\ y\ ,\ z)\\ \vec{r}(\theta,z) = \langle 1\cos\theta,\ 1\sin\theta,\ z\rangle \leftarrow r=1\end{array}$$

$$0\leq\theta\leq 2\pi \qquad 0\leq z\leq 3$$

② compute $dS = \left\|\frac{\partial\vec{r}}{\partial\theta}\times\frac{\partial\vec{r}}{\partial z}\right\|d\theta\,dz$

$$dS = \left\|\begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix}\times\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\right\|d\theta\,dz = d\theta\,dz$$

③ Evaluate

$$\iint_R x\,dS = \int_0^{2\pi}\int_0^{3}\cos\theta\,d\theta\,dz = \dots$$

Tutorial 7 Oct 22, 2024

Scalar. Integral:

$$\int_a^b f(x)\,dx$$

↖ scalar

Vector Integral

↳ Vector field types are 2

↙ ↘

Radial field Irotational field

↳ integral types:

↙ ↘

contour integral surface integrals

→ Contour Integral: tangent vector in 2d and 3d
normal vector in 2d only

$$\underbrace{\int_C \vec{F}\cdot\hat{n}_s\,dO}_{\text{flux}},\ \underbrace{\oint_C \vec{F}\cdot\hat{n}_s\,dO}_{\text{flux}},\ \underbrace{\int_C \vec{F}\cdot\hat{T}dO}_{\text{circulation}},\ \underbrace{\oint_C \vec{F}\cdot\hat{T}dO}_{\text{circulation}}$$

→ Surface Integral: normal vector in 3d

$$\iint_S \vec{F}\cdot\hat{n}_s\,ds\ ,\ \oiint_S \vec{F}\cdot\hat{n}_s\,ds$$

Calculating Integral: for contour integral

1) Describe C in math. way (parametrize)

$\vec{r}(t)$ ← position vector

2) Define tangent vector: $\boxed{\vec{T} = \dfrac{\dot{\vec{r}}(t)}{\|\dot{\vec{r}}(t)\|}} = \dfrac{\langle \dot{x}(t), \dot{y}(t)\rangle}{\|\dot{\vec{r}}(t)\|}$

Normal vector: $\boxed{\vec{n} = \dfrac{\langle \dot{y}(t), -\dot{x}(t)\rangle}{\|\dot{\vec{r}}(t)\|}}$

3) $dO = |\vec{dO}| = \|\dot{\vec{r}}(t)\|\,dt$

$$\oint_C F\cdot T\,dO,\ \oint_C F\cdot n\,dO,\ \int_C F\cdot T\,dO,\ \int_C F\cdot n\,d\theta$$

Ex 1: Compute line integral $\int_C \sqrt{16y^2+x^2}\,dO$ where C is portion of the ellipse $x^2+4y^2=16$ in Quad. 1

1) parametrize contour: ellipse: $r(t) = \langle 4\cos t, 2\sin t\rangle$
$0 \le t \le \frac{\pi}{2}$

3) $dO = |\dot{\vec{r}}(t)|\,dt$.

$$\Rightarrow \int_C \sqrt{16y^2+x^2}\,dO = \int_0^{\pi/2} \sqrt{64\sin^2 t + 16\cos^2 t}\ \sqrt{16\sin^2(t) + 4\cos^2 t}\ dt \quad = 10\pi$$

Ex 2: $\vec{F}(x,y,z) = \langle z, -y, x \rangle$

C is line segment from (5,0,2) to (5,3,4) with tangent vector. Compute $\int_C \vec{F}\cdot\hat{T}\,ds$

1) $r(t) = (5,0,2) + t(5-5, 0-3, 2-4)$

$= \langle 5,0,2 \rangle + t\langle 0,-3,-2 \rangle$

$\therefore \vec{r}(t) = \langle 5, 3t, 2t+2 \rangle \quad 0 \le t \le 1$

2) $\vec{\dot{r}}(t) = \langle 0,3,2 \rangle \Rightarrow T = \frac{\dot{r}(t)}{\|\dot{r}(t)\|} = \frac{\langle 0,3,2 \rangle}{\sqrt{13}}$

3) $ds = |\dot{r}(t)|\,dt = \sqrt{13}\,dt$

4) $\int_C \vec{F}\cdot\vec{T}\,ds = \int_0^1 \langle z,-y,x \rangle \cdot \frac{\langle 0,3,2 \rangle}{\sqrt{13}} \sqrt{13}\,dt$

Surface Integral

1) position vector $\vec{r}(t) = \langle x(u,v) \; y(u,v) \; z(u,v) \rangle$

2) calulate $t_u = \frac{\partial r}{\partial u}$, $t_v = \frac{\partial r}{\partial v}$

3) $\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S \vec{F}\cdot \underbrace{\frac{t_u \times t_v}{|t_u \times t_v|}}_{\vec{n}_S} \underbrace{|t_u \times t_v|\,du\,dv}_{dS}$

Ex: $\iint_S y\,dS$, S is in the portion of the cylinder $x^2+y^2=3$ that lies b/n $z=0$, $z=6$

1) $\vec{r}(u,v) = \langle \sqrt{3}\cos u, \sqrt{3}\sin u, v \rangle \quad 0 \le u \le 2\pi, \; 0 \le v \le 6$

2) $t_u = \frac{\partial r}{\partial u} = \langle -\sqrt{3}\sin u, \sqrt{3}\cos u, 0 \rangle$

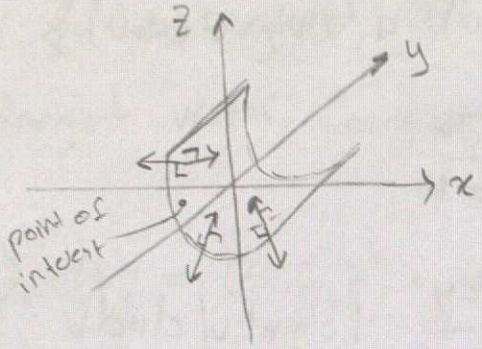
$t_v = \frac{\partial r}{\partial v} = \langle 0,0,1 \rangle$

$t_u \times t_v = \begin{vmatrix} i & j & k \\ -\sqrt{3}\sin u & \sqrt{3}\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \sqrt{3}\cos u, \sqrt{2}\sin u, 0 \rangle$

$|t_u \times t_v| = \sqrt{3}$

Week 8: Lecture 2 Oct 23, 2024

Normal Vectors to Surfaces

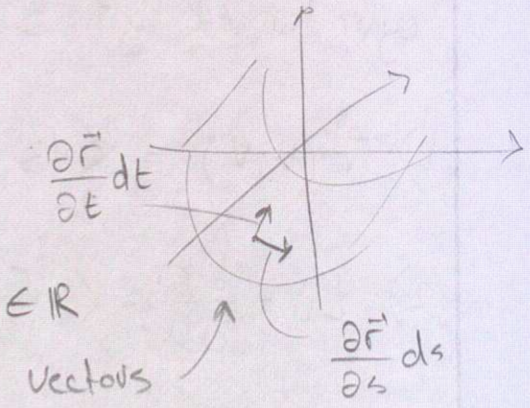


A unit normal at point $\vec{p}$ on surface is unit vector perpendicular to surface

- typically a surface will have 2 unit normal vectors of opposite direction
- The orientation of a surface is a consistent choice of normal vector.

  ex: oriented outward, upwards etc.

How to Compute



- given parametrization $\vec{r}(s,t)$, $(s,t) \in \mathbb{R}$
- the two small vectors are two target vectors
- the normal vector will be orthogonal to both tangents

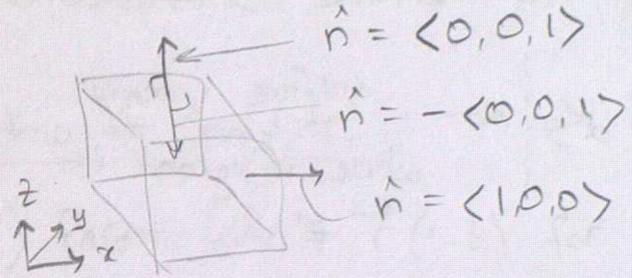
$$\boxed{\hat n = \frac{\frac{\partial \vec r}{\partial s} \times \frac{\partial \vec r}{\partial t}}{\left\| \frac{\partial \vec r}{\partial s} \times \frac{\partial \vec r}{\partial t} \right\|}}$$

← gives us one unit normal (other one is negative of this)

Method 2: if given as a level curve $f(x,y,z) = C$

$$\boxed{\hat n = \pm \frac{\nabla f}{\|\nabla f\|}}$$

Method 3: → observation

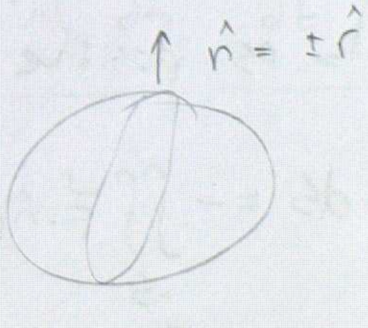


Ex: unit normal of $\underbrace{x^2 + y^2 + z^2 = 9}_{f(x,y,z) = C}$

$$\hat n = \pm \frac{\nabla f}{\|\nabla f\|} = \pm \frac{\langle 2x, 2y, 2z \rangle}{\|\langle 2x, 2y, 2z \rangle\|} = \pm \frac{2 \langle x, y, z \rangle}{2\sqrt{x^2+y^2+z^2}}$$

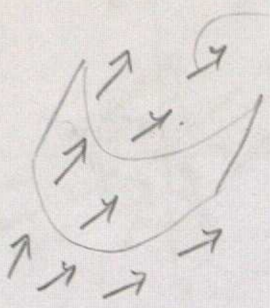
$$= \pm \frac{2}{2(3)} \langle x, y, z \rangle = \pm \frac{1}{3} \langle x, y, z \rangle$$

$$\boxed{\hat n = \pm \hat \rho = \pm \hat r}$$



$$\hat r = \frac{\vec r}{\|\vec r\|}$$

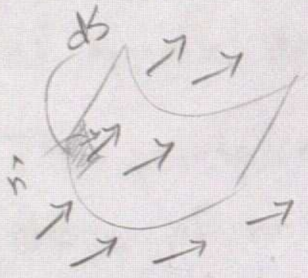
Flux Integrals (Integrals of Vector Fields over Surfaces)



Vector field $\vec{F}$

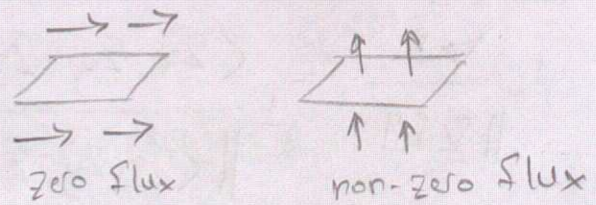
The flux of a vector field $\vec{F}$ through the surface S is defined

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,dS} \leftarrow$$ surface integral where integrand is $\vec{F}\cdot\hat{n}$



$\vec{F}\cdot\hat{n}$ → measures how much of the vector field goes thru surface

- Flux tells us how much of the vector field flows through S.
- think of $\vec{F}(x,y,z)$ representing fluid movement, than flux is total flow through S

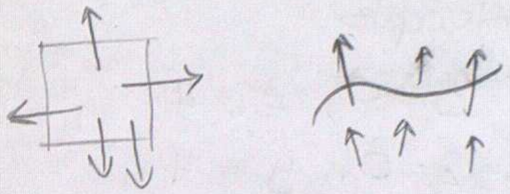


- the choice of orientation of S matters (which direction counts as positive flow)

$$\iint_S \vec{F}\cdot\hat{n}\,dS = -\iint_{-S}\vec{F}\cdot\hat{n}\,dS$$

opposite orientation

→ The flux of a vector field $\vec{F}$ through a curve $\underline{C \text{ in } \mathbb{R}^2}$ is: $\int_C \vec{F}\cdot\hat{n}\,ds$



← measures flow through curve (only defined for curves in $\mathbb{R}^2$)

How to Compute

1) Parametrize $\vec{r}(s,t)$ for $(s,t)\in\mathbb{R}$

2) Compute $\hat{n}\,dS$

$$\hat{n} = \frac{\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}}{\left\|\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\right\|} \qquad dS = \left\|\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\right\|dsdt$$

check orientation ⟶

$$\boxed{\Rightarrow \hat{n}\,dS = \left(\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\right)dst}$$

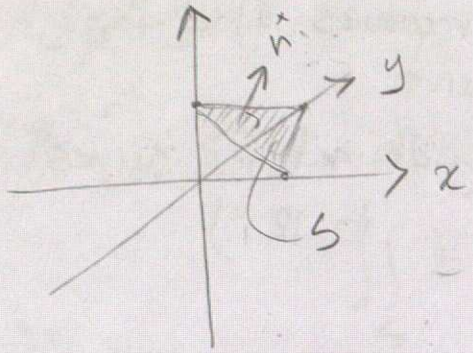
3) Evaluate

$$\boxed{\iint_R f(x(s,t),y(s,t),z(s,t))\left(\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\right)dst}$$

Week 8: Lecture 3

Oct 25, 2024

Ex: flux of $\vec{F} = \langle -1, 0, 0 \rangle$ across $z = 4 - x - y$ in 1^{rst} quad. oriented upwards



→ intercepts

$x = y = 0, \ z = 4$

$x = z = 0, \ y = 4$

$y = z = 0, \ x = 4$

$z = 4 - x - y$, let $x = s$, $y = t$

$\vec{r}(s,t) = \langle s, t, 4-s-t \rangle$ ↓ we can just leave as x and y

$\vec{r}(x,y) = \langle x, y, 4-x-y \rangle$ ← explicit parametrization

the values for (x,y) are in the shadow triangle below S

$0 \leq x \leq 4$

$0 \leq y \leq 4 - x$

$$\hat{n}dS = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \langle 1, 0, -1 \rangle \times \langle 0, 1, -1 \rangle = \langle 1, 1, 1 \rangle dxdy$$

$$\Rightarrow \iint_S \vec{F} \cdot \hat{n} dS = \iint_R \langle -1, 0, 0 \rangle \cdot \langle 1, 1, 1 \rangle dxdy$$

$$= \int_0^4 \int_0^{4-x} -1 \, dydx = \boxed{-8}$$

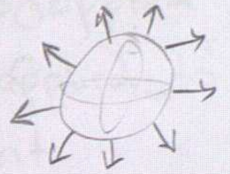
Ex: $\vec{F} = \frac{r}{|r|^3}$ over sphere radius a @ origin oriented outward

① $\vec{r}(\theta, \phi) = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi \rangle$

② $\hat{n}dS = \hat{r}a^2\sin\phi \, d\phi d\theta$ $\quad 0 \leq \theta \leq 2\pi \quad 0 \leq \phi \leq \pi$

→ recall $dS = a^2\sin\phi \, d\phi d\theta$ (general spherical dS formula)

→ dermine $\hat{n}$



directly outwards, normalized

$$\hat{n} = \frac{\vec{r}}{\|\vec{r}\|} = \hat{r}$$

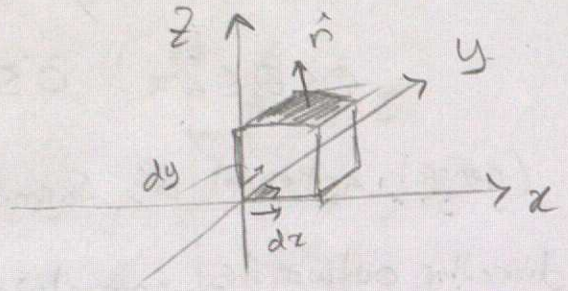
③ $$\Rightarrow \iint_S \vec{F} \cdot \hat{n}dS = \int_0^{2\pi} \int_0^{\pi} \frac{\vec{r}}{\|\vec{r}\|^3} \cdot \hat{r}a^2\sin\phi \, d\phi d\theta$$

$$\vec{r} \cdot \hat{r} = \vec{r} \cdot \frac{\vec{r}}{\|\vec{r}\|} = \frac{\|\vec{r}\|^2}{\|\vec{r}\|}$$

$$\Rightarrow \int_0^{2\pi} \int_0^{\pi} \frac{1}{\|\vec{r}\|^2} a^2 \sin\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\theta \boxed{= 4\pi}$$

↖ $\|\vec{r}\| = a$ on sphere

Ex: $\vec{F} = \langle xy + z, z - xy, xz + y^2 \rangle$ over top face of cube $[0,1]^3$ oriented upwards



Parametrize:

$\vec{r}(x,y) = \langle x, y, 1 \rangle$

$0 \leq x \leq 1$

$0 \leq y \leq 1$

② $\hat{n}\,dS = \langle 0,0,1 \rangle\, dx dy$

$\hat{n} = \langle 0,0,1 \rangle$ by inspection

$dS = dx dy$ by inspection

③ $\Rightarrow \iint_S \vec{F} \cdot \hat{n}\, dS = \int_0^1 \langle xy + 1, 1 - xy, x \cdot 1 + y^2 \rangle \cdot \langle 0,0,1 \rangle dx dy$

$= \int_0^1 \int_0^1 (x + y^2)\, dx dy \; \ldots \; = \frac{1}{2} + \frac{1}{3}$

Nabla (∇) and the gradient

Define ∇ to be the operator (nabla)

$\boxed{\nabla := \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle}$ $\quad \nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

- an operator is applied to a function
- nabla operator applied to a scalar function gives you the gradient

$\nabla : \text{scalar function} \longrightarrow \text{vector field}$

- apply nabla operator to vector fields via dot product and cross product

Divergence Operator

Given vector field $\vec{F} = \langle F_x, F_y, F_z \rangle$

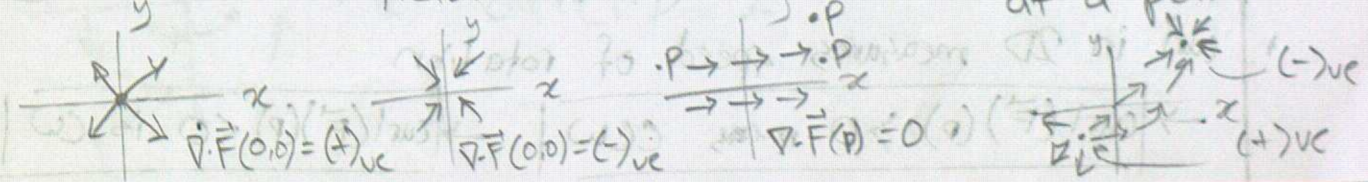
$\nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle F_x, F_y, F_z \rangle$

$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

also works in 2D

The divergence of $\vec{F}$ is $\nabla \cdot \vec{F}$, represents how much vector field is "created" at a point

- divergence $\nabla \cdot$: Vector field $\longrightarrow$ scalar function



Week 9: Lecture 1 Nov 4, 2024

Curl Operator

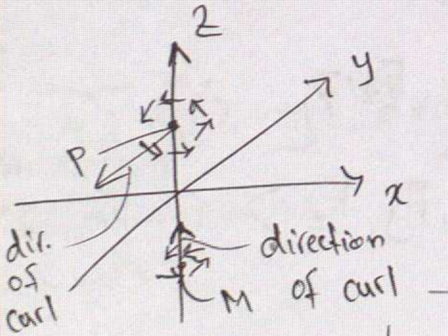
- The curl operator is defined as:

$$\nabla \times \vec F = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \langle F_x, F_y, F_z \rangle$$

$$\boxed{\nabla \times \vec F = \langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \rangle}$$

- curl is operator curl: $\begin{pmatrix}\text{Vector}\\ \text{field } \mathbb{R}^3\end{pmatrix} \longmapsto \begin{pmatrix}\text{Vector}\\ \text{field } \mathbb{R}^3\end{pmatrix}$
- curl measures rotation of vector field at a point
  - ↳ direction of curl is axis of rotation
  - ↳ magnitude is speed (right hand rule)

curl of $\vec F$ at point P



~~curl~~(P) = (-y) by RHR = curl($\vec F$)(P)

~~curl~~(M) = (+z) by RHR = curl($\vec F$)(M)

- Curl in 2D : $\boxed{\text{curl}(\vec F) = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}}$

equivalent to extending $\vec F$ in 3D by taking $F_z = 0$

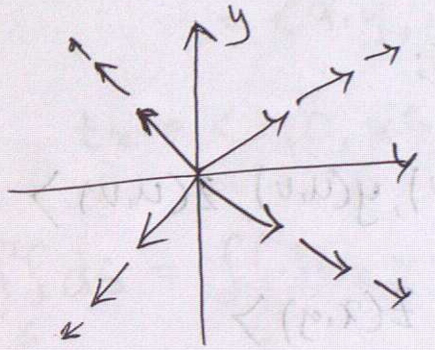
- curl: $\begin{pmatrix}\text{vector field}\\ \text{in } \mathbb{R}^2\end{pmatrix} \longmapsto \begin{pmatrix}\text{scalar}\\ \text{function}\end{pmatrix}$
- curl in 2D measures speed of rotation

→ $\boxed{\text{curl}(\vec F)(P) > 0 \text{ means CCW}}$ → $\boxed{\text{curl}(\vec F)(P) < 0 \text{ is CW}}$

Ex: $\vec F = \cancel{\langle y, -x \rangle}\ \frac{1}{x^2+y^2}\langle x, y \rangle = \langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \rangle$

$$\nabla \cdot \vec F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

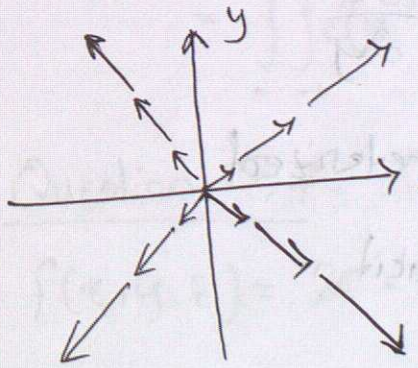
$$\frac{\partial F_x}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right] = \frac{(1)(x^2+y^2) - (x)(2x+y^2)}{(x^2+y^2)^2}$$



$$\text{div}(\vec F) = \begin{cases} 0 & (x,y) \neq 0 \\ \text{undefined} & (x,y) = 0 \end{cases}$$

$$\text{curl}(\vec F) = \{\ 0 \quad \leftarrow \text{no rotation anywhere}$$

Ex: $\vec F = \langle x, y \rangle$

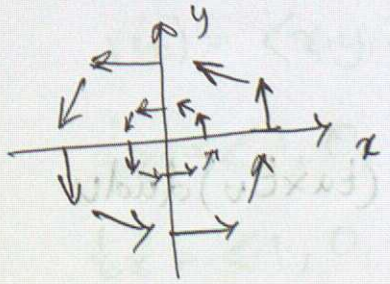


$$\text{div}(\vec F) = \frac{\partial F_x}{\partial y} + \frac{\partial F_y}{\partial y} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = \underline{\underline{2}}$$

$$\text{curl}(\vec F) = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0 - 0 = \underline{\underline{0}}$$

at every point, 2 units of the vector field is being "created", and there is no rotation anywhere

Ex: $\vec F = \overset{\langle -y, x \rangle}{\cancel{\langle y, x \rangle}}$



$$\text{div}(\vec F) = \cancel{\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}} = 0$$

$$\text{curl}(\vec F) = 2$$

Tutorial 8 Nov 5, 2024

$\iint_S ds$ vs. $\iint_S \vec{F}\cdot\vec{ds} = \iint_S \cancel{d}\vec{F}\cdot\hat{n}ds^{**}$

$\vec{ds} = \hat{n}ds$

Session 8: Compute Surface Integral

surface int: ① position vector:

parameterized: $r(t) = \langle x(u,v), y(u,v), z(u,v)\rangle$

explicit: $r(t) = \langle x, y, z(x,y)\rangle$

$\hookrightarrow$ calculate $t_u = \frac{\partial r}{\partial u}$, $t_v = \frac{\partial r}{\partial v}$

$\hookrightarrow$ calculate $t_x = \frac{\partial z}{\partial x}$, $t_y = \frac{\partial z}{\partial y}$

② $|\vec{ds}| = |t_u \times t_v|\,du\,dv$: parameterized

$|\vec{ds}| = |t_x \times t_y|\,dx\,dy$: explicit

③ $\vec{n_s}$ = unit vector to the tangential planes

$$\vec{n_s} = \frac{t_u \times t_v}{|t_u \times t_v|}$$

④ $$\iint_S \vec{F}\cdot\vec{ds} = \overbrace{\iint_S \vec{F}\cdot\vec{n_s}ds}^{\text{flux}} = \iint_S \vec{F}\cdot(t_u \times t_v)\,du\,dv$$

Lecture 20 (extracted). problems:

Section 17.6 Q.32: find area of the following surface using explicit: the trough $z = \frac{1}{2}x^2$ for $-1 \le x \le 1$, $0 \le y \le 4$

Sol: $r(t) = \langle x, y, z(x,y)\rangle$

$= \langle x, y, \frac{1}{2}x^2\rangle$

$t_x = \langle 1, 0, x\rangle$ $t_y = \langle 0, 1, 0\rangle$

$$\iint_S ds = \iint_S |t_x \times t_y|\,dx\,dy = \iint_S \begin{vmatrix} i & j & k \\ 1 & 0 & x \\ 0 & 1 & 0 \end{vmatrix} dx\,dy$$

$$= \iint_S |\langle -x, 0, 1\rangle|\,dx\,dy$$

$$= \int_0^4\int_{-1}^1 \sqrt{(x^2+1)}\,dx\,dy = 4\sqrt{2} + 4\ln(1+\sqrt{2})$$

Question 37: find $\iint_S f(x,y,z)dS$ using explicit

$f(x,y,z) = 25 - x^2 - y^2$, S is hemisphere with $r = 5$ and $z \ge 0$

Sol: $x^2 + y^2 + z^2 = 25$

$\hookrightarrow \cancel{P=}\; z = \sqrt{25 - x^2 - y^2}$

$r(t) = \langle x, y, z(x,y)\rangle = \langle x, y, \sqrt{25 - x^2 - y^2}\rangle$

$t_x = \langle 1, 0, \frac{\partial z}{\partial x}\rangle = \langle 1, 0, \frac{-x}{\sqrt{25 - x^2 - y^2}}\rangle$

$t_x = \langle 1, 0, \frac{-x}{z}\rangle$

$$t_y = \langle 0, 1, \frac{\partial z}{\partial y} \rangle = \langle 0, 1, -\frac{y}{z} \rangle$$

$$ds = |t_x \times t_y|\,dxdy = \left| \langle \frac{x}{z}, \frac{y}{z}, 1 \rangle \right|$$

$$= \sqrt{\frac{x^2+y^2+z^2}{z^2}}$$

$$\iint_S f(x,y,z)\,ds = \iint_S \overbrace{(25-x^2-y^2)}^{z^2} \sqrt{\frac{x^2+y^2+z^2}{z^2}}\,dxdy$$

$$= \iint_S 5\left(\sqrt{25-x^2-y^2}\right)dxdy$$

$$= 5\int_0^{2\pi}\int_0^{5} r\sqrt{25-r^2}\,drd\theta = \frac{1250\pi}{3}$$

Question 43: find flux of $\vec{F} = \langle 0,0,-1 \rangle$, S is $z = 4-x-y$ in the first coordinate and normal vector points outward

Sol: $\vec{F} = \langle 0,0,1 \rangle$

$z = 4-x-y \Longrightarrow r(t) = \langle x,y,z(x,y) \rangle$

$\Longrightarrow t_x = \langle 1,0,-1 \rangle \quad \Longrightarrow t_y = \langle 0,1,-1 \rangle$

$\vec{n}_s$, $ds = |t_x \times t_y|\,dxdy$

$\vec{n}_s = \frac{t_x \times t_y}{|t_x \times t_y|}$ $\quad \vec{n}_s ds = t_x \times t_y$

$= \langle -1,1,1 \rangle$

$$\iint_S \vec{F}\cdot\vec{n}_s ds = \iint_S \overset{\langle 0,0,-1 \rangle}{\langle 0,0,-1 \rangle} \cdot \langle -1,1,1 \rangle\,dxdy$$

$$= \int_0^4\int_0^{4-y} (-1)\,dxdy = -8$$

Question 56: find flux of field $\vec{F} = \langle x,y,z \rangle$ across the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ in first octant

$\longrightarrow z = -\frac{c}{a}x - \frac{c}{b}y + c$

$r(t) = \langle x,y,z(x,y) \rangle$

$\Longrightarrow t_x = \langle 1,0,-\frac{c}{a} \rangle$

$\Longrightarrow t_y = \langle 0,1,-\frac{c}{b} \rangle$

$t_x \times t_y = \langle \frac{c}{a}, \frac{c}{b}, 1 \rangle$

$$\iint_S \vec{F}\cdot\vec{n}_s ds = \iint_S \langle x,y,z \rangle \cdot \langle \frac{c}{a}, \frac{c}{b}, 1 \rangle\,dxdy$$

$$= \iint \frac{c}{a}x + \frac{c}{b}y + z\,dxdy$$

$z = -\frac{c}{a}x - \frac{c}{b}y + c$

$$= \iint_S c\,dxdy$$

flux equals to "c" multiplied by the area of the region

Notation: → ∂R denotes boundary of region R
→ $\oint_C$ and $\oiint_S$ is the same as $\int_C$ and $\iint_S$ ↱ but used when C or S are closed

Week 9: Lecture 2

Nov 6, 2024

Divergence Theorem

Let R be a region in $\mathbb{R}^3$ and $S = \partial R$ be the boundary surface, then:

$$\boxed{\oiint_{\partial R} \vec{F}\cdot\hat{n}\,dS = \iiint_R \nabla\cdot\vec{F}\,dV}$$

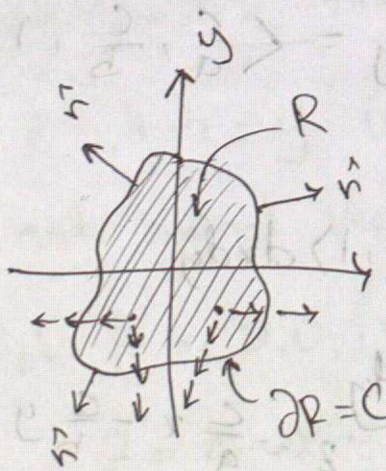
⟸ $\partial R = S$ oriented outwards

Green's Theorem (2D Version)

Let R be region in $\mathbb{R}^2$ and $C = \partial R$ be the boundary curve, then:

$$\boxed{\oint_{\partial R} \vec{F}\cdot\hat{n}\,ds = \iint_R \nabla\cdot\vec{F}\,dA}$$

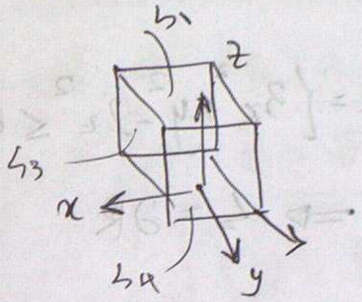
⟸ $C = \partial R$ oriented outwards



↑ ∂R oriented outward

$$\underbrace{\oint_{\partial R} \vec{F}\cdot\hat{n}\,ds}_{\text{how much vector field leaves the edges}} = \underbrace{\iint_R \nabla\cdot\vec{F}\,d\cancel{V}A}_{\text{measures how much vector field is created inside}}$$

Ex: $\vec{F} = \langle -x, -y, -z \rangle$ and $R : [-1,1] \times [-1,1], \times [-1,1]$



$$\iiint_R \nabla\cdot\vec{F}\,dV = \oiint_{\partial R} \vec{F}\cdot\hat{n}\,dS$$

→ R is a cube, ∂R are the six faces

$$\iiint_R \nabla\cdot\vec{F}\,dV = LHS \qquad \nabla\cdot\vec{F} = \frac{\partial}{\partial x}[-x] + \frac{\partial}{\partial y}[-y] + \frac{\partial}{\partial z}[-z]$$

$$= -1 - 1 - 1 = -3$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} -3\,dV = \boxed{-24}$$

$$\oiint_{\partial R} \vec{F}\cdot\hat{n}\,dS = RHS = \iint_{S_1} \vec{F}\cdot\hat{n}\,dS + \iint_{S_2} \vec{F}\cdot\hat{n}\,dS + \cdots + \iint_{S_6} \vec{F}\,\hat{n}\,dS$$

→ start with S_1 : $\langle x, y, 1 \rangle$ for $(x,y) \in [-1,1] \times [-1,1]$

$\hat{n} = \langle 0, 0, 1 \rangle$ ⟸ since oriented outward

$dS = dxdy$ $\quad \iint_{S_1} \vec{F}\cdot\hat{n}\,dS = \int_{-1}^{1}\int_{-1}^{1} (-z)\,dxdy = \int_{-1}^{1}\int_{-1}^{1} -1\,dxdy = -4$

→ repeat for $S_2 \cdots S_6$ ⟹ $RHS = \oiint_{\partial R} \vec{F}\cdot\hat{n}\,dS = -4 -4 -4 -4 -4 -4$ $\boxed{= -24}$

Since LHS = RHS, Divergence Theorem Proved ✓

Ex: Outward Flux of $\vec{F} = \langle y+z, x+z, x+y \rangle$ over ellipsoid $3x^2+y^2-2z^2=4 \leftarrow S$

$$R = \{3x^2+y^2-2z^2 \leq 4\} \Rightarrow S = \partial R$$

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_R \nabla\cdot\vec{F}\,dV$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}[y+z] + \frac{\partial}{\partial y}[x+z] + \frac{\partial}{\partial z}[x+y] = 0$$

RHS: $\iiint_R \nabla\cdot\vec{F}\,dV = \iiint_R 0\,dV = 0$

LHS: $\iint_{\partial R} \vec{F}\cdot\hat{n}\,dS = 0$, meaning outward flux is zero

Ex: $\vec{F} = \langle x, y, z \rangle$ over $\overset{S}{\boxed{\text{surface}}}$ of cone $z = \sqrt{x^2+y^2}$, $0 \leq z \leq 4$ plus its top surface in plane $z=4$

$\leftarrow$ let R be the solid cone, then $S = \partial R$

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_R \nabla\cdot\vec{F}\,dV$$

$$= \iiint_R 3\,dV$$

$$= 3\int_0^{2\pi}\int_0^{2}\int_0^{r} r\,dz\,dr\,d\theta$$

Week 9: Lecture 3 Nov 8, 2024

Stoke's Theorem

Let S be a surface in $\mathbb{R}^3$ (with some orientation) and let $C = \partial S$ be the boundary curve (with induced orientation):

$$\boxed{\iint_S \nabla\times\vec{F}\cdot\hat{n}\,dS = \oint_{C=\partial S} \vec{F}\cdot\hat{T}\,ds}$$

∂S is the part of surface that can give you a papercut

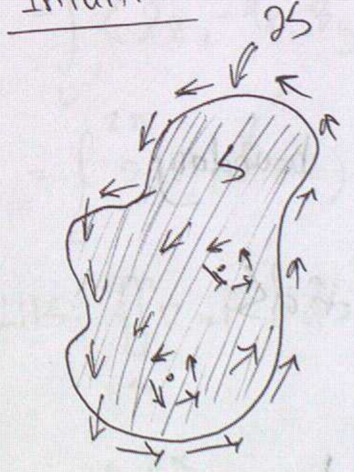
Green's Theorem (2D Version)

Let S be a region in $\mathbb{R}^2$ then:

$$\boxed{\iint_S \text{curl}(\vec{F})\,dA = \oint_{\partial S} \vec{F}\cdot\hat{T}\,ds}$$

∂S has induced orientation

Intuition



$$\underbrace{\iint_S \text{curl}(\vec{F})\,dA}_{\text{total rotation generated in } S} = \underbrace{\oint_{\partial S} \vec{F}\cdot\hat{T}\,ds}_{\text{flow of vector field along } \partial S}$$

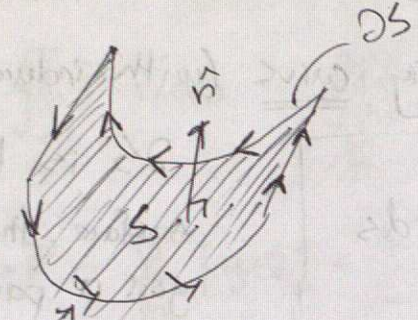
→ (+)ve curl implies CCW rotation

∴ induced orientation is also CCW

∂S is combination of two boundries

⟸ induced orientation CCW on outside and CW on inside

Stove's in 3D



induced orientation

→ induced orientation on ∂S

→ right hand rule (thumb in $\hat{n}$ and fingers curl in $\hat{T}$)

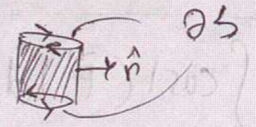
→ walk on ∂S with head pointed $\hat{n}$, which direction for $\hat{T}$ keeps left hand on inside?

ex// $x^2+y^2+z^2=1$

← boundary DNE (0)

$x^2+y^2=1$

$0 \leq z \leq 1$



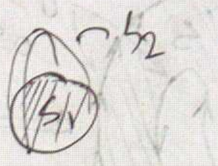
→ consequence: if S has no boundary, then

$$\iint_S \nabla \times \vec{F} \cdot \hat{n} \, dS = 0$$

if S_1 and S_2 have same boundary

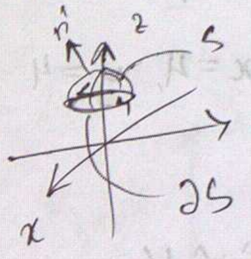
$$\iint_{S_1} \nabla \times \vec{F} \cdot \hat{n} \, dS = \pm \iint_{S_2} \nabla \times \vec{F} \cdot \hat{n} \, dS$$

ex//



both have circle as boundary

Ex: $\vec{F} = \langle 2z, -4x, 3y \rangle$ over S: the cap of the sphere $x^2+y^2+z^2=169$ above plane $z=12$ oriented upward



CCW viewed from top ← orientation of ∂S

S = spherical cap

∂S = circle at z = 12

$x^2+y^2+z^2=169 \Rightarrow x^2+y^2=169-144$

$x^2+y^2=25=5^2$

$\underline{\underline{x^2+y^2=25, \ z=12}}$

$$\iint_S \nabla \times \vec{F} \cdot \hat{n} \, dS = \oint_{\partial S} \vec{F} \cdot \hat{T} \, ds$$

RHS: $\oint_{\partial S} \vec{F} \cdot d\vec{r}$ → parametrize: $\vec{r}(t) = \langle \overset{5}{\cos t}, \overset{5}{\sin t}, 12 \rangle \quad 0 \leq t \leq 2\pi$

$\vec{r}'(t) = \langle -\overset{5}{\sin t}, \overset{5}{\cos t}, 0 \rangle$

$= \int_0^{2\pi} \langle 2z, -4x, 3y \rangle \cdot d\vec{r}$

$= \int_0^{2\pi} 2(12)(-\overset{5}{\sin t}) - 4(5\cos t)(5\cos t) + 0 \, dt = -100\pi$

LHS: $\iint_S \nabla \times \vec{F} \cdot \hat{n} \, dS = \iint_{x^2+y^2 \leq 25} 3\frac{x}{z} + 2\frac{y}{z} - 4) \, dx dy$

$\nabla \times \vec{F} = \langle 3, 2, -4 \rangle$

$\vec{r}(t) = \langle x, y, \sqrt{169-x^2-y^2} \rangle$

x,y ↗ $(x,y) \in \{x^2+y^2 \leq 25\}$

$= \int_0^5 \int_0^{2\pi} \cancel{15} \frac{3r\cos\theta}{\sqrt{169-r^2}} + \frac{2r\sin\theta}{\sqrt{169-r^2}} - 4 \; r dr d\theta$

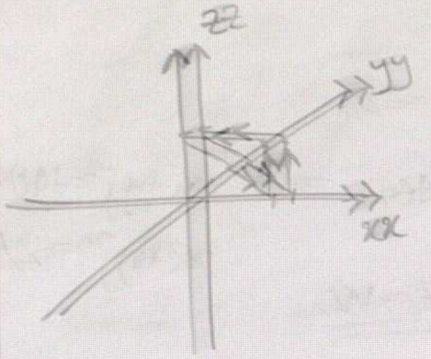
$\hat{n} dS = \langle \frac{x}{\sqrt{169-x^2-y^2}}, \frac{y}{''}, 1 \rangle$

z ↗

$= -100\pi$

Week 10: Lecture 1 Nov 11, 2024

Ex: $F = \langle x^2 - z^2, y, 2xz\rangle$ over C: the boundary of the region plane $z = 4 - x - y$ in 1st Octant, CCW



→ intercepts: $z = 4$, $x = 4$, $y = 4$

→ use Stoke's Thm

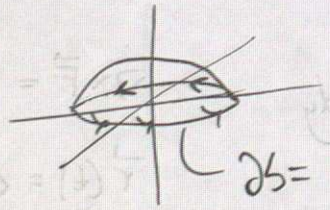
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,dS$$

$\nabla\times\vec{F} = \langle 0, -4z, 0\rangle$, $\vec{r}(\overset{x,y}{t}) = \langle x, y, 4-x-y\rangle$, $0 \le x \le 4$, $0 \le y \le 4-x$

$\hat{n}dS = \begin{bmatrix}1\\1\\1\end{bmatrix}dxdy$ ← correct orientation

$$\therefore \oint_C \vec{F}\cdot dr = \iint_S \nabla\times\vec{F}\cdot\hat{n}dS = \int_0^4\int_0^{4-x} -4(4-x-y)\,dxdy = \boxed{-128}$$

Ex: $\iint_S \nabla\times\vec{F}\cdot\hat{n}dS$ for $\vec{F} = \langle ze^{x^2y^2}+y, \sin(z)+xy, \log(x^2y^2+1)\rangle$ and S: upper half of ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}\underset{\nwarrow\text{"+"}}{+}z^2=1$, $z \ge 0$ oriented upwards



→ use Stoke's $\iint_S \nabla\times\vec{F}\cdot\hat{n}dS = \oint_{\partial S}\vec{F}\cdot d\vec{r}$

$\partial S = \left\{\frac{x^2}{4}+\frac{y^2}{9} = 1\right\}$ $\vec{r}(\theta) = \langle 2\cos\theta, 3\sin\theta, 0\rangle$ $0 \le \theta \le 2\pi$

$\Rightarrow \oint_{\partial S}\vec{F}\cdot d\vec{r}$ $d\vec{r} = \langle -2\sin\theta, 3\cos\theta, 0\rangle dt$

$$= \oint_{\partial S}\langle y, xy, \log(x^2y^2+1)\rangle\cdot\langle -2\sin\theta, 3\cos\theta, 0\rangle dt$$

$$= \int_0^{2\pi}(-6\sin^2\theta + 18\cos^2\theta\sin\theta)\,d\theta = \cdots$$

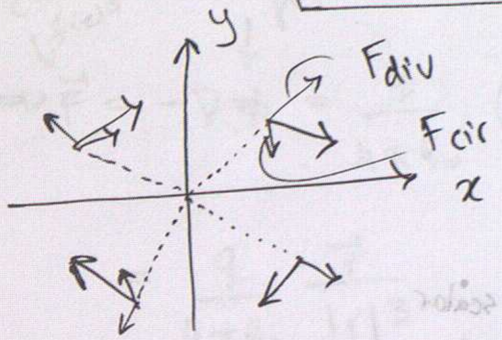
Helmholtz Decomposition Theorem

→ Given a vector field $\vec{F}$, we can write it as:

$$\boxed{\vec{F} = \vec{F}_{circ} + \vec{F}_{div}}$$

$\boxed{\nabla\cdot\vec{F}_{circ} = 0}$ (soure-free field)

$\boxed{\nabla\times\vec{F}_{div} = 0}$ (rotation-free component)



→ moreover, there exists scalar function ϕ and vector field $\vec{A}$ s.t. $\boxed{\begin{matrix}\vec{F}_{div} = -\nabla\phi \\ \text{and} \\ \vec{F}_{cir} = \nabla\times\vec{A}\end{matrix}}$

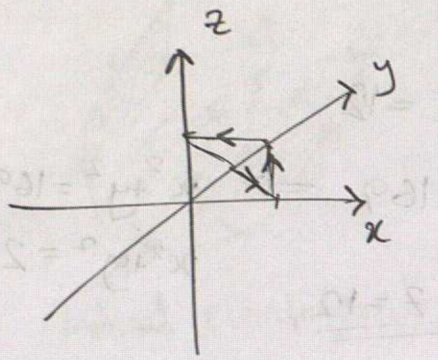
→ Given a vector field $\vec{F}$ s.t. $\vec{F}\to 0$ as $(x,y,z)\to\infty$ (decaying), then F is uniquely determined by $\nabla\cdot\vec{F}$ and $\nabla\times\vec{F}$

Convention:

$$\left.\begin{aligned}\nabla\cdot\vec{F} &= \rho(x,y,z)\\ \nabla\times\vec{F} &= \vec{J}(x,y,z)\end{aligned}\right\}$$ given ρ and $\vec{J}$, there is only one corresponding vector field

Week 10: Lecture 1 Nov 11, 2024

<u>Ex:</u> $F = \langle x^2 - z^2, y, 2xz \rangle$ over C : the boundary of the region plane $z = 4-x-y$ in 1st Octant, CCW



→ intercepts: $z=4,\ x=4,\ y=4$

→ use Stoke's Thm

$$\oint_C \vec{F}\cdot dr = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,dS$$

$$\nabla\times\vec{F} = \langle 0, -4z, 0\rangle,\quad \vec{r}(\overset{x,y}{\cancel{t}}) = \langle x, y, 4-x-y\rangle,\quad \begin{matrix} 0\le x\le 4 \\ 0\le y\le 4-x \end{matrix}$$

$\hat{n}\,dS = \begin{bmatrix}1\\1\\1\end{bmatrix} dxdy$ ← correct orientation

$$\therefore \oint_C \vec{F}\cdot dr = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,dS = \int_0^4\int_0^{4-x} -4(4-x-y)\,dxdy = \boxed{-128}$$

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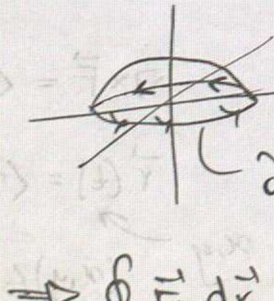

→ use Stoke's $\iint_S \nabla\times\vec{F}\cdot\hat{n}\,dS = \oint_{\partial S}\vec{F}\cdot d\vec{r}$

$\partial S = \left\{\frac{x^2}{4}+\frac{y^2}{9} = 1\right\}$

$\vec{r}(\theta) = \langle 2\cos\theta, 3\sin\theta, 0\rangle \quad 0\le\theta\le 2\pi$

$d\vec{r} = \langle -2\sin\theta, 3\cos\theta, 0\rangle dt$

$$\Rightarrow \oint_{\partial S}\vec{F}\cdot d\vec{r}$$

$$= \oint_{\partial S}\langle y, xy, \log(x^2y^2+1)\rangle\cdot\langle -2\sin\theta, 3\cos\theta, 0\rangle dt$$

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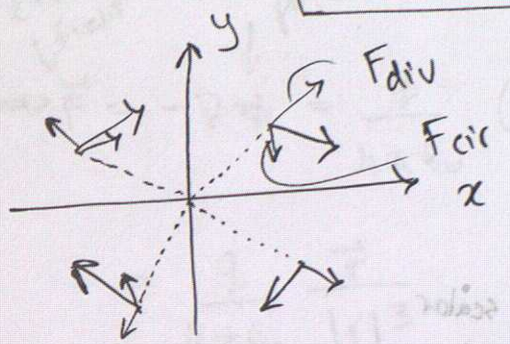

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Convention:

$$\left.\begin{matrix}\nabla\cdot\vec{F} = \rho(x,y,z) \\ \nabla\times\vec{F} = \vec{J}(x,y,z)\end{matrix}\right\}$$ given ρ and $\vec{J}$, there is only one corresponding vector field

Tutorial 9 W, Nov 12, 2024

Divergence, Curl, Gradient

	Divergence	Curl	Gradient
input:	vector	vector	scalar
output:	scalar	vector	vector

$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \qquad \nabla \overset{\substack{\text{scalar} \\ \downarrow}}{\phi} = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle$$

- if $\vec{F}$ is irrotational, $\nabla \times \vec{F} = 0$, $\overset{\underline{\text{example}}}{\nabla \times \vec{E} = 0 \Rightarrow \vec{E} = -\nabla \phi}$
- if $\vec{F}$ is rotational, $\nabla \cdot \vec{F} = 0$, $\nabla \cdot \vec{B} = 0 \Rightarrow \vec{B} = \nabla \times \vec{A}$

17.5 : Question 60 (Lec 21, Q. 13)

$$\phi = \frac{1}{4\pi\epsilon_0} \frac{q}{|r|} \quad \text{where } r = \langle x, y, z \rangle$$

a) Compute the $\vec{F} = -\nabla \phi$

b) Show $\nabla \times \vec{F} = 0 \longrightarrow$ show $\vec{F}$ is irrotational

a) $$\vec{F} = -\nabla \phi = \frac{q}{4\pi\epsilon_0} \left\langle \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right], \frac{\partial}{\partial y}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right], \frac{\partial}{\partial z}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right] \right\rangle$$

$$\Rightarrow \overset{\substack{\text{electric field} \\ \downarrow}}{\vec{F}} = -\nabla \overset{\substack{\text{electric potential} \\ \downarrow}}{\phi} = \frac{q}{4\pi\epsilon_0} (x^2+y^2+z^2)^{-3/2} \langle x, y, z \rangle$$

$$= \frac{q}{4\pi\epsilon_0} \frac{\vec{r}}{|r|^3}$$

b) $$\nabla \times \vec{F} = 0 \Longrightarrow \nabla \times \vec{F} = \frac{q}{4\pi\epsilon_0} (x^2+y^2+z^2)^{-5/2} \langle (-3yz, +3yz), (-3xz + 3xz), (-3xy + 3xy) \rangle$$

$$\nabla \times \vec{F} = 0$$

$$\frac{\partial}{\partial y} F_z - \frac{\partial}{\partial z} F_y = \left(\frac{q}{4\pi\epsilon_0} (x^2+y^2+z^2)^{-5/2} (-3yz) \right) - \left(\frac{q}{4\pi\epsilon_0} (x^2+y^2+z^2) (+3yz) \right) = 0$$

Q.14 Lec. 21

$$\nabla \times \vec{H} = C \frac{\partial \vec{E}}{\partial t} \quad \text{and} \quad E(z,t) = A\sin(kz - \omega t)\hat{i}$$

$$H(z,t) = A\sin(kz - \omega t)\hat{j}$$

and $\omega = \frac{k}{c}$

$$L.H.S = \nabla\times\vec{H} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$= -\frac{\partial}{\partial z}\left(A\sin(kz-\omega t)\right)\hat{i} + 0\hat{j} + \frac{\partial}{\partial x}\left(A\sin(kz-\omega t)\right)\hat{k}$$

$$= -Ak\cos(kz-\omega t)\,\hat{i} + 0 + 0$$

$$RHS: \; c\frac{\partial\vec{E}}{\partial t} = c\frac{\partial}{\partial t}\left[A\sin(kz-\omega t)\hat{i}\right]$$

$$= c\left(-A\omega\cos(kz-\omega t)\right)\hat{i}$$

$\therefore$ Same

$$= -\cancel{c}\,\frac{k}{\cancel{k}}A\cos(kz-\omega t)\hat{i}$$

$$= -Ak\cos(kz-\omega t)\,\hat{i}$$

Divergence Theorem $\displaystyle \oiint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$

Excersice 29 (lec 23?)

$\vec{F} = \langle x^2, -y^2, z^2\rangle$ calculate flux thru first octant b/n planes $z = 4-x-y$

$z = 2-x-y$

$$\oiint_S \vec{F}\cdot\hat{n}\,dS$$

$$\nabla\cdot\vec{F}\,\cancel{d} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(z^2) = 2(x-y+z)$$

$$\rightarrow \iiint_V 2(x-y+z)\,dV = 2\int_0^4\int_0^{4-x}\int_0^{4-x-y}(x-y+z)\,dz\,dy\,dx$$

$$+ 2\int_0^2\int_0^{2-x}\int_0^{2-x-y}(x-y-z)\,dz\,dy\,dx$$

$$= 2(\cdots$$

Question 10 (Lec 23) $\displaystyle E = \frac{Q}{4\pi\epsilon_0}\frac{\vec{r}}{|\vec{r}|^3} \leftarrow r = \langle x, y, z\rangle$

region is sphere $(r=a)$

show $\displaystyle \oiint_S \vec{E}\cdot\hat{n}\,dS = \frac{Q}{\epsilon_0}$

$$L.H.S: \oiint_S \vec{E}\cdot\hat{n}\,dS = \oiint_S \frac{Q}{4\pi\epsilon_0}\frac{\vec{r}}{|\vec{r}|^3}\cdot\frac{\langle \frac{x}{z}, \frac{y}{z}, 1\rangle}{|\langle \frac{x}{z}, \frac{y}{z}, 1\rangle|}\,dS$$

$$= \frac{Q}{4\pi\epsilon_0}\oiint_S \frac{\langle x,y,z\rangle\cdot\langle \frac{x}{z}, \frac{y}{z}, 1\rangle}{|\vec{r}|^4}\,dS$$

$$= \frac{Q}{4\pi\epsilon_0}\oiint_S \frac{|\vec{r}|^2}{|\vec{r}|^4}\,dS = \frac{Q}{4\pi\epsilon_0}\frac{1}{|r|^2}\underbrace{\oiint_S dS}_{4\pi r^2} = \frac{Q}{\epsilon_0} \;//$$

Stoke's Theorem $\oint_C \vec{F} \cdot \hat{T} ds = \iint_S \nabla \times \vec{F} dS$

Q.9 lec 24

$x^2+y^2+z^2=16$ above

$\vec{F} = \langle y-z, z-x, x-y \rangle$ plane $z = \sqrt{7}$

$\nwarrow$ our region

$$\oint_C \vec{F} \cdot \hat{T} ds = \int_0^{2\pi} \langle 3\sin t - \sqrt{7}, \sqrt{7} - 3\cos t, 3\cos t - 3\sin t \rangle \cdot \langle -3\sin t, 3\cos t, 0 \rangle dt$$

$$= \int_0^{2\pi} (-9 + 3\sqrt{7}\sin t + 3\sqrt{3}\cos t) dt = -18\pi \,//$$

$$\iint_S \nabla \times \vec{F} \cdot \hat{n} dS = \iint_S \langle -2, -2, -2 \rangle \cdot \langle \frac{x}{y}, \frac{y}{z}, 1 \rangle dS$$

$$= -2 \int_0^{2\pi} \int_0^3 r \left(\frac{r\cos\theta}{\sqrt{16-r^2}} + \frac{r\sin\theta}{\sqrt{16-r^2}} + 1 \right) dr d\theta = -18\pi \,//$$

Week 10: Lecture 2 Nov 13, 2024

- Goal: Given $\left.\begin{array}{l} \nabla \cdot \vec{F} = \rho(x,y,z) \\ \nabla \times \vec{F} = \vec{J}(x,y,z) \end{array}\right\}$ determine vector vield $\vec{F}$
  - * very difficult in general
- Motivation is electromagnetism

$$\left.\begin{array}{ll} \nabla \cdot \vec{E} = \rho/\epsilon_0 & \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = 0 & \nabla \times \vec{B} = \mu_0 \vec{J} \end{array}\right\}$$ time-$\overset{\text{ind}}{\downarrow}$dependent (time-independent) Maxwell Equation

- consider only the following:
  - ① irrotational fields where $\nabla \times \vec{F} = 0$ (electric field)
  - ② source-free fields where $\nabla \cdot \vec{F} = 0$ (magnetic field)

Irrotational Fields

Problem Statement: $\left.\begin{array}{l} \nabla \times \vec{F} = 0 \\ \nabla \cdot \vec{F} = \rho(x,y,z) \end{array}\right\}$ determine $\vec{F}$

$\leftarrow$ is scalar charge density with units [Coulombs/m^3]

- also difficult to solve in general.
- consider problems with symmetries only
  - ① spherical symmetry
  - ② cylindrical symmetry
  - ③ planar symmetry

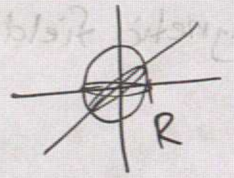
Spherically Symmetrical

$\rho(x,y,z)$ is spherically symmetric if it only depends on

$r = \sqrt{x^2+y^2+z^2} \Rightarrow \rho(x,y,z) = f(\sqrt{x^2+y^2+z^2})$ ← some $f: \mathbb{R} \to \mathbb{R}$

- ρ doesn't depend on ϕ and θ in spherical coordinates

ex// ① uniform sphere



$$\rho = \begin{cases} \rho_0 & r \le R \\ 0 & r > R \end{cases}$$

② non-uniform sphere

$$\rho = \begin{cases} r & r \le R \\ 0 & > R \end{cases}$$

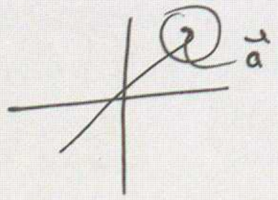
③ spherical shell of inner radius R_1 and outer R_2

$$\rho = \begin{cases} 1/r & R_1 \le r \le R_2 \\ 0 & r < R_1 \text{ or } r > R_2 \end{cases}$$

Variations: shape: sphere, spherical shell (with thickness), point charge, infinitely thin shell ...

charge: uniform, non-uniform, dirac-delta

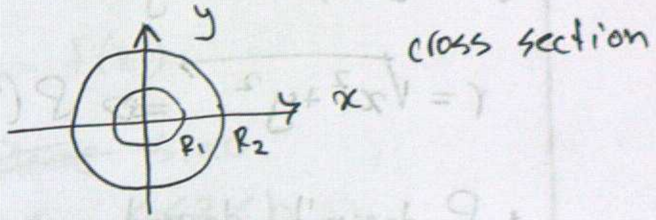
location: move center from origin to $\vec{a}$



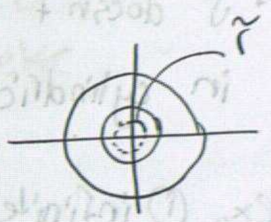
(general tequnique: shift $\vec{a}$ to origin and compute)

Ex: $\underbrace{\text{sphere radius } \tilde{r}}_{S_{\tilde r}}$, $\rho(r) = \begin{cases} 0 & , r \le R_1 \\ \rho_0/r^2 & , R_1 \le r \le R_2 \\ 0 & , R_2 \le r \end{cases}$

$$\begin{matrix}\text{Total}\\\text{charge}\end{matrix} = \iiint_{S_{\tilde r}} \rho(r)\, dV$$

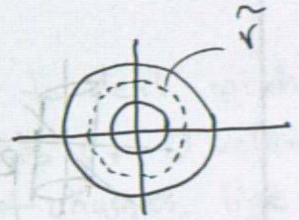


Case 1: if $\tilde{r} \overset{<}{\cancel{\le}} R_1$



$\Rightarrow \rho(r) = 0$ $\quad \begin{matrix}\text{total}\\\text{charge}\end{matrix} = \iiint_{S_{\tilde r}} 0\, dV = 0$

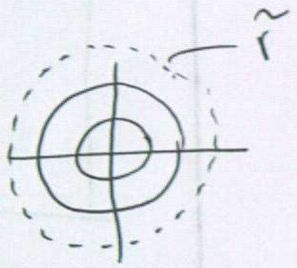
Case 2: if $\tilde{r}$ is : $R_1 \le \tilde{r} \le R_2$



$$\begin{matrix}\text{total}\\\text{charge}\end{matrix} = \iiint_{\sqrt{x^2+y^2+z^2} \le R_1} 0\, dV + \iiint_{R_1 \le \sqrt{x^2+y^2+z^2} \le \cancel{R}\, \tilde r} \rho_0/r^2\, dV$$

$$= 0 + \int_0^{2\pi}\int_0^{\pi}\int_{R_1}^{\tilde r} \frac{\rho_0}{r^2} r^2 \sin\phi \, dr\, d\phi\, d\theta = 4\pi\rho_0(\tilde{r} - R_1)$$

Case 3: $\tilde{r} > R_2$



$$\begin{matrix}\text{total}\\\text{charge}\end{matrix} = \iiint_{r < R_1} 0\, dV + \iiint_{R_1 \le r \le R_2} \rho_0/r^2\, dV + \iiint_{R_2 \le r \le \tilde r} 0\, dV$$

$$= 0 + 4\pi\rho_0(R_2 - R_1) + 0$$

$$\boxed{\begin{matrix}\text{total}\\\text{charge}\end{matrix} = \begin{cases} 0 & \text{if } \tilde{r} < R_1 \\ 4\pi\rho_0(\tilde{r} - R_1) & \text{if } R_1 \le \tilde{r} \le R_2 \\ 4\pi\rho_0(R_2 - R_1) & \text{if } \tilde{r} > R_2 \end{cases}}$$

Cylindrically Symmetrical

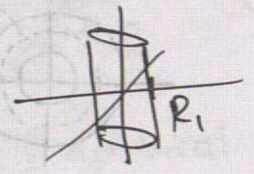
$\rho(x,y,z)$ is cylindrically symmetric if it only depends on

$$r=\sqrt{x^2+y^2} \Rightarrow \rho(x,y,z) = f(\sqrt{x^2+y^2})$$

$f: \mathbb{R} \to \mathbb{R}$

- ρ doesn't depend on z or θ in cylindrical coordinates

ex., ① infinite cylinder

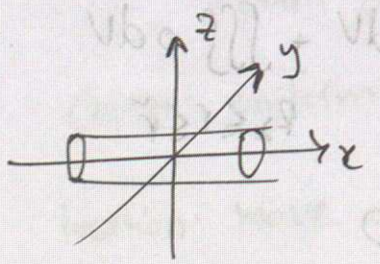


$$\rho = \begin{cases} \rho_0 & r \le R_1 \\ 0 & r > R_1 \end{cases}$$

Variations: shape: cylinder, cylindrical shell, wire ...

charge: uniform, non-uniform, dirac-delta

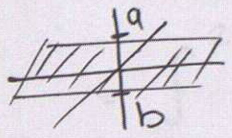
location: shift axis, rotate axis



Planar Charge Density

$\rho(x,y,z)$ is planar symmetric if it only depends on z (or x,y) $\Rightarrow \rho(x,y,z) = f(z)$

ex.,



Infinite ~~Plane~~ Slab

$$\rho(z) = \begin{cases} \rho_0, & a \le z \le b \\ 0 & z \not\lessgtr b \text{ etc} \end{cases}$$

Week 10: Lecture 3

Dirac Distribution (Dirac Delta)

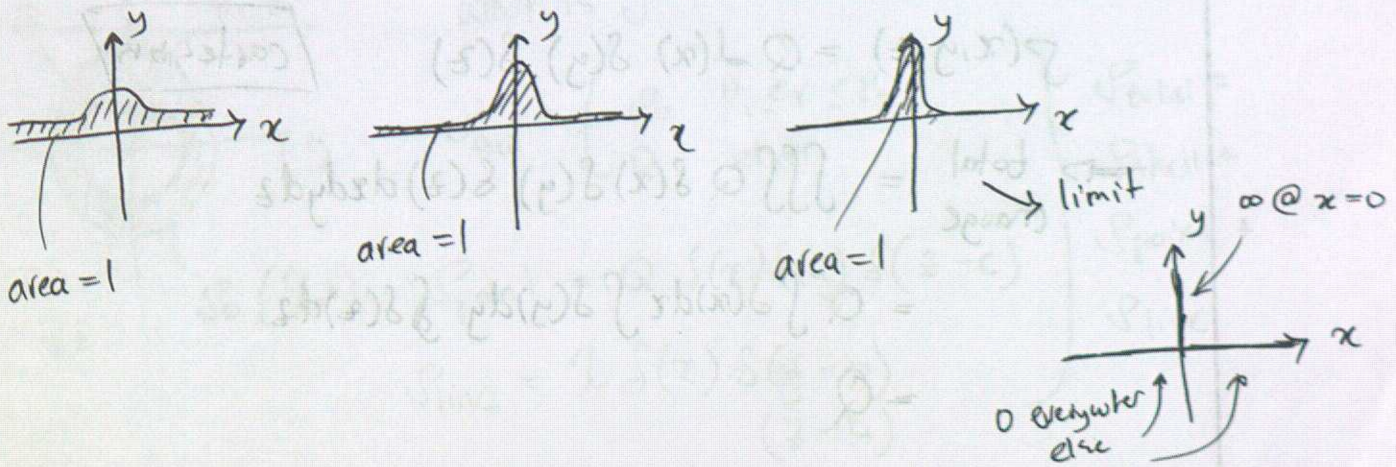
Motivation: $\text{total charge} = \iiint_{\text{object}} \rho \, dV$ ← $[m^3]$, ↑ $[C/m^3]$

How to deal with objects of 0 volume? (point charges, line charges etc.)

The dirac delta function is defined as:

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x) = 1$$

→ dirac delta function is a limit of



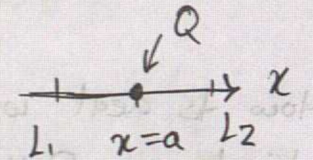
Properties

$$\int_{-\varepsilon_1}^{\varepsilon_2} \delta(x) = 1 \quad \text{for all } \varepsilon_1, \varepsilon_2 > 0$$

$$\int_{0}^{\varepsilon_1} \delta(x) = \frac{1}{2} \quad \text{for } \varepsilon_1 > 0$$

charge

- dirac delta gives distribution for a point charge in 1D



$$\rho(x) = \delta(x-a)\,Q = Q\,\delta(x-a)$$

$$\Rightarrow \text{total charge} = \int_{L_1}^{L_2} \rho(x)\,dx = Q$$

$$\int_{-\varepsilon_1}^{\varepsilon_2} f(x)\,\delta(x)\,dx = f(0) \quad \text{for } \varepsilon_1, \varepsilon_2 > 0$$

Dirac Delta in 3D

① Point charge at $\vec{r} = (a,b,c) = (0,0,0)$

$$\rho(x,y,z) = Q\,\delta(x)\,\delta(y)\,\delta(z)$$ $\boxed{\text{cartesian}}$

$$\Rightarrow \text{total charge} = \iiint Q\,\delta(x)\delta(y)\,\delta(z)\,dx\,dy\,dz$$

$$= Q\int\delta(x)dx\int\delta(y)dy\int\delta(z)dz$$

$$= Q$$

$$\rho(r,\theta,\phi) = \frac{Q}{2\pi r^2}\,\delta(r)$$ $\boxed{\text{spherical}}$

$$\Rightarrow \text{total charge} = \int_0^\infty\int_0^{2\pi}\int_0^{\pi} \frac{Q}{2\pi r^2}\,\delta(r)\,r^2\sin\phi\,d\phi\,d\theta\,dr$$

$$= \underbrace{\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta}_{4\pi}\int_0^\infty \frac{Q}{2\pi r^2}\,\delta(r)\,r^2\,dr$$

$$= Q$$

② Line charge with density λ (C/m) located along z-axis

cartesian $\rho(x,y,z) = \lambda\,\delta(x)\,\delta(y)$

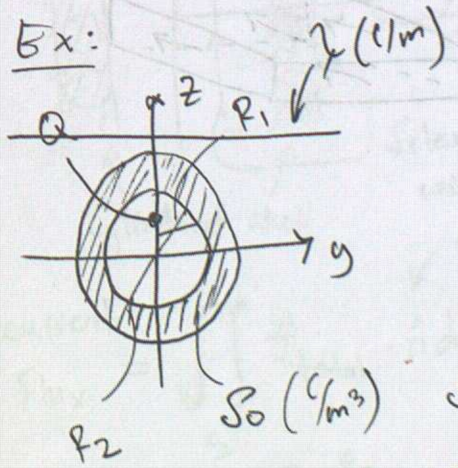
cylindrical $\rho(r,\theta,z) = \frac{\lambda}{\pi r}\,\delta(r)$

charge density (C/m^2)

③ Infinite Plane: $\rho(z) = \sigma\,\delta(z)$

④ Spherical Shell: $\rho(r) = \sigma\,\delta(r-R) \Leftarrow$ shell of radius R

Ex:



What is ρ?

$$\rho_{shell} = \begin{cases} \rho_0 & R_1 \leq r \leq R_2 \\ 0 & \text{else} \end{cases}$$

$$\rho_{point} = Q\,\delta(x)\delta(y)\,\delta(z-c)$$

$$\rho_{line} = \lambda\,\delta(x)\,\delta(y-b)$$
(z-b)

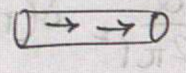
$\rho_{total} = \rho_{shell} + \rho_{point} + \rho_{line}$

Week 11: Lecture 1 Nov 18, 2024

Current Flux Density

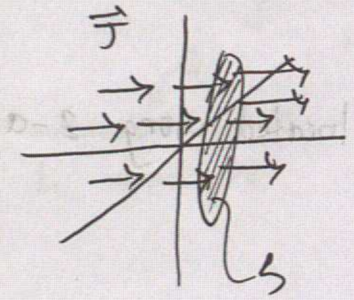
→ solving for source-free fields: $\nabla \cdot \vec{F} = 0 \quad \nabla \times \vec{F} = \vec{J}$

→ $\vec{J}$ is called "current flux density" (represents the movement of charges through space) units: A/m2



⟸ total flow = current

→ current $\vec{J}$ is a vector field



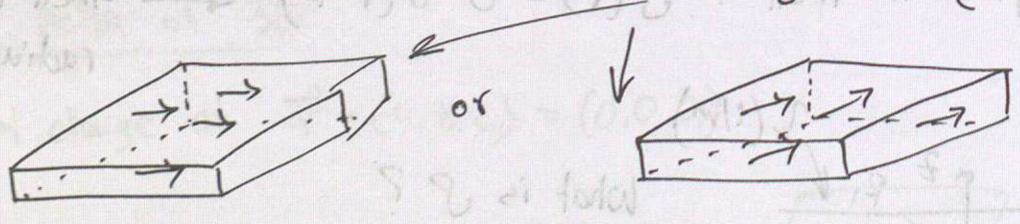
total current thru S $= \iint_S \vec{J} \cdot \hat{n}\, dS$

→ assume current is steady (no time-dependence)

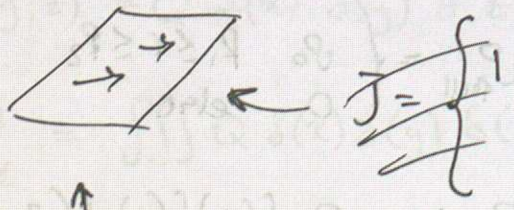
- We will consider 4 symmetries:

① Infinite plane

$$J = \begin{cases} I_0 \hat{x} & a \le z \le b \\ 0 & \text{else} \end{cases}$$



or



$J = I_0 \delta(z) \hat{x}$

② Cylindrical (axial)

$$\vec{J} = \begin{cases} \underset{\text{function of } r}{I_0(r)} \hat{z} & r < R \\ 0 & \text{else} \end{cases}$$

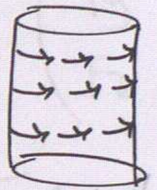
or

$\vec{J} = I_0\, \delta(x)\, \delta(y)\, \hat{z}$

or

$= I_0 \frac{1}{\pi r} \delta(r) \hat{z}$

③ Cylindrical (rotational)



⟸ $\vec{J}$ in $\hat{\theta}$ direction

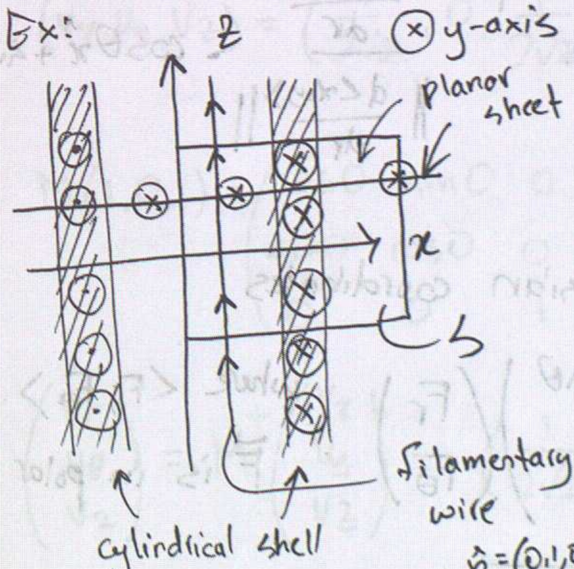
$\vec{J} = I_0\, \delta(r - R)\, \hat{\theta}$

(infinitely thin shell)

$$\vec{J} = \begin{cases} I_0 \hat{\theta} & R_1 \le r \le R_2 \\ 0 & \text{else} \end{cases}$$

④ Toroid (later in course)

☆ Ex:



what is total current flux density?

$$\vec{J}_{cyl} = \begin{cases} J_0 \hat{\theta} & R_1 \le r \le R_2 \\ 0 & \text{else} \end{cases}$$

$\vec{J}_{plane} = J_1\, \delta(z-1)\, \hat{y}$

$\vec{J}_{wire} = I_0\, \delta(x-1)\, \delta(y)\, \hat{z}$

$\hat{n} = (0,1,0),\ dS = dx\,dz$

$$\text{current flux} = \iint_S \vec{J}_{total} \cdot \hat{n}\, dS = \iint_S \vec{J}_{cyl} \cdot \hat{n}\, dS + \iint_S \vec{J}_{wire} \cdot \hat{n}\, dS + \iint_S \vec{J}_{wire} \cdot \hat{n}\, dS$$

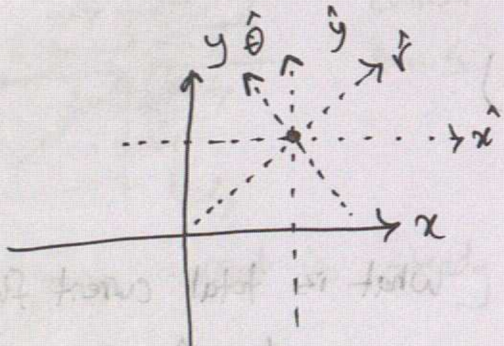
$$= \int_{-2}^{2} \int_{R_1}^{R_2} J_0 \hat{\theta} \cdot \hat{j}\, dx\, dz + \iint \cdots = 4(R_2 - R_1) J_0 + 0 + 5 J_1$$

Principle of Superposition

Given a charge distibution $\rho = \rho_1 + \rho_2$, the field generated is $\vec{F} = \vec{F_1} + \vec{F_2}$ where F_i is generated by ρ_i

→ we can solve fields by considering each shape at a time : $\vec{F} = \vec{F_1} + \vec{F_2}$

but what if diff. coordinate systems?



⟸ two different bases

How to convert?

$$\hat{r} = \frac{\frac{d\langle x,y\rangle}{dr}}{\left\| \frac{d\langle x,y\rangle}{dr} \right\|} - \cos\theta\hat{x} + \sin\theta\hat{y}$$

$$\hat{\theta} = \cdots = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

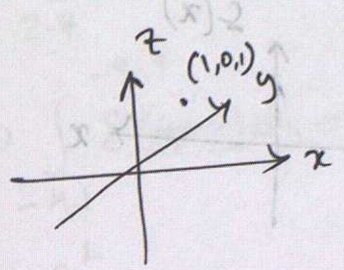
→ given $\vec{F} = \langle F_x, F_y \rangle$ in cartesian coordinates

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{M} \begin{pmatrix} F_r \\ F_\theta \end{pmatrix}$$

where $\langle F_r, F_\theta \rangle$ $\vec{F}$ is in polar coord.

M is called coordinate transform matrix

Ex: $\vec{F} = \frac{1}{r^2}\hat{r}$ in spherical



Point $= (1,0,1) \Rightarrow (r, \theta, \phi) = (\sqrt{2}, 0, \frac{\pi}{4})$

$$\vec{F} = \frac{1}{r^2}\hat{r} + 0\hat{\theta} + 0\hat{\phi}$$

$$\vec{F}(\text{point}) = \frac{1}{(\sqrt{2})^2}\hat{r} = \frac{1}{2}\hat{r}$$

$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_r \\ F_\theta \\ F_\phi \end{pmatrix}$$

$$\vec{F}(\text{point}) = \frac{1}{2}\hat{r} = \_\_\,\hat{x} + \_\_\,\hat{z} = \frac{1}{2}\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{2}\frac{1}{\sqrt{2}}\hat{z}$$

Now cartesian ⟼ cylindrical

Point = ~~$(\sqrt{2}, 0, \frac{\pi}{4})$~~

Point $= (r, \theta, z) = (1,0,1)$

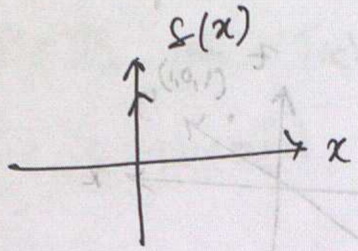
$$(V_x, V_y, V_z) = \left(\frac{1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}}\right)$$

$$M(1,0,1) = \begin{pmatrix} \cos 0 & \sin 0 & 0 \\ \sin 0 & \cos 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} V_r \\ V_\theta \\ V_z \end{pmatrix} = M^T \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \left(\frac{1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}}\right) = (V_r, V_\theta, V_z)$$

Tutorial 10 Nov 19, 2024

Dirac Delta : $\rightarrow \begin{cases} \delta(x) = \infty & x=0 \\ \delta(x)=0 & x\neq 0 \end{cases}$



$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

$\Rightarrow$ Acts like a sampler

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

Q.1 lec 25B $\rho(x,y,z) = \underset{\substack{\uparrow \\ \text{charge density}}}{\lambda}\,\delta(\underset{\substack{\uparrow \\ x_0=3}}{x-3})\,\delta(\underset{\substack{\uparrow \\ y_0=-4}}{y+4})$

a) total charge Q in $0.2\,m \times 0.2\,m \times L\,m$ box Centered at $x=3$, $y=-4$

$$Q = \iiint_{\underset{\nwarrow \text{box}}{V}} \rho(x,y,z)\,dx\,dy\,dz = \iiint_V \lambda\,\delta(x-3)\,\delta(y+4)\,dx\,dy\,dz$$

$x_{min} = 3-0.1 = 2.9$ $y_{min} = -4-0.1 = -4.1$

$x_{max} = 3+0.1 = 3.1$ $y_{max} = -4+0.1 = -3.9$

$z \rightarrow z_{min} = 0$, $z_{max} = L$

$$\iiint_V \lambda\,\delta(x-3)\,\delta(y+4)\,dx\,dy\,dz = \lambda\int \delta(x-3)\,dx \int \delta(y+4)\,dy \int dz$$

① $\int_{2.9}^{3.1} \delta(x-3)\,dx = 1$ sine $x_0 = 3$ is in $[2.9, 3.1]$

② $\int_{-4.1}^{-3.9} \delta(y+4)\,dy = 1$ sine $y_0 = -4$ is in $[-4.1, -3.9]$

③ $\int_0^L dz = L$ $\boxed{Q = \lambda \times 1 \times 1 \times L = \lambda L}$

b) Q in $x=3$, $y=0$, $\rho(x,y,z) = \lambda\,\delta(x-3)\,\delta(y+4)$

$\left.\begin{aligned} y_{min} &= 0-0.1 = -0.1 \\ y_{max} &= 0+0.1 = 0.1 \end{aligned}\right\} \int_{-0.1}^{0.1} \delta(\underset{\substack{\downarrow \\ y_0=0}}{y})\,dy = 1$ $\boxed{\therefore Q = \lambda L}$

Quiz 9: 2023 ① cylindrical shell radius 2, $J_s = 2$ (flows in $\left\langle -\frac{z}{\sqrt{x^2+z^2}}, 0, \frac{x}{\sqrt{x^2+z^2}} \right\rangle$

symmetry axis passes through (3,0,0)

② filamentory conductor parallel to z-axis passing through $(-3,2,0)$, $I=4$ (direction given by $\langle 0,0,-1 \rangle$

③ planar S: $\{-4 \leq x \leq 2,\ 1 \leq y \leq 3,\ z=0\}$ (in x-y plane)

a) $J_t = J_{cylindrical} + J_{conductor}$

compute $\hat{n}\cdot dS \Rightarrow \hat{n} = \langle 0,0,1 \rangle \Rightarrow \langle 0,0,1 \rangle\,dx\,dy$

$\uparrow$ related to plane S

$dS = dx\,dy$

conductor : $\vec{J}_{conductor} = I\,\delta(x+3)\,\delta(y-2)\cdot\langle 0,0,-1\rangle$

$\vec{J}_{cylindrical\ shell} = J_S\,\delta\left(\sqrt{\cancel{x}^2+z^2}-2\right)\cdot\left\langle \frac{-z}{\sqrt{x^2+z^2}},0,\frac{x}{\sqrt{x^2+z^2}}\right\rangle$

(x-3)   only consider all the points on the cylinder

b) formulate double integral for computing current flux

$$\iint_S \vec{J}_t\cdot dS = \iint_S 2\,\vec{J}_{cylind.}\cdot dS + \cancel{\iint}$$

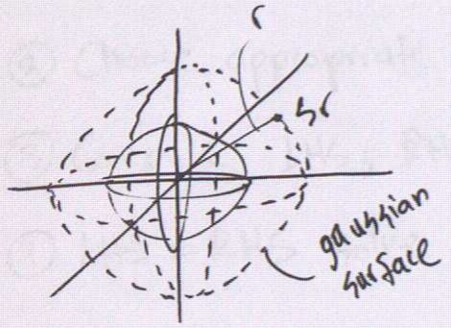
Week 11: Lecture 2

Solving for Irrotational Fields

→ given charge density, solve for electric field

Engineer: Determine the field generated by a sphere with radius R centered at the origin and a uniform charge density ρ_0

→ exploit symmetry :



$\nabla\cdot\vec{F} = \rho(r) \leftarrow$ only depends on r (not θ or ϕ)

$\Rightarrow$ field depends only on r

$\vec{F} = F_r(r)\,\hat{r} + F_\theta(r)\,\hat{\theta} + F_\phi(r)\,\hat{\phi}$

→ we can also note that $\vec{F}$ points only in $\hat{r}$ direction :

→ Intuition: no curl, only divergence

→ Mathematical : $0 = \nabla\times\vec{A}$

$\therefore$ By symmetry $\Rightarrow \vec{F} = F(r)\,\hat{r}$

→ use divergence theorem : $\iiint_V \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,dS$, $S = \partial V$

→ pick gaussian surface (appropriate hypothetical surface) where RHS easy to compute

$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S F(r)\,\hat{r}\cdot\hat{n}\,dS \leftarrow$ pick S so $\hat{n} = \hat{r}$ (sphere in this case)

RHS: $= \iint_{S_r} F(r)\,\hat{r}\cdot\hat{r}\,dS = \iint_{S_r} F(r)\,dS = F(r)\iint_{S_r} dS$ since F(r) constant on Sr

$\boxed{= F(r)4\pi r^2}$

$$LHS := \underset{\substack{\text{interior} \\ \text{of } S_r}}{\iiint} \rho \, dV \qquad \nabla \cdot \vec{F} = \rho = \begin{cases} \rho_0, & r < R \\ 0, & r > R \end{cases}$$

Case 1: $r < R$

$$LHS = \iiint \rho_0 \, dV = \int_0^{\pi} \int_0^{2\pi} \int_0^{r} \rho_0 r^2 \sin\phi \, dr \, d\theta \, d\phi$$

$$= \frac{4}{3} \pi r^3 \rho_0$$

Case 2: $r > R$

$$LHS = \cancel{\iiint_{0<r<R} \rho_0 dV} =$$

$$= \iiint_{0 \le \sqrt{x^2+y^2+z^2} \le R} \rho_0 \, dV + \iiint_{R \le \sqrt{x^2+y^2+z^2} \le r} 0 \, dV$$

$$= \frac{4}{3} \pi R^3 \rho_0$$

→ evaluate LHS = RHS

$$F(r) \, 4\pi r^2 = \begin{cases} \cancel{\pi} \frac{4}{3} \pi r^3 \rho_0 & r < R \\ 4/3 \, \pi R^3 \rho_0 & r > R \end{cases}$$

$$\vec{F} = \begin{cases} 1/3 \, r \rho_0 \hat{r} & r < R \\ 1/3 \frac{\rho_0 R^3}{r^2} \hat{r} & r > R \end{cases}$$

Week 11: Lecture 3

Nov 22, 2024

Summary of General Tequnique

① Exploit symmetry :

- → spherical ⟹ $\vec{F} = F(r)\hat{r}$ $r = \sqrt{x^2+y^2+z^2}$ (spherical)
- → cylindrical ⟹ $\vec{F} = F(r)\hat{r}$ $r = \sqrt{x^2+y^2}$ (cylindrical)
- → planar ⟹ $\vec{F} = F(z)\hat{z}$

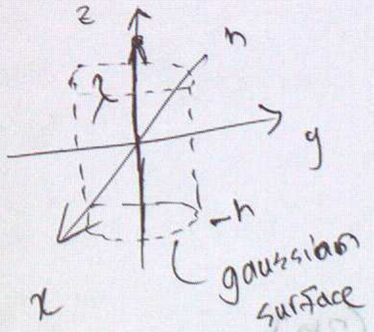
② Choose appropriate Gaussian surface

③ Compute LHS = RHS of divergence theorem

④ LHS = RHS solve for $\vec{F}$

Infinite Line Charge

What is field generated by infinite line charge along z-axis with line charge density λ



$$\nabla \cdot \vec{F} = \rho(r) = \frac{\lambda}{\pi r} \delta(r)$$

• cylindrical symmetry, for problem: $\vec{F}$ depends only on r since ρ depends only on r, and direction is $\hat{r}$

$\vec{F} = F(r)\hat{r}$, now choose a gaussian surface (a cylinder) and call it $S_{r,h}$ for r radius and h height

→ compute $\text{RHS} = \underset{int(S_{r,h})}{\iiint} \nabla\cdot\vec{F}\,dV$ $\quad \nabla\cdot\vec{F} = \rho = \frac{\lambda}{\pi r}\delta(r)$

$$= \int_0^{2\pi}\int_{-h}^{h}\int_0^{r} \frac{\lambda}{\pi r}\delta(r)\, r\,dr\,dz\,d\theta$$

$$= Q_{enc} = \lambda\cdot 2h$$

→ compute $\text{LHS} = \underset{S_{r,h}}{\oiint} \vec{F}\cdot\hat{n}\,dS$

$$= \underset{\substack{cyl\\body}}{\iint} \vec{F}\cdot\hat{n}\,dS + \underset{top}{\iint}\vec{F}\cdot\hat{n}\,dS + \underset{bottom}{\iint}\vec{F}\cdot\hat{n}\,dS$$

$$= \underset{\substack{cyl\\body}}{\iint} \vec{F}(r)\,\hat{r}\,\hat{r}\,dS + \underset{top}{\iint} F(r)\underbrace{\hat{r}\cdot\hat{z}}_{0}\,dS + \underset{bott}{\iint} F(r)\underbrace{\hat{r}\cdot(-\hat{z})}_{0}\,dS$$

$$= F(r)\int_{-r}^{h}\int_0^{2\pi} r\,d\theta\,dz$$

constant

$$\boxed{= F(r)\,2\pi r\,(2h)}$$

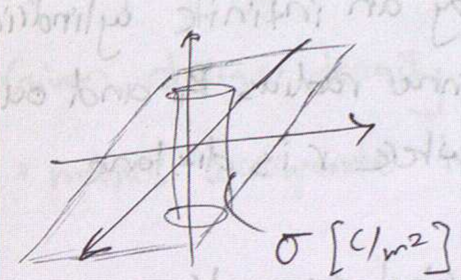
$\text{LHS} = \text{RHS} \Longrightarrow F(r)\,2\pi r\,(2h) = \lambda(2h)$

$$F(r) = \frac{\lambda}{2\pi r}$$

$$\therefore \vec{F}(r) = \frac{\lambda}{2\pi r}\hat{r}$$ //

Infinite Plane Charge

what is field generated by an infinite plane charge along xy-plane with surface charge density σ



$\nabla\cdot\vec{F} = \rho = \sigma\,\delta(z)$

→ exploit symmetry, since $\vec{F}$ depends only on z since ρ depends on $|z|$, and direction of $\vec{F}$ is (+)ve $\hat{z}$ on top and (−)ve $\hat{z}$ on bottom

$$\vec{F} = F(z)\begin{cases} +\hat{z} & z>0 \\ -\hat{z} & z<0\end{cases}$$

→ choose gaussian surface as cylinder $S_{r,h}$

→ compute $\text{LHS} = \underset{int(S_{r,h})}{\iiint}\rho\,dV = Q_{enc} = \sigma(\text{area of base})$

$$= \sigma\pi r^2$$

→ compute $\text{RHS} = \underset{S_{r,h}}{\iint}\vec{F}\cdot\hat{n}\,dS$

$$= \underset{\substack{cyl\\body}}{\iint}\underbrace{\vec{F}\cdot\hat{n}}_{=0}\,dS + \underset{top}{\iint}\vec{F}\cdot\hat{n}\,dS + \underset{bott}{\iint}\vec{F}\cdot\hat{n}\,dS$$

$$= \underset{top}{\iint} F(\overset{h}{z})\,\hat{z}\cdot\hat{z}\,dS + \underset{bott}{\iint} F(-h)(-\hat{z})(-\hat{z})\,dS$$

$F(-h) = F(h)$

$$= F(h)\left(\begin{matrix}\text{area of}\\\text{base}\end{matrix}\right) + F(h)\left(\begin{matrix}\text{area of}\\\text{base}\end{matrix}\right)$$

$$= 2F(h)(\text{area of base})\ \boxed{= 2F(h)\,\pi r^2}$$

LHS = RHS

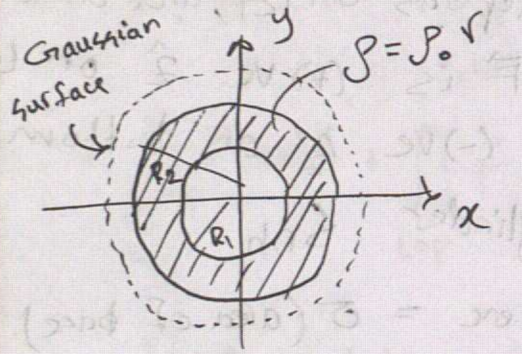
↓

$\sigma\pi r^2 = 2F(h)\pi r^2 \Longrightarrow \vec{F}(h) = \begin{cases}\sigma/2\ \hat{z} & z>0\\ -\sigma/2\ \hat{z} & z<0\end{cases}$ //

Week 12: Lecture 1 Nov 25, 2024

For divergence thm, LHS is same standard symmetry, and by LHS we mean the surface integral portion.

Ex: Determine the field generated by an infinite cylindrical shell centered at the z axis with inner radius R_1 and outer R_2. Shell charge distrib. $\rho = \rho_0 r$ where r is distance from z-axis.



$\rho(r) \rightarrow$ cylindrical symmetry

$\Longrightarrow \vec{F} = F(r)\hat{r}$, $r = \sqrt{x^2+y^2}$

→ choose Gaussian surface: cylinder with radius r, height h.

→ use divergence thm / Gauss Law

$$LHS = \oiint_S \vec{F} \cdot \hat{n} dS = F(r) 2\pi r h$$

RHS = Qenc

case 1: $r < R_1$

RHS = 0

case 2: $R_1 < r < R_2$

$$RHS = \int_0^{2\pi}\int_0^{h}\int_{R_1}^{r} \rho_0 r r dr dz d\theta$$

$$= 2\pi h \rho_0 \frac{1}{3}(r^3 - R_1^3)$$

case 3: $r > R_2$

$$RHS = \int_0^{2\pi}\int_0^{h}\int_{R_1}^{R_2} \rho_0 r r dr dz d\theta$$

$$= 2\pi h \rho_0 \frac{1}{3}(R_2^3 - R_1^3)$$

$$RHS = \begin{cases} 0, & r < R_1 \\ \rho_0 \frac{2}{3}\pi h (r^3 - R_1^3), & R_1 < r < R_2 \\ \rho_0 \frac{2}{3}\pi h (R_2^3 - R_1^3), & r > R_2 \end{cases}$$

LHS = RHS

$$\vec{F} = \begin{cases} 0 & r < R_1 \\ \frac{\rho_0}{3}\frac{r^3 - R_1^3}{r}\hat{r} & R_1 < r < R_2 \\ \frac{\rho_0}{3}\frac{R_2^3 - R_1^3}{r}\hat{r} & r > R_2 \end{cases}$$

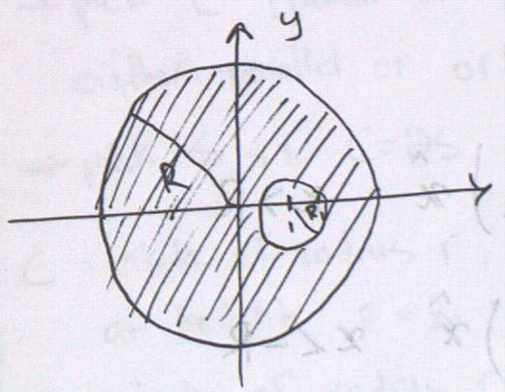
Principle of Superposition Cont'd

→ Given charge density $\rho = \rho_1 + \rho_2$ then field generated is $\vec{F} = \vec{F}_1 + \vec{F}_2$ where $\vec{F}_1$ generated by ρ_1, $\vec{F}_2$ generated by ρ_2

→ Given charge density $\rho = \rho_1 - \rho_2$, $\vec{F} = \vec{F}_1 - \vec{F}_2$

+ minus corresponds to

Ex: A sphere of radius R centered at the origin with uniform charge density / distrib. ρ_0 has a hollowed out sphere of radius R_1 centered at (1,0,0), and $R > 2$, $R_1 < 1$. What B field generated outside larger sphere along x-axis



$\rho = \rho_0 - \rho_1 \Longrightarrow \rho = \rho_{sph} - \rho_{void}$

ρ_0 ↑ sphere of big R

ρ_1 ↑ hollow portion

ρ_{void} : density sphere R_1 centered at (1,0,0) and uniform charge ρ_0

ρ_{sphe} : charge density of sphere w/h radius R and uniform charge ρ_0

→ compute field from each object:

sphere: LHS = RHS

$$F(r) 4\pi r^2 = \iiint_{gaussian} \rho_0 dV$$

$$= \rho_0 \frac{4}{3}\pi R^3$$

$$F(r) = \frac{1}{3}\rho_0 \frac{R^3}{r^2}$$

$$\rightarrow \vec{F} = \frac{\rho_0}{3}\frac{R^3}{r^2}\hat{r}$$

Void: assume void centered at origin

same case as shere, but different radius

$$\vec{F} = \frac{\rho_0}{3}\frac{R_1^3}{r^2}\hat{r}$$

→ translate objects as needed and compute along x-axis $(x,0,0)$

$$\vec{F} = \vec{F}_{sph} - \vec{F}_{void}$$

$$= \underbrace{\frac{\rho_0}{3}\frac{R^3}{x^2}(\pm\hat{x})}_{} - \frac{\rho_0}{3}\frac{R_1^3}{(x-1)^2}$$

$r = x$ along x-axis

$$\hat{r} = \begin{cases} \hat{x}, & x>1 \\ -\hat{x}, & x<1 \end{cases}$$

$$\therefore \hat{r} = \begin{cases} \hat{x}, & x>0 \\ -\hat{x}, & x<0 \end{cases}$$

$$\Rightarrow \vec{F} = \begin{cases} \frac{\rho_0}{3}\left(\frac{R^3}{x^2} - \frac{R_1^3}{(x-1)^2}\right)\hat{x} & x>R \\ -\frac{\rho_0}{3}\left(\frac{R^3}{x^2} - \frac{R_1^3}{(x-1)^2}\right)\hat{x} & x<-R \end{cases}$$

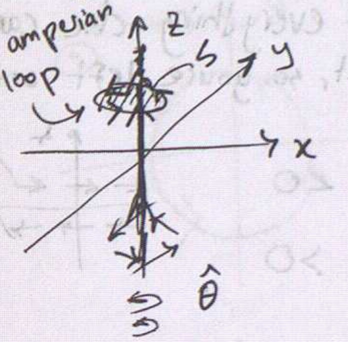
$\nearrow$ $\vec{F}(x,0,0)$

Week 12: Lecture 2 Nov 27, 2024

Infinite Line Current

Find $\vec{F}$ where $\vec{F}$ is source-free and $\nabla\times\vec{F} = \frac{I_0}{\pi r}\delta(r)\hat{z}$

Determine the field generated by an infinite line current along the z axis flowing upwards



→ exploit symmetry; current depends only on $r = \sqrt{x^2+y^2} \Rightarrow$ field depends only on r

→ use the right-hand-rule to determine direction

$\therefore \vec{F} = F(r)\hat{\theta}$, use stoke's thm

→ use Stoke's Thm: $\iint_S \nabla\times\vec{F}\cdot\hat{n}\,dS = \oint_{C=\partial S} \vec{F}\cdot\hat{T}\,ds$

→ pick C (called an Amperian loop) s.t. $\vec{F}$ and $\hat{T}$ are eigther parallel or orthogonal

→ pick S s.t. $C=\partial S$ and $\overset{RHS}{\oint_C \vec{F}\cdot\hat{T}ds}$ easy to compute ($\hat{n}$ and $\nabla\times\vec{F}$) ← are parallel

C = circle of radius r, oriented CCW (viewed from above) at height $z = z_0$

S = disk of radius r

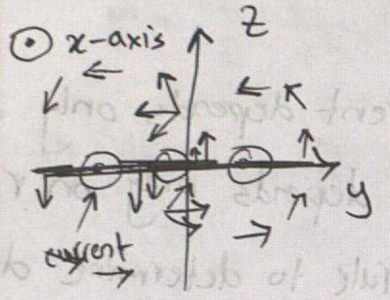
$$RHS = \oint_C \vec{F}\cdot\hat{T}ds = \oint_C F(r)\underbrace{\hat{\theta}\cdot\hat{\theta}}_{=1}ds = F(r)\oint_C ds = \boxed{F(r)2\pi r}$$

$$LHS = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,dS = \int_0^{2\pi}\int_0^r\left(\frac{I_0}{\pi r}\delta(r)(\hat{z})\right)\cdot\hat{z}\,dS = I_0$$

$$LHS = RHS \Rightarrow F(r) = \frac{I_0}{2\pi r} \Rightarrow \boxed{\vec{F} = \frac{I_0}{2\pi r}\hat{\theta}}$$

Infinite Plane Current

Determine the field generated by an infinite sheet along the xy plane with surface current flux density $J_0 \hat{x}$



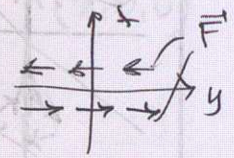
→ exploit symmetry: field must depend only on z

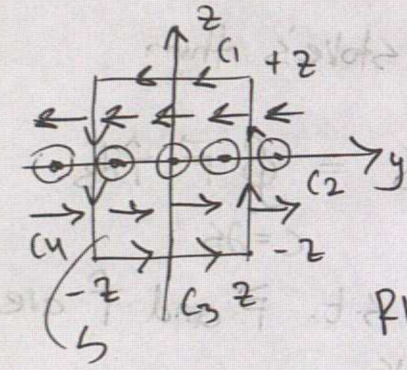
* everything else cancels out, so you're left with

→ direction:

$+\hat{y}$ for $z<0$

$-\hat{y}$ for $z>0$





→ amperian loop: square

→ surface: solid (filled in) square

$$RHS = \oint_C \vec{F}\cdot d\vec{r} = \oint_{C_1} \vec{F}\cdot dr + \underbrace{\oint_{C_2} \vec{F}\cdot dr}_{=0} + \oint_{C_3} \vec{F}\cdot dr + \underbrace{\oint_{C_4} \vec{F}\cdot dr}_{=0}$$

$$\therefore RHS = F(z)\int_{C_1} (-\hat{y})\cdot(-\hat{y})\,ds + F(z)\int_{C_3} (+\hat{y})\cdot(+\hat{y})\,ds$$

$$= 2F(z)\left(\text{width of square}\right)$$

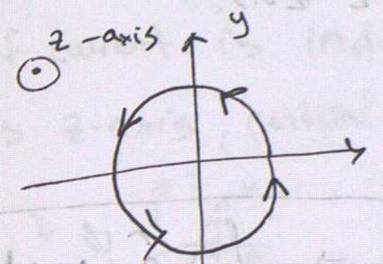
$$LHS = \iint \nabla\times\vec{F}\cdot\hat{n}\,dS = I_{enc} = J_0\left(\text{width of square}\right)$$

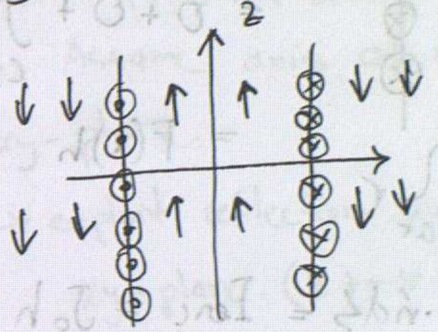
$$LHS = RHS \implies \vec{F} = \frac{J_0}{2}\begin{cases}\hat{y} & z<0 \\ -\hat{y} & z>0\end{cases}$$

Week 12: Lecture 3 Nov 29, 2024

Solenoid

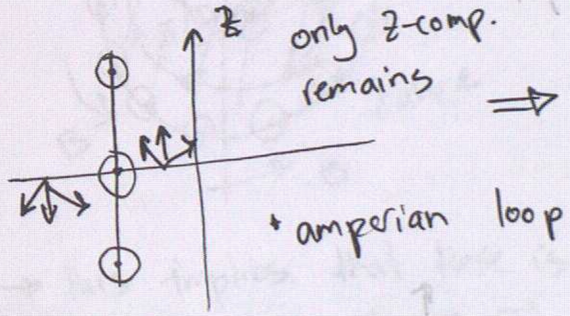
Determine the field generated by an infinite cylinder sheet of radius R with axis of symm. alligned w/h z axis. and surface current density $J_0 \hat{\theta}$





→ exploit symmetry ⟹ cylindrical symm, so $\vec{F}$ depends on $r = \sqrt{x^2+y^2}$

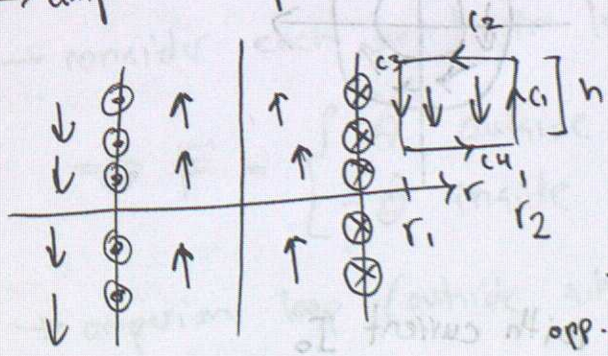
→ for direction:



$$\implies \vec{F}\text{-direction}\begin{cases}+\hat{z} & \text{inside} \\ -\hat{z} & \text{outside}\end{cases}$$

* amperian loop differs depending on outside or inside

→ amperian loop for outside: square outside solenoid



$$LHS = \oint_C \vec{F}\cdot\hat{T}\,ds$$

$$= \int_{C_1} \vec{F}\cdot\hat{T}\,ds + \underbrace{\int_{C_2} \vec{F}\cdot\hat{T}\,ds}_{=0} + \int_{C_3} \vec{F}\cdot\hat{T}\,ds + \underbrace{\int_{C_4} \vec{F}\cdot\hat{T}\,ds}_{=0}$$

$\hat{T}$ and $\vec{F}$ opp. dir.

$$= -F(r_2)h + F(r_1)h$$

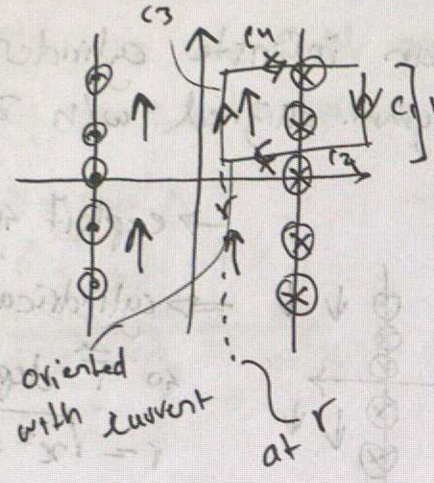
$$RHS = \iint_S \vec{J}\cdot\hat{n}\,dS = I_{enc} = 0$$

$$LHS = RHS \implies 0 = -F(r_2)h + F(r_1)h \implies F(r_1) = F(r_2)$$

this means $\vec{F}_{\text{outside}}$ is constant ⟹ since $\vec{F}(\infty) = 0$,

$$\boxed{\vec{F} = 0 \text{ outside}}$$

→ Amperian loop inside + including current + outside



$$LHS = \oint_C \vec{F}\cdot\hat{T}\,ds$$

$$= 0 + 0 + \int_{C_3} F(r)\,\hat{z}\cdot\hat{z}\,ds$$

$$= F(r)h$$

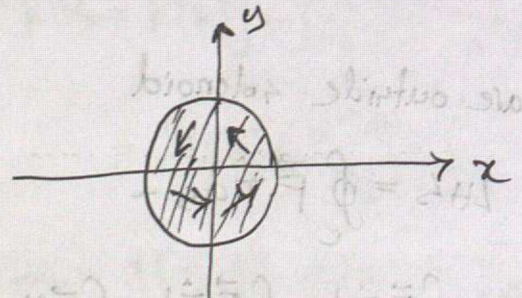
$$RHS = \iint \vec{J}\cdot\hat{n}\,dS = I_{enc} = J_0 h$$

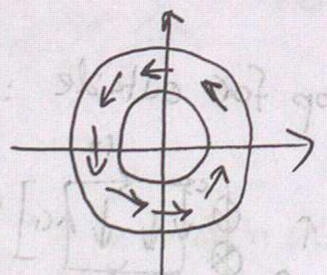
$$\boxed{\vec{F} = \begin{cases} J_0\hat{z} & \text{inside} \\ 0 & \text{outside} \end{cases}}$$

$$LHS = RHS \Longrightarrow J_0\cdot h = F(r)h$$

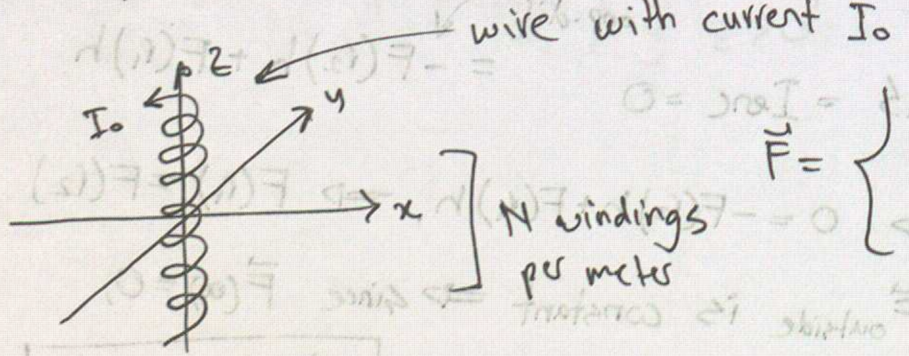
$$F(r) = J_0$$

Variations





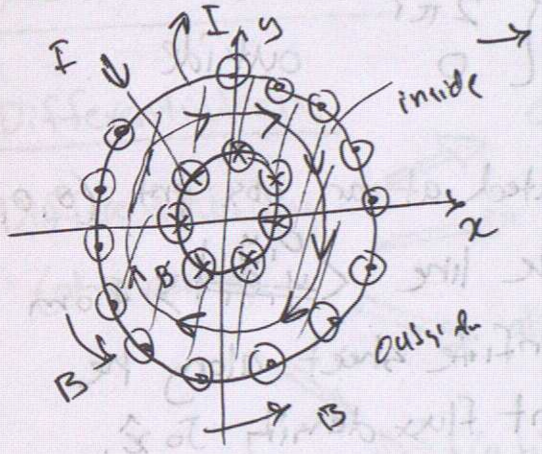
→ loop wire in helix



$$\vec{F} = \begin{cases} I_0 N\hat{z} & \text{inside} \\ 0 & \text{outside} \end{cases}$$

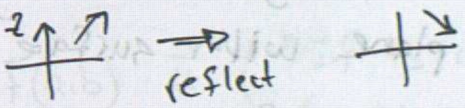
Toroid

Determine the field generated by a conducting wire with a current I wrapped into a toroidal helix of inner radius a and outer radius b with N total windings. The dir. of current is indicated. Assume axis of rotational symm. is z-axis, centered at xy-plare.



→ exploit reflection symmetry along xy-plane ($z \mapsto -z$) ... you flip it

→ toroid will be same, but field line of $\vec{F}$ will point opposite dir. w.r.t. to z



→ this implies that there is no z-component of $\vec{F}$

→ consider each point in loop, draw field lines, and find that

$$\Longrightarrow \vec{F} = \begin{cases} \hat{\theta} & \text{outside} \\ -\hat{\theta} & \text{inside} \end{cases}$$

→ amperian loop (outside whole toroid)

$$\Longrightarrow \int \vec{F}\cdot\hat{T}\,ds = F(r)\,2\pi r$$

*consider only mid-slice of toroid $\Longrightarrow z = 0$

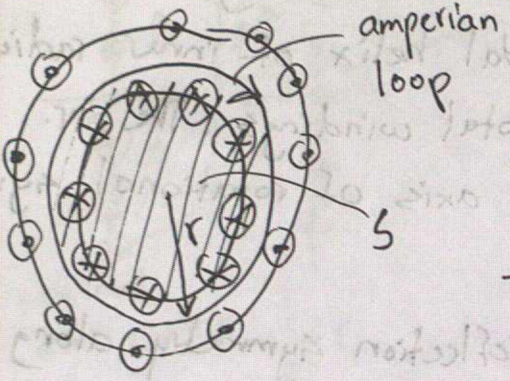
$$\Longrightarrow \iint \vec{J}\cdot\hat{n}\,ds = I_{enc} = 0$$

$$\therefore F(r) = 0 \text{ outside}$$

Week 13: Lecture 1

Dec 2, 2024

→ for inside solenoid:

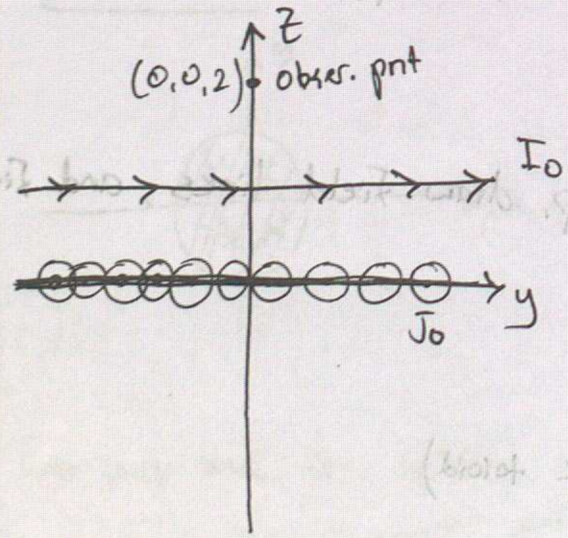


$$RHS = \oint \vec{F} \cdot d\vec{r} = F(r)2\pi r$$

$$LHS = \iint_S J \cdot \hat{n} \, dS = I_{enc} = I_0 N$$

$$\Rightarrow \vec{F} = \begin{cases} -\frac{I_0 N}{2\pi r} \hat{\theta} & \text{inside} \\ 0 & \text{outside} \end{cases}$$

Ex: Determine the field generated at an obs. pnt (0,0,2) by current I_0 flowing along the line $\langle$ ~~y, 0, 1~~ 0, y, 1 $\rangle$ from $y = -\infty$ to $y = \infty$ and by an infite sheet along the xy plane with surface current flux density $J_0 \hat{x}$.



→ find field from each individally

I_0: circle amperian loop

LHS = RHS (stoke's thm)

$$I_{enc} = I_0 = F(r) 2\pi r$$

$$\Rightarrow \vec{F}(r) = \frac{I_0}{2\pi r} \hat{\theta}$$

J_0: LHS = RHS

$$J_0 (\text{width}) = 2 \cdot F(z) (\text{width})$$

$$\Rightarrow \vec{F} = \frac{J_0}{2} \hat{\pm y}$$

→ determine field at obs. pnt

$$\vec{F}(obs) = \vec{F}_{line} + \vec{F}_{plane}$$

$$= \frac{I_0}{2\pi(1)} \hat{x} + \frac{J_0}{2}(-\hat{y}) = \left\langle \frac{J_0}{2\pi}, -\frac{J_0}{2}, 0 \right\rangle$$

Week 13: Lecture 2

Differentiation:

Differentiability (at a point) ⇒ continuous (at a point)

Differentiability (at a point) ⇒ partials exist (at a point)

Differentiability (at a point) ⇒ all directional derivatives exist (at a point)

Differentiability (at a point) ⇒ $D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$ ($D_{\vec{u}} f$ is a linear function of $\vec{u}$)

ex// $\begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

differentiable at (0,0)

→ not cont. ∴ not differentiable

ex// $\begin{cases} \frac{x^3+xy}{x+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

b) differentiable?

→ if it was ⇒ $D_{\vec{u}} f(0,0) = \nabla f(0,0) \cdot \vec{u}$,

$= \langle a_1, a_2 \rangle \cdot \langle u_1, u_2 \rangle$

$= a_1 u_1 + a_2 u_2$

a) $$D_u f(0,0) = \lim_{t \to 0} \frac{f(0+tu_1, 0+tu_2) - f(0,0)}{t} = \begin{cases} u_2 & u_1 \neq 0 \\ 0 & \text{if } u_1 = 0 \end{cases}$$

ex: $f(x,y) = |xy|$

$\frac{\partial f}{\partial x}(0,0) = \frac{d}{dx}|x\cdot 0| = \frac{d}{dx}(0) = 0$

y held constant

$\frac{\partial f}{\partial y}(0,0) = \frac{d}{dy}|0\cdot y| = \frac{d}{dy}(0) = 0$

we see partials are defined at (0,0), but is not defined nowhere else

$\frac{\partial f}{\partial x}(0,0.1) = \frac{d}{dx}|x(0.1)| = 0.1\frac{d}{dx}|x| \longrightarrow$ DNE at $x=0$

Differentiable ? $\underbrace{\text{tangent plane}} = f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0)$

$= 0$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L_{(0,0)}(x,y)}{\sqrt{(x-0)^2+(y-0)^2}} = 0 \ ?$$

$$= \lim_{(x,y)\to(0,0)} \frac{|xy| - 0}{\sqrt{x^2+y^2}}$$

$\therefore$ Yes, differentiable

$$= \lim_{r\to 0} \frac{r^2|\cos\theta\sin\theta|}{r} = 0$$

☆ **Stoke's** $\vec{F} = \langle xe^{\sin x^2} + yz,\ xz - \frac{1}{\log(y^2+2)},\ z^2 + yx \rangle$

a) $\nabla \times \vec{F} = \langle 0,0,0 \rangle$

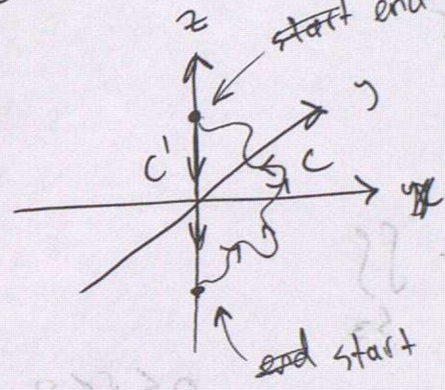
$$\oint \vec{F}\cdot\hat{T}ds = \iint_S \nabla\times\vec{F}\cdot\hat{n}dS$$

closed $\rightarrow C = \partial S$

b) $\int_C \vec{F}\cdot\hat{T}ds$

$\oint \vec{F}\cdot\hat{T}ds = 0$

any culved that is closed



$\Rightarrow$ CUC' is closed

$\Rightarrow \int_{CUC'} \vec{F}\cdot\hat{T}ds = 0$

$\Rightarrow \int_C \vec{F}\cdot\hat{T}ds = -\int_{C'} \vec{F}\cdot\hat{T}ds$

$C': \vec{r}(t) = \langle 0,0,t \rangle$

$t = 1$ to

$t = -1$

$= -\int_{-1}^{1} +t^2\,dt$

Divergence Thm $F = \langle x, y, z \rangle$

$$\iiint_V \nabla \cdot \vec{F}\, dV = \oiint_{S = \partial V} \vec{F} \cdot \hat{n}\, dS$$

$$LHS = \int_0^2 \int_{-\pi}^0 \int_0^{-y = -r\sin\theta} 3\, r\, dz\, d\theta\, dr$$

$$RHS: \iint_{\partial V} \cdots = \iint_{S_1} \cdots + \iint_{S_2} + \iint_{S_3}$$

R_1, $z = 0 \Rightarrow \langle 2r\cos\theta, 2r\sin\theta, 0 \rangle$ $0 \leq r \leq 2$

R_2, cylindrical coord. $\Rightarrow r = 2, 0 < \theta < 2\pi, 0 \leq z \leq 2\sin\theta$

R_3,