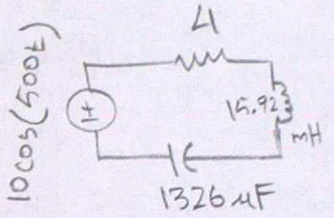
Week 1 Sep 3, 2024

Lecture 1

- tutorials week of Sep 16
- quizzes Thursday Sep 19



KUL: $V_R + V_L + V_C = V_s \rightarrow i'' + \frac{R}{L} i'' + \frac{1}{LC} i = V_s'$

ECE110 → Phasor Domain

$\bar{I}R + j\omega L\bar{I} + \frac{1}{j\omega C}\bar{I} = \bar{V}_s$ ← algebra solving

$\bar{I} = \frac{\bar{V}_s}{\bar{Z}_{tot}} = 1.32\angle{-58.2}$ [A] → $i(t) \doteq 1.32\cos(500t - 58.2^\circ)$ [A]

MAT290/ECE212 → Go into s-domain ($s = \sigma + j\omega$)

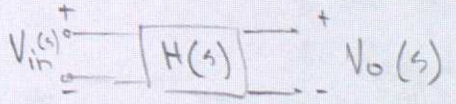
$\omega = 2\pi f$

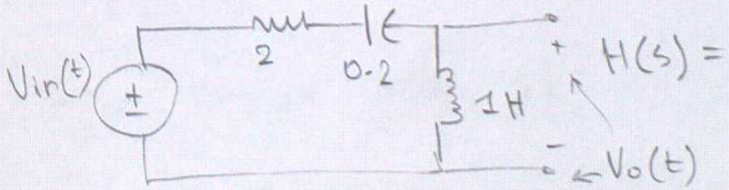
KUL: $s^2 I(s) + s\frac{R}{L} I(s) + \frac{1}{LC} I(s) = \frac{-5000}{s^2 + (500)^2}$ ← $V_s(s)$ after Laplace transform

$$I(s) = \frac{-5000}{s^2 + (500)^2}\left[\frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\right]$$

$$i(t) = 1.32\cos(500t - 58.2)$$

- transfer function



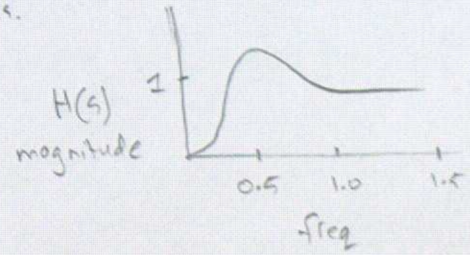


$H(s) = G_v(s) = \frac{V_o(s)}{V_{in}(s)}$

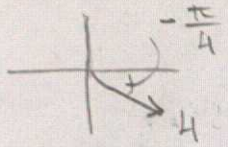
$= \frac{s^2 LC}{s^2 LC + RLC + 1}$

high pass filter (HPF)

kills off lower freq. signals.



Practice $V = 4\angle -\frac{\pi}{4}$



$$4\cdot\frac{\sqrt{2}}{2} - 4\frac{\sqrt{2}}{2}i = \boxed{2\sqrt{2} - 2\sqrt{2}i}$$

$$\boxed{Z = x+jy = |Z|\angle\theta = |Z|\cos\theta + j|Z|\sin\theta}$$

Practice: $Z = -8-6i$,

$$\frac{z\bar{z}}{z+\bar{z}} = \frac{(-8-6i)(-8+6i)}{(-8-6i)+(-8+6i)} = \frac{64 - 48i + 48i - 36i^2}{-16+0i} \qquad i^2 = -1$$

$$= \frac{64+36}{-16} = -\frac{25}{4}$$

$$\boxed{\frac{z\bar{z}}{z+\bar{z}} = -\frac{25}{4}}$$

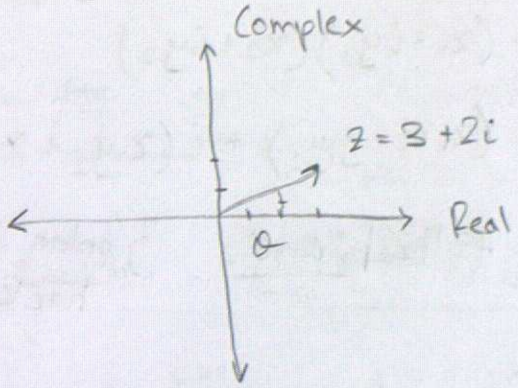
Chapter 17.1 — Complex Numbers

Lecture 2: Complex Numbers Sep 5, 2024

$i = \sqrt{-1}$

Complex Plane

$$Z = x + iy = \underbrace{(x,y)}_{\text{vector in complex plane}}$$



Eg: $Z = 3+2i$

$$\left.\begin{aligned} \mathrm{Re}\{z\} &= x = |z|\cos\theta \\ \mathrm{Im}\{z\} &= y = |z|\sin\theta \end{aligned}\right\} \text{Real numbers}$$

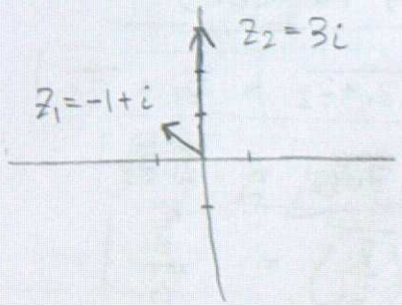
$$Z = x+iy = |z|\angle\theta = |z|\cos\theta + i|z|\sin\theta$$

- i can be thought of as a rotational operator

↳ If I apply i to a number, that number is rotated CCW by 90° ↺

Eg:

$$\left.\begin{aligned} 1i &= i = i(i^4) = i^5 \\ i(i) &= -1 = i^6 \\ i(i^2) &= -i = i^7 \\ i(i^3) &= 1 = i^8 \end{aligned}\right\} \begin{aligned} &\text{all on unit circle} \\ &|z| = 1 \end{aligned}$$



Complex Number Arithmetic

- Addition (subtraction):

$$\begin{aligned} Z_1 + Z_2 &= x_1 + iy_1 + x_2 + iy_2 \\ &= (x_1+x_2) + i(y_1+y_2) \\ &= Z_1 + Z_2 \end{aligned}$$

- Multiplication:

$$\underline{i^2 = -1}$$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

$$z_1 z_2 = |z_1||z_2| \angle \theta_1 + \theta_2 \quad \Big\} \text{ polar form hack}$$

- Division:

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2}$$

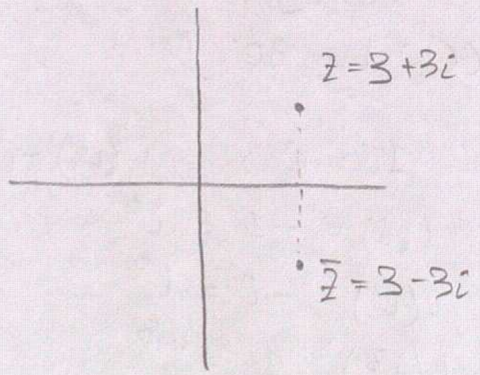
$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_1|} \angle \theta_1 - \theta_2 \quad \Big\} \text{ polar form hack}$$

Complex Conjugate

- reflection across real axis

$$z = x + iy$$

$$\bar{z} = z^* = x - iy$$



5 Properties

1. $\boxed{\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2}$
2. $\boxed{\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2}$
3. $\boxed{\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}}$
4. $\boxed{z\bar{z} = |z|^2}$
5. Triangular identity

$$\boxed{|z_1 + z_2| \leq |z_1| + |z_2|}$$

↑ not related to complex plane

Practice: simplify $z = \frac{i}{1+i}$

$$\frac{\pi}{2} = \frac{2\pi}{4} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$z = \frac{1 \angle \pi/2}{\sqrt{2} \angle \pi/4} = \frac{1}{\sqrt{2}} \angle \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} \angle \pi/4$$

$$\boxed{\therefore z = \frac{1}{\sqrt{2}} \angle \frac{\pi}{4} = \frac{1}{2} + \frac{1}{2} i}$$

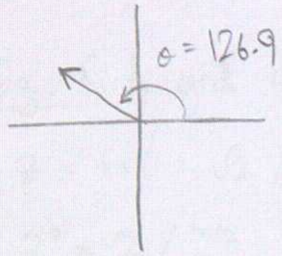
$$z = \frac{i}{1+i} \cdot \frac{1-i}{1-i} = \frac{i - i^2}{1-(-1)} = \frac{i+1}{2} = \frac{1}{2} + \frac{1}{2} i$$

Polar Form : Principal Argument

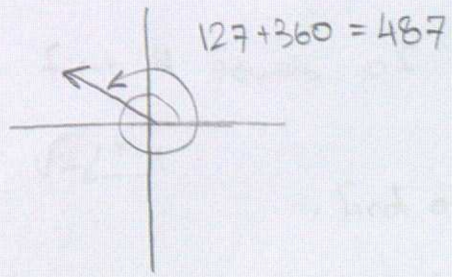
- Rectangular : $z = x + iy$
- Polar : $z = |z| \angle \theta$, $\underbrace{|z| = \sqrt{x^2 + y^2}}_{\text{magnitude/modulus}}$, $\underbrace{\theta = \arctan\left(\frac{y}{x}\right)}_{\text{argument}}$ ← careful

Eg: $z = -3 + 4i$

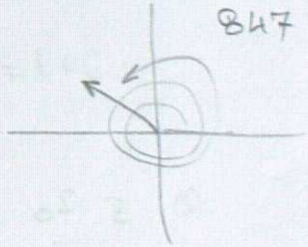
$$|z| = \sqrt{3^2 + 4^2} = 5 \quad , \quad \theta = \arctan\left(-\frac{4}{3}\right) = -53.1 + 180^\circ = 126.9^\circ$$



$z = 5 \angle 127$



$z = 5 \angle 487^\circ$



$z = 5 \angle 847^\circ$

$$\boxed{\text{Principal Argument}: -180^\circ < \text{Arg}(z) = \theta_0 \leq 180^\circ}$$

$$\boxed{\text{General Argument}: \quad \arg(z) = \theta_0 + 2\pi k, \quad k = 0, \pm 1, \pm 2}$$

Week 2 Sep 7, 2024

17.2 Powers and Roots

- for n'th power of z : $\boxed{z^n = r^n(\cos n\theta + i\sin n\theta)}$
- De Moivre's formula : $\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$
- To find n'th roots of $z = r(\cos\theta + i\sin\theta)$

$$\boxed{w_k = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]}$$

where $k = 0, 1, 2, \cdots, n-1$

- every (non-zero) complex number has n complex n-th roots

Week 2: Lec 1 Sep 9, 2024

- Taylor series of e^x is:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

infinit sum of complex vectors

- what happens if $x \Rightarrow i\theta$

$$e^{i\theta} = 1 + i\theta + \underbrace{\frac{i^2\theta^2}{2!}}_{-1} + \underbrace{\frac{i^3\theta^3}{3!}}_{-i} + \underbrace{\frac{i^4\theta^4}{4!}}_{1} + \underbrace{\frac{i^5\theta^5}{5!}}_{i \;\cdots} + \cdots$$

$$= \underbrace{1 - \frac{\theta^2}{2i} + \frac{\theta^4}{4!}}_{\cos\theta} - \cdots + i\underbrace{\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right]}_{\sin\theta}$$

group real and im parts

- Thus we prove Euler's formula $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

Powers of a Complex Number

$$\boxed{z^n = |z|^n \angle n\theta}$$

Eg: find and draw first 4 powers of $z = 1+i$

$z = 1+i = \sqrt{2}\angle 45^\circ = \sqrt{2}\angle \pi/4$

$z^2 = 2\angle \pi/2$

$z^3 = 2\sqrt{2}\angle 3\pi/4$

$z^4 = 4\angle \pi$

- find angle of z θ
- you rotate by θ every power of z

Sep 9, 2024

Practice: given $z = i$, show $z^{12} = 1$

- We know $z = 1\angle\pi/2$, and we know each power rotates z by $\frac{\pi}{2}$, so

$$\frac{\pi}{2}\cdot 12 = 6\pi \Longrightarrow 1e^{6\pi i} = \cos 6\pi + i\sin 6\pi = \boxed{1}$$

Roots of a Complex Number

- For n'th root of z, solve $w^n = z$
- n'th root of z:

① All lie on circle radius $|z|^{1/n} = r^{\frac{1}{n}}$

② All are equally spaced on circle by $\frac{2\pi}{n}$ starting at $\frac{\theta_0}{n}$

$$\boxed{w_k = z^{\frac{1}{n}} = \left(re^{i\theta}\right)^{\frac{1}{n}} = r^{\frac{1}{n}}\angle\frac{\theta_0 + 2\pi k}{n} \qquad k = 0, 1, 2, \dots, n-1}$$

Week 2: lecture 2

Sep 10, 2024

→ Complex Exponential and Logarithmic Functions

Complex Exponential

$z = x + iy$

recall $e^{iy} = \cos y + i\sin y$
$= 1\angle y$

$$e^z = e^{x+iy} = e^x e^{iy} = e^x\angle y$$

$$= e^x(\cos y + i\sin y)$$

$$= \underbrace{e^x\cos y}_{u(x,y)} + i\underbrace{e^x\sin y}_{v(x,y)}$$

- what of magnitude and phase?

$$e^z = e^x\angle y$$

phase

magnitude

- $f(z) = u(x,y) + iv(x,y)$

every complex valued function

real-valued function

$$e^z = |e^z|\angle\theta = |e^z|e^{i\theta}$$

Magnitude: $|e^z| = \sqrt{u^2 + v^2} = |e^x|\overset{1}{|e^{iy}|} = e^x$

$|e^z| \neq 0$ and always positive in the finite complex plane

Phase: $\arg(e^z) = \arg(e^x e^{iy})$
$= \arg(e^x) + \arg(e^{iy})$

only true for general argument

$2\pi k$ (on real axis)

$y + 2\pi k$

$$\boxed{\arg(e^z) = y + 2\pi k, \quad k = 0, \pm 1, \pm 2, \dots}$$

Properties of Complex Exponential

$$\boxed{e^{z} = e^{x} \angle y+2\pi k \quad k=0,\pm1,\pm2,\cdots}$$

1) $\dfrac{de^{z}}{dz} = e^{z}$

2) $|e^{z}| = e^{x}$

3) $\arg(e^{z}) = y+2\pi k, \ (k=0,\pm1,\pm2,\cdots)$

↳ periodic with period $2\pi i$

I **Exercise:** $f(z) = \dfrac{e^{z_1}}{e^{z_2}}$, $\quad z_1 = 2 - i\pi$, $\quad z_2 = -1 - i\dfrac{3\pi}{4}$

$$f(z) = e^{z_1 - z_2} = e^{\left(3 + i\frac{7\pi}{4}\right)} = e^{3} \angle \tfrac{7\pi}{4}$$

$$z_1 - z_2 = (2 - i\pi) - \left(-1 - i\tfrac{3\pi}{4}\right)$$

$$= 2 - i\pi + 1 + i\tfrac{3\pi}{4}$$

$$\tfrac{3\pi}{4} - \tfrac{4\pi}{4} = -\tfrac{\pi}{4}$$

$$= 3 + i\left(\tfrac{3\pi}{4} - \pi\right) = 3 + i\tfrac{7\pi}{4}$$

$$f(z) = e^{3}\left(\cos\tfrac{7\pi}{4} + i\sin\tfrac{7\pi}{4}\right)$$

$$= e^{3}\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right) = \boxed{\frac{e^{3}}{\sqrt{2}} - i\frac{e^{3}}{\sqrt{2}}}$$

The Complex Logarithm

- Real numbers: $\quad w = \ln x \Longrightarrow x = e^{w}$

  $\quad w = \log_{10} x \Longrightarrow x = 10^{w}$

↓

Complex numbers:

$w = \ln z \Longrightarrow z = e^{w}$

- To find w let $z = x + iy = re^{i\theta}$

$$\underline{r}\underline{\underline{e^{i\theta}}} = e^{w} \longrightarrow \text{Let } w(x,y) = u(x,y) + iv(x,y) = \ln(z)$$

$$= \underline{e^{u}}\,\underline{\underline{e^{iv}}}$$

$$\therefore r = e^{u} \longrightarrow \text{Therefore } \overbrace{u(x,y) = \ln(r) = \log_e|z|}^{\text{real valued natural log}}$$

$$e^{i\theta} = e^{iv} \longrightarrow v(x,y) = \theta = \text{Arg}(z) + 2\pi n \quad (n=0,\pm1,\pm2,\cdots)$$

$$\boxed{\therefore \ln(z) = w(x,y) = \log_e|z| + i\,\underbrace{[\text{Arg}(z) + 2\pi n]}_{\arg(z)}}$$

imaginary part can give many answers

multivalued function for a single value of z

- set specific "n" value

- principal branch of $\ln(z)$: $n=0$

$$\boxed{\text{Ln}(z) = \log_e|z| + i\,\text{Arg}(z)}$$

Recall

$x^{n} = e^{n\ln x}$

$\therefore \boxed{z^{\alpha} = e^{\alpha \ln z}}$

Week 2: Lec 2 Sep 10, 2024

Properties of Complex Logarithm

1) $e^{\ln z} = z \quad (z \neq 0)$

2) $\ln(z_1 z_2) = \ln(z_1) + \ln(z_2)$

3) $\ln\left(\frac{z_1}{z_2}\right) = \ln(z_1) - \ln(z_2)$

Note: these two don't necessarily hold for $\mathrm{Ln}(z)$

I **Eg:** $\mathrm{Ln}(3) = \log_e(3) + i\overset{0}{\mathrm{Arg}(3)} = \ln(3) = 1.0986\ldots$

II **Eg:** $\mathrm{Ln}(-3) = \log_e(3) + i\underbrace{\mathrm{Arg}(-3)}_{\pi} = 1.0986\ldots + i\pi$

III **Eg:** 2^π

$2^\pi = e^{\pi \ln 2} = e^{\pi(\log_e(2) + i2\pi n)}$ $n = 0, \pm 1, \pm 2, \ldots$

$\ln(2) = \log_e(2) + i[\underset{0}{\mathrm{Arg}(2)} + 2\pi^2 n] = \log_e(2) + i2\pi n$

$\therefore 2^\pi = e^{\pi \log_e(2)}$

Choose principal branch, $n=0$

IV **Eg:** i^i

$i^i = e^{i\ln(i)} = e^{i(e + \frac{\pi}{2}i)} = e^{0 + \frac{\pi}{2}i^2} = e^{-\frac{\pi}{2}}$

$\ln(i) = \cancel{\ln(1)}^{0} + i[\mathrm{Arg}(i) + 2\pi n]$

$= 0 + i[\pi/2 + 2\pi n]$, $n = 0, \pm 1, \pm 2, \ldots$

$\therefore i^i = e^{-\frac{\pi}{2}}$

pick principal branch: $\ln(i) = 0 + \frac{\pi}{2}i$

Week 2: Lec 2 Sep 10, 2024

II **Excersize:** Find all values of z: $e^{z-1} = -ie^2$

$e^{z-1} = -ie^2$

$\ln(e^{z-1}) = \ln(-ie^2)$

$z - 1 = \ln(-ie^2)$

$z = 1 + \ln(-ie^2)$

$\therefore z = 3 + i(-\frac{\pi}{2} + 2\pi n)$ for $n = 0, \pm 1, \pm 2, \ldots$

$\ln(-ie^2) = \log_e(|-ie^2|) + i[\mathrm{Arg}(-ie^2) + 2\pi n]$ $(n = 0, \pm 1, \ldots)$

$= \log_e(e^2) + i[-\frac{\pi}{2} + 2\pi n]$

$= 2 + i(-\frac{\pi}{2} + 2\pi n)$, $n = 0, \pm 1, \pm 2, \ldots$

Notes from H.W.

- Power rule for logs doesn't work for complex numbers
- $(z^\alpha)^n = z^{n\alpha}$ doesn't hold if n is complex

Week 2: Lecture 3

Sep 12, 2024

→ Complex Trig and Hyperbolic Functions

$$e^{i\theta} = \cos\theta + i\sin\theta = \exp(i\theta) = 1\angle\theta$$

$\underline{\theta = \pi}$

$$e^{i\pi} = \cos\pi + i\sin\pi$$
$$= -1 + i(0) = -1$$

$\boxed{e^{i\pi} + 1 = 0}$ ← 5 fundamental constants $0, +1, i, e, \pi$

Euler's Identity

Starting with $e^{i\pi} = \cos\theta + i\sin\theta$ ①

and $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$ ②

① + ② $= 2\cos\theta = e^{i\theta} + e^{-i\theta}$

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

θ is real

① − ② : $2i\sin\theta = e^{i\theta} - e^{-i\theta}$

$$\boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

Sep 12, 2024

I Ex: Prove $\cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$$\cos A\cos B = \left(\frac{e^{iA} + e^{-iA}}{2}\right)\left(\frac{e^{iB} + e^{-iB}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{e^{i(A+B)} + e^{i(A-B)} + e^{-i(A-B)} + e^{-i(A+B)}}{2}\right]$$

$$= \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

Complex Hyperbolic Func.

$$\boxed{\cosh(z) = \frac{e^{z} + e^{-z}}{2}} \qquad \boxed{\sinh(z) = \frac{e^{z} - e^{-z}}{2}}$$

Properties: $z = x + iy$

$$\sinh(z) = \sinh(x)\cos(y) + i\left[\cosh(x)\sin(y)\right]$$

$$\cosh(z) = \cosh(x)\sin(y) + i\left[\sinh(x)\sin(y)\right]$$

$$\boxed{\sinh(z) = -i\sin(iz)} \qquad \boxed{\cosh(z) = \cos(iz)}$$

Complex Trig Func.

From $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$, Let $\theta = z = x+iy$

$$\cos(z) = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}$$

see details

$$= \frac{e^{ix}e^{-y} + e^{-ix}e^{+y}}{2} \dots$$

$$= \left(\frac{e^{ix}+e^{-ix}}{2}\right)\left(\frac{e^{-y}+e^{+y}}{2}\right) - i\left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{y}-e^{-y}}{2}\right)$$

$$\boxed{\cos(z) = \cos x \cosh y - i \sin x \sinh y}$$

$$\boxed{\sin(z) = \sin x \cosh y + i \cos x \sinh y}$$

unbounded

$|\sin(z)| \longrightarrow \pm\infty$

$|\cos(z)| \longrightarrow \pm\infty$

Exersize: find z for $\cos z = \sin z$

Ordinary Differential Equations

Week 3: Lecture 1

Sep 16, 2024

- Know ODEs, PDEs, order, and linearity

Ex: $y' = 6x(y-1)^{2/3}$, $y(x) = 1+(x^2+c)^3$

$$y' = 3(x^2+c)^2(2x) = 6x(x^2+c)^2$$

Week 3 Tutorial

14. $\left(\frac{1}{2}-\frac{1}{4}i\right)\left(\frac{2}{3}+\frac{5}{3}i\right) = \frac{1}{2}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{5}{3}i - \frac{1}{4}i\cdot\frac{2}{3} - \frac{1}{4}\cdot\frac{5}{3}i^2$

$$= \frac{1}{3} + \frac{5}{6}i - \frac{1}{6}i + \frac{5}{12} = \frac{4}{12} + \frac{5}{12} + \frac{4}{6}i = \boxed{\frac{9}{12} + \frac{2}{3}i}$$

30. $\frac{1}{(1+i)(1-2i)(1+3i)} = \frac{1}{(1+3i)(1-2i+i-2i^2)} = \frac{1}{(1+3i)(3-i)}$

$$= \frac{1}{3-i+9i-3i^2} = \frac{1}{6+8i}\cdot\frac{6-8i}{6-8i} = \frac{6-8i}{36+64} = \boxed{\frac{6}{100} - \frac{8}{100}i}$$

20. $\frac{\sqrt{2}+\sqrt{6}i}{-1+\sqrt{3}i}$ $\quad r = \sqrt{8},\ \theta = \pi/3$ $\quad r = 2,\ \theta = 2\pi/3$ $\quad \frac{2\sqrt{2}\angle \pi/3}{2\angle 2\pi/3}$

$$\hookrightarrow \sqrt{2}\angle\left(\frac{\pi}{3}-\frac{2\pi}{3}\right) = \sqrt{2}\angle -\frac{\pi}{3} = \sqrt{2}\angle 5\pi/3$$

$$\boxed{= \sqrt{2}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)}$$

26. $\left[\sqrt{3}\left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right]^6 = 3^3\left(\cos\frac{12\pi}{9} + i\sin\frac{12\pi}{9}\right)\right.$

$$= 27\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = \boxed{-\frac{27}{2} - \frac{27\sqrt{3}}{2}i}$$

30. $(-1+i)^{1/3} \longrightarrow r=\sqrt{2},\ \theta=\frac{3\pi}{4}$

$$w_0 = 2^{1/6}\left[\cos\frac{3\pi}{4\cdot 3} + i\sin\frac{3\pi}{4\cdot 3}\right]$$

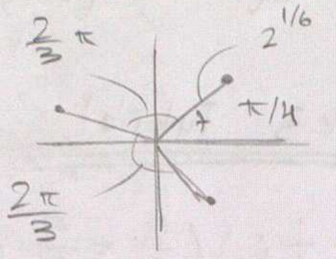
$$= 2^{1/6}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$w_1 = 2^{1/6}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$$

$$w_2 = 2^{1/6}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$$

$$\frac{\frac{3\pi}{4}+2\pi k}{3}$$

$$\frac{\pi}{4}+\frac{2}{3}\pi k$$



6. $e^z,\ z=-\pi+\frac{3\pi}{2}i$

$$e^z = e^{-\pi}e^{\frac{3\pi}{2}i} = e^{-\pi}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = \boxed{-e^{-\pi}i}$$

12. $$\frac{e^{2+3\pi i}}{e^{-3+\frac{\pi}{2}i}} = e^{2+3\pi i+3-\frac{\pi}{2}i} = e^{5+\frac{5\pi}{2}} = e^5e^{\frac{5\pi}{2}}$$

$$= e^5\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = \boxed{e^5 i}$$

$$3\pi-\frac{\pi}{2}$$

$$\frac{6\pi}{2}-\frac{\pi}{2}=\frac{5\pi}{2}$$

24. $\ln(-ei) = \log_e(e) + i\left(-\frac{\pi}{2}+2\pi n\right)$

$$= \boxed{1 + i\left(-\frac{\pi}{2}+2\pi n\right) \quad n=\pm1,\pm2,0,\pm3,\cdots}$$

32. $\mathrm{Ln}(3-4i) = \log_e(5) + i\left(\arctan\left(-\frac{4}{3}\right)\right)$

$$\boxed{= 1.6094 - 0.9273i}$$

44 $(1-i)^{2i} = e^{2i(\ln(1-i))} = e^{2i\log_e\sqrt{2}}e^{i\frac{\pi}{4}} = e$

$\ln(1-i) = \log_e(\sqrt{2}) + i\left(\frac{\pi}{4}+2\pi n\right) \Rightarrow \log_e\sqrt{2} + i\frac{\pi}{4}$

Week 3: Lecture 2 Sep 17, 2024

General and Particular Solutions

$y(x) = 1+(x^2+c)^3$ is sol.ⁿ to ODE $y' = 6x(y-1)^{3/2}$

separable: $$\int\frac{dy}{\underbrace{(y-1)^{3/2}}} = \int 6x\,dx$$

Valid solution $y(x)=1$ is not in this set

$$\underbrace{y(x)=1}_{\text{called a "singular solution"}} \rightarrow y'=0 \overset{\checkmark}{=} y' = 6x(y-1)^{2/3} = 6x(1-1)^{2/3} = 0$$

Linear 1st Order IVP and Integrating Factor

$xy'-y=2x^2$ and $y(x_0)=y_0 \Rightarrow y'-\frac{1}{x}y = 2x$

- standard form: $y'+Py = f(x)$
- integrating factor: $f(x) = e^{\int P(x)\,dx}$

$$\boxed{y(x) = \frac{1}{e^{\int P dx}}\left[\int e^{\int P dx} f(x) + C\right]}$$

$$f(x) = e^{-\int\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$-\int\frac{1}{x}dx = -\ln x$$

$$\frac{1}{x}\left(y'-\frac{1}{x}y\right) = \frac{1}{x}(2x)$$

$$\underbrace{\left(\frac{1}{x}y'-\frac{1}{x^2}y\right)}_{} = 2$$

$$\left(\frac{1}{x}y\right)' = 2$$

$$\int\left(\frac{1}{x}y\right)'dx = \int 2\,dx$$

$$\frac{1}{x}y = 2x+C$$

$$\underline{y = 2x^2+Cx}$$

$y(x_0)=y_0$

$y_0 = 2(x_0)^2 + Cx_0$

$C = \frac{y_0-2x_0^2}{x_0}$

$$\boxed{\therefore y = 2x^2 + x\left(\frac{y_0-2x_0^2}{x_0}\right)}$$

$y(1)=1$ ← Yes sol.

$y(0)=0$ ← ∞ sol.

$y(0)=2$ ← no sol.

Linear 1st Order IVP: Existence and Uniqueness

1. Existence: does IVP have a sol.?
2. Uniqueness: if yes, does IVP have only one sol.?

Theorem 1.2.1: Unique Sol.

- given $y' = f(x,y)$ and $y(x_0) = y_0$
- if $\boxed{f(x,y)}$ and $\boxed{\frac{\partial f}{\partial y}}$ are cont. then unique sol. exists
  - ↳ unique solution $y(x)$

sufficient but not necessary

Linear IVP

- continuity of $P(x)$ and $f(x)$ guarantees unique sol. $y(x)$

  $y' + P(x)y = f(x)$ and $y(x_0) = y_0$

1.2.1 $y' = f(x,y) \longrightarrow$ continuity of $f(x,y)$ and $\frac{\partial f}{\partial y}$

1.2.1' $y' + P(x)y = f(x) \Longrightarrow y' = f(x,y) = \underbrace{-P(x)y + f(x)}_{f(x,y)}$

- Continuity of $f(x,y)$ depends on $P(x)$ and $f(x)$
- $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[-P(x)y + f(x)\right] = -P(x) + \overset{0}{\cancel{\frac{\partial f(x)}{\partial y}}} = -P(x)$

continuity of $\frac{\partial f}{\partial y}$ follows from continuity of $P(x)$

Ex: $y' = 3y^{2/3}$ and $y(2) = 0$ Q: Am I guaranteed a unique sol. to IVP?

$y' = \underbrace{3y^{2/3}}_{f(x,y)}$ think about 2 things

1) $f(x,y) = 3y^{2/3} = 3\sqrt[3]{y^2}$
- this is always continuous even though it's >0
- nowhere does it blow up

2) $\frac{\partial f}{\partial y} = 3\left(\frac{2}{3}\right)y^{-1/3} = 2y^{-1/3} = \frac{2}{\sqrt[3]{y}}$
- this is a problem since func. invalid at $y = 0$

∴ We are not guaranteed a unique solution since argument 2 fails → there can still be unique sols. but I'm not guaranteed.

Linear 1st Order IVP: Existence and Uniqueness

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∴ We are not guaranteed a unique solution since argument 2 fails → these can still be unique sols. but I'm not guaranteed.

Week 3: Lecture 3

Sep 19, 2024

- normal form of 1st order ODE is:

$y' = f(x,y) \longleftarrow$ slope of $y(x)$ as both x and y vary

Autonomous DEs

- indep. var. doesn't appear explicitly

Ex: general population eq: $\frac{dP}{dt} = (\beta - \alpha)P$

$$\frac{dP}{dt} = \overbrace{(\beta - \alpha)P}^{f(P)} \longleftarrow [\text{no } t]$$

birth rate

death rate

Ex: velocity of falling object

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \underbrace{g - \frac{k}{m}v}_{f(v)} \quad [\text{no } t]$$

const.

mass

- given $x' = f(x)$, critical points when $f(x) = 0$

since $x' = f(x) = \text{slope} = 0$ then $x = c$ is a sol^n

$\hookleftarrow$ equilibrium sol^ns

Ex: critical pnts for $x' = x(4-x)$ are:

$x = 0 \quad x = 4 \longleftarrow$ equilibrium sol^ns

Stability of Critical Points

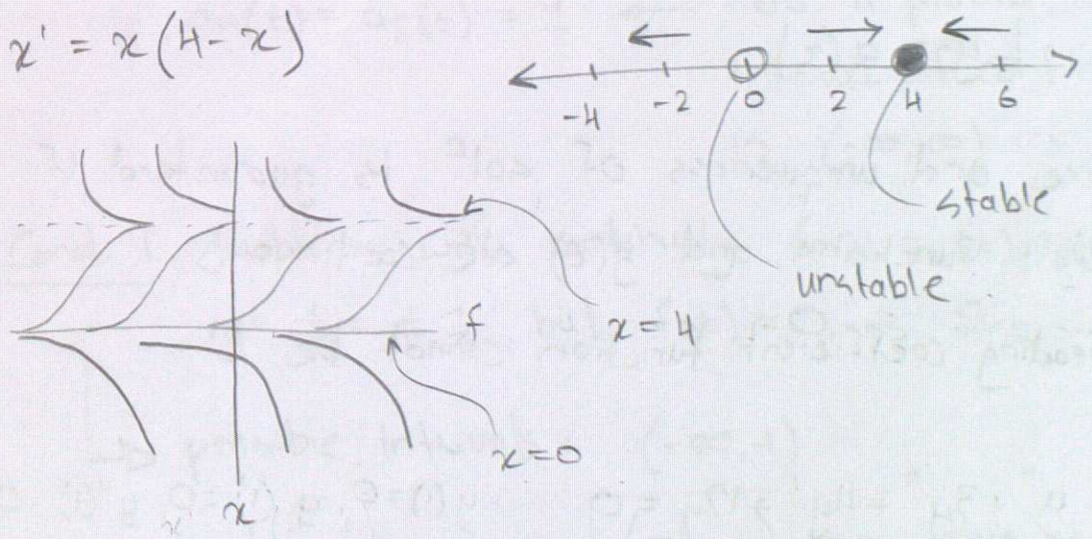
$$y' = \frac{y(y-1500)}{3200} \qquad y' = \frac{y(y-1500)}{3200} - 150$$

$y = 0$ or $y = 1500$ $\qquad$ $y = 0$ or $y = 450$ or $y = 1000$

- controlling pop. can affect the critical points as well as their types
- types: stable, semistable, unstable

Phase Portrait Diagrams

$x' = x(4-x)$



Theory of Differential Equations

1) Separable DEs $\boxed{\frac{dy}{dx} = g(x)h(y)}$ seperable methods

2) Linear 1st DEs $\boxed{\frac{dy}{dx} + P(x)y = f(x)}$ integrating factors method

- homogerous, nonhomogerous
- constant coeffients, variable coefficients
- linear differential operator L : a placeholder for the coefficients and the derivative operator s.t.

  $$Ly = g(x)$$

- existence and uniqueness of sol^n is guaranteed if coefficient functions and $g(x)$ are continuous

  ↳ leading coefficient function cannot be 0

Ex: $y''' + 3y'' + 4y' + 12y = 0$ $\quad y(1) = 0,\ y'(1) = 0,\ y''(1) = 0$

- there is a unique solution since all coefficients and $g(x)$ are continuous, and $a_n(x) \neq 0$
- solution is trivial solution : $y(x) = 0$

Week 4: Lecture 1

Sep 23, 2024

Ex: $(1-t^2)y'' - 2ty' + 2y = 0$ $\quad y(0) = 1 \quad y'(0) = 2$

Standard form : $\overbrace{y''}^{a_2(t)} - \overbrace{\frac{2t}{1-t^2}}^{a_1(t)} y'' + \overbrace{\frac{2}{1-t^2}}^{a_0(t)} y = \overbrace{0}^{f(t)}$

Q: what is largest interval over which we have a unique solution? ⟶ $\boxed{\therefore \text{ largest interval is } (-1, 1)}$

2 sufficient conditions:

1) continuity of $f(t)$ and $a(t)$'s
2) $a_n(t) \neq 0$ on interval of interest

Cond. 2: $a_n(t) \neq 0$, so let's check our case:

$a_n(t) = a_2(t) = 1$ ← never a problem so interval of continuity is $(-\infty, \infty)$

Cond. 1: Problems with continuity for $a_1(t), a_0(t)$ at $t = \pm 1$, but $f(t) = 0$ is fine

↳ possible intervals : $(-\infty, -1)$

$\boxed{(-1, 1)}$ ← this is the one that matters since $\left.\begin{array}{l} y(0) = 1 \\ y'(0) = 2 \end{array}\right\}$ we plug in $\underline{\underline{t = 0}}$

$(1, \infty)$

if we had $y(2) = 4$ $y'(2) = 1$, then we pick this interval

- fundamental set of solutions: for n'th order DE, we have n lin. indep. sol^{n}s

  ↖ basis for set is $\{y_1(x), y_2(x), \cdots, y_n(x)\}$

Linearly Independent Functions

- can I express any of the functions in terms of the others? } → Yes : dep → No : indep

- if sol. is only the trivial sol., then lin. indep.

Ex: functions $\{1, x, x^2\}$ are linearly independent since, for example, I can't get 1 and x to become x^2.

Ex: set of func. $\{1-x, 1+x, 1-3x\}$

Q: are they linearly indep.?

Sol^n: find sol to $c_1(1-x) + c_2(1+x) + c_3(1-3x) = 0$

$$\left.\begin{aligned} &\text{consts terms: } c_1+c_2+c_3=0 \quad (I) \\ &x \text{ terms: } -1c_1+c_2-3c_3=0 \quad (II) \end{aligned}\right\} \text{solve}$$

$$\underbrace{c_1=1 \quad c_2=1 \quad c_3=-2}_{\text{these non-zero } c\text{'s get a valid sol.}}$$

$\boxed{\text{Linearly Dependent}}$

Ex: set of functions $\{e^{m_1x}, e^{m_2x}\}$

Q: Are they linearly indep?

Sol^n: (I) $c_1e^{m_1x} + c_2e^{m_2x} = 0$ → differentiate

(II) $c_1m_1e^{m_1x} + c_2m_2e^{m_2x} = 0$ → put into matrix

$$\begin{bmatrix} e^{m_1x} & e^{m_2x} \\ m_1e^{m_1x} & m_2e^{m_2x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- take determinant, so if det is nonzero, then we have line. indep.

$$\det \rightarrow m_2e^{(m_1+m_2)x} - m_1e^{(m_1+m_2)x} = 0$$

← $\det(A)=0$ when $m_1=m_2$, so if $m_1 \neq m_2$ then lin. indep.

- As long as $m_2 \neq m_1$, then $\det(A) \neq 0$, so then we'd have e^{mx} and e^{m_2x} are lin. indep.

Wronskian

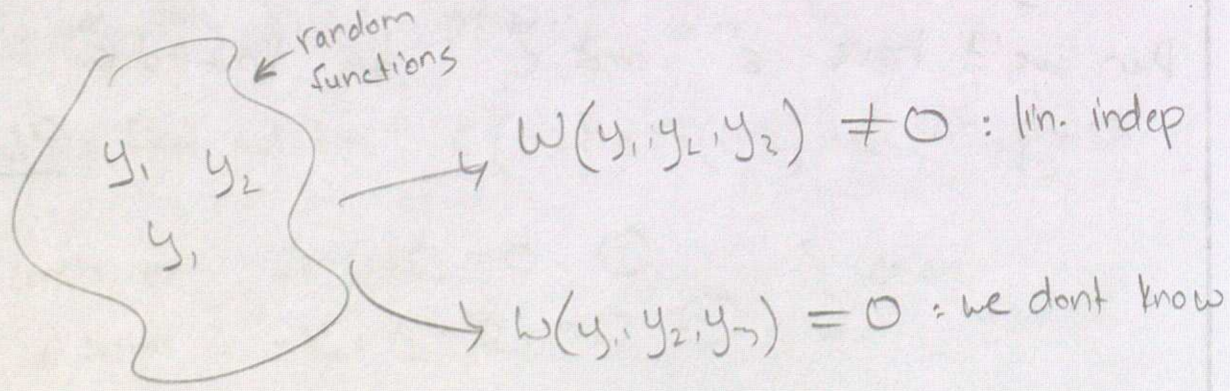
- the second way to see if functions are lin. indep.

$$W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f'_1 & f'_2 & \cdots & f'_n \\ f''_1 & f''_2 & \cdots & f''_n \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}
\begin{array}{l} \leftarrow \text{functions} \\ \leftarrow 1^{st} \text{ derivatives} \\ \leftarrow 2^{nd} \text{ derivative} \\ \\ \\ \end{array}$$

- If W is non-zero, so function lin. indep.
- If W is zero, then functions lin. dep.

Solutions

- linearly dependent solutions, then $W(y_1, y_2, \dots, y_n) = 0$



sol^{n}s to DE

$$\boxed{y_1 \;\; y_2 \;\; y_3} \begin{array}{l} \longrightarrow W(y_1 y_2 y_3) \neq 0 \rightarrow \text{lin. indep.} \\ \longrightarrow W(y_1, y_2, y_3) = 0 \rightarrow \text{lin. dep.} \end{array}$$

BATU: $y''' - y' = 0$, is set $\{e^x, e^{-x}, \cosh x\}$ the fundamental set of solutions

1) they are all sol^{n}s to $y''' - y' = 0$

2) they are lin. indep.

① $C_1 e^x + C_2 e^{-x} + C_3 \cosh x = 0$

$$C_1 e^x + C_2 e^{-x} + C_3 \left[\frac{e^x + e^{-x}}{2} \right] = 0$$

(II) $C_1 e^x + C_2(-e^{-x}) + C_3 \sinh x = 0$

(III) $C_1 e^x + C_2 e^{-x} + C_3 \cosh x = 0$

$$W = \begin{vmatrix} e^x & e^{-x} & \cosh x \\ e^x & -e^{-x} & \sinh x \\ e^x & e^{-x} & \cosh x \end{vmatrix} = \begin{array}{l} e^x(-e^{-x}\cosh x - \sinh x e^{-x}) \\ - e^{-x}(e^x \cosh x - \sinh x e^x) \\ + \cosh x(\end{array}$$

Tutorial 2

Sep 23, 2024

17.7.15

* I did not write the sin z correctly

$\sin z = 2$

$\frac{e^{iz} - e^{-zi}}{2i} = 2$

$e^{zi} - e^{-zi} = 4i \rightarrow e^{zi}(e^{zi} - e^{-zi}) = 4e^{z} i$

$e^{2zi} - 1 = 4e^{zi} i$

$e^{2zi} - 4ie^{z} - 1 = 0$

Let $a = e^{z}$

$a^2 - 4a - 1 = 0$

$$a = \frac{-(-4i) \pm \sqrt{(-4i)^2 - 4(1)(-1)}}{2} = \frac{4i \pm \sqrt{16+4}}{2} = \frac{4 \pm \sqrt{20}}{2} = 2 \pm \frac{\sqrt{20}}{2}$$

$$e^{iz} = 2 \pm \frac{\sqrt{20}}{2} \leftarrow r = \sqrt{\left(2 \pm \frac{\sqrt{20}}{2}\right)^2 + (0)^2} = 2 \pm \frac{\sqrt{20}}{2}$$

$\theta = 0$

$$iz = \ln\left(2 \pm \frac{\sqrt{20}}{2}\right)$$

1.2.25

$$y' = \sqrt{y^2 - 9} = \sqrt{(y+3)(y-3)} = f(x,y)$$

Q: On passing point (1,4), do we guarantee unique solution?

$y^2 - 9 \geqslant 0 \implies (y+3)(y-3) \geqslant 0$

$\sqrt{y^2} = |y|$

$y^2 \geqslant 9$

$y + 3 \geqslant 0 \quad y - 3 \geqslant 0$

$y \geqslant -3 \quad y \geqslant 3$

$\sqrt{y^2} \geqslant 3$

$|y| \geqslant 3 \longrightarrow -3 \leq y \leq 3$

$\boxed{\therefore (1,4) \rightarrow \text{can guarantee solution}}$

Week 4: Lecture 1

Sep 24

Reduction of Order

$y'' + 4y' + 4y = 0, \quad y_1(x) = e^{-2x}$

$y_2(x) = u(x) y_1(x) = u(x) e^{-2x}$

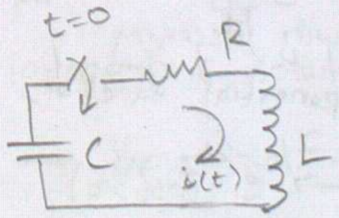
$\left.\begin{array}{l} \hookrightarrow y_2'(x) \\ \hookrightarrow y_2''(x) \end{array}\right\} \quad u''(x) = 0, \quad u'(x) = C_1, \quad u(x) = C_1 x + C_2$

$\therefore y(x) = C_3 y_1(x) + C_4 y_2(x) = e^{-2x}(C_3 + xC_4)$

---

1. Put into standard form $y'' + P(x)y' + Q(x)y = 0$
2. See if $P(x)$ and/or $Q(x)$ are zero: see pg. 124 ← sol^n may be on integral-defined function
3. Use known solution $y_1(x)$ to get $y_2(x) = u(x) y_1(x)$
4. Solve it: second sol^n $\boxed{y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx}$
5. The general sol^n: $y(x) = C_1 y_1(x) + C_2 y_2(x)$

Ex: $R = 700\,\Omega$, $L = 100\,mH$, $C = 1\mu F$, $V_C(0-) = 100\,V$



Observe: $i'' + \frac{R}{L} i' + \frac{1}{LC} i = 0$

$i(0-) = i(0+) = 0\,[A]$

$i'(0-) = i'(0+) = \frac{V_C(0-)}{L} = 1000\ A/s$

We know $i_1(t) = e^{-2000t}$ is a sol^n

$$\text{Sol}^n:\ i_2(t) = i_1(t) \int \frac{e^{-\int P(t)dt}}{i_1^2(t)} = e^{-2000t} \int \frac{e^{-\int 7000 dt}}{(e^{-2000t})^2} dt = \frac{e^{-5000t}}{-3000}$$

The Auxillary (characteristic) Equations

$ay''' + by'' + cy' + dy = 0$, sol^n $y(x) = e^{mx}$

$y'(x) = me^{mx}$, $y''(x) = m^2e^{mx}$, $y'''(x) = m^3e^{mx}$

plug back in : $a(m^3\cancel{e^{mx}}) + b(m^2\cancel{e^{mx}}) + c(m\cancel{e^{mx}}) + d(\cancel{e^{mx}}) = 0$

$$\boxed{am^3 + bm^2 + cm + d = 0}$$

we factor this: $(m-m_1)(m-m_2)(m-m_3) = 0$

this means $e^{m_1x}, e^{m_2x}, e^{m_3x}$ are sol^n to this ODE !!!

↳ they are lin. indep. if $m_1 \neq m_2 \neq m_3$

$$\boxed{\therefore \text{Fundamental Set: } \{e^{m_1x}, e^{m_2x}, e^{m_3x}\}}$$

3 types of roots

1) Distinct Real Roots : e^{-2t}, e^{3x} ... growing / decaying exponential functions

2) Distinct Complex Roots : $e^{(2+3i)t}, e^{(2-3i)t} \rightarrow e^{-2t}\cos(3x)$

3) Repeated Roots : $e^{-2x}, xe^{-2x}, x^2e^{-2x}$

- General Sol^n can combine ALL of them

Ex: $y'' - y' - 2y = 0$

$m^2 - m - 2 = 0$ $\quad \therefore m_1 = 2 \quad m_2 = -1$

$(m-2)(m+1) = 0$ $\quad y(x) = C_1e^{2x} + C_2e^{-x}$

60-Second Review : Higher Order ODE's

$$a_n(x)y^{(n)} + \dots + a_2(x)y' + a_1(x)y = g(x)$$

$$L(y) = g(x)$$

a) ODE homogenous if $g(x) = 0$

b) ODE has constant coefficients if all $a_k(x)$'s constants

c) IVP based on ODE has unique sol^n if all the $a_k(x)$'s and $g(x)$ are continuous on an interval surrounding the point of interest

d) An n^{th} order homogenous ODE will have n linearly independent solutions

→ these sol^{n}s are lin. indep. IF their Wronskian is nonzero at least one point the sol^n int.

Week 4: Lecture 2

Sep 26, 2024

Solutions to Characteristic Equations

1) **Distinct Real Roots**

- linear comb. of growing or decaying exponentials

$$\boxed{y = C_1 e^{m_1 t} + C_2 e^{m_2 t}}$$

2) **Distinct Complex Roots** : oscillation with growing and decaying exponential

- $$\left.\begin{aligned} m_k &= \alpha_k + i\beta_k \\ m_{k+1} &= \alpha_{k+1} + i\beta_{k-1} \end{aligned}\right\}$$ 2 roots, complex roots come in conjugate pairs

$m_k, m_{k+1} = \alpha_k + i\beta_k$ so $(\beta_{k+1} = -\beta_k)$

$e^{m_u x}$ and $e^{m_{k+1} x}$

$e^{\alpha_k} \underbrace{e^{i\beta_k x}}_{\cos\beta_k x + i\sin\beta_k x}$ and $e^{\alpha_{k+1} x} e^{i\beta_{k+1} x} = e^{\alpha_k x} \overbrace{e^{-\beta_k x}}^{\cos\beta_k x - i\sin\beta_k x}$

our general

One part of solution becomes

$$\boxed{C_k e^{\alpha_k x}\left[\cos\beta_k x + i\sin\beta_k x\right] + C_{k+1} e^{\alpha_{k+1} x}\left[\cos\beta_{k+1} x - i\sin\beta_{k+1} x\right]}$$

$$\downarrow$$

$$\boxed{C_k' e^{\alpha_k x} \cos\beta_k x + C_{k+1}' e^{\alpha_k x} \sin\beta_k x}$$

Ex: When will we have oscillation? $i'' + \frac{R}{L} i' + \frac{1}{LC} i = 0$

Let $i = e^{mt}$, $i' = me^{mt}$, $i'' = m^2 e^{mt}$

$m^2 + \frac{R}{L} m + \frac{1}{LC} = 0$ $\quad m = \frac{-\left(\frac{R}{L}\right) \pm \sqrt{\left(\frac{R}{L}\right)^2 - 4(1)\left(\frac{1}{LC}\right)}}{2}$

$\left(\frac{R}{L}\right)^2 - \frac{4}{LC} < 0 \implies \frac{R^2}{L^2} < \frac{4}{LC} \implies \frac{R^2}{4L^2} < \frac{1}{LC}$

$$\boxed{\frac{R}{2L} < \frac{1}{\sqrt{LC}}}$$

3) **Repeated Roots** : $(m - m_1)(m - m_2)^k = 0$ $\quad k = n-1$ (a bunch of repeated roots)

if the root m_2 has multiplicity k, then

$$\boxed{(C_1 + C_2 x + C_3 x^2 + \cdots + C_k x^{k-1}) e^{mx}}$$

Ex: $y'' + y' + \frac{1}{4} y = 0$, $y(0) = 3$, $y'(0) = -3.5$

$m^2 + m + \frac{1}{4} = 0$ $\quad m = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1/4)}}{2} = -\frac{1}{2}$

$y = C_1 e^{-\frac{x}{2}} + C_2 x e^{-\frac{x}{2}}$, $y' = -\frac{C_1}{2} e^{-\frac{x}{2}} + C_2 e^{-\frac{x}{2}} - \frac{C_2 x}{2} e^{-\frac{x}{2}}$

$3 = C_1$ $\quad -3.5 = -\frac{3}{2} + C_2 \implies C_2 = -\frac{7}{2} + \frac{3}{2} = \frac{-4}{2} = -2$

$-2 = C_2$

$$\boxed{\therefore y = 3e^{-\frac{x}{2}} - 2xe^{-\frac{x}{2}}}$$

Section 3.5

Variation of Parameters

- nonhomogenous linear ODEs $L(y) = \overset{g(x) \neq 0}{g(x)}$
- consider general sol^{n} y and particular sol^{n} y_P.

$$\underbrace{L(y-y_P) = L(y) - L(y_P)}_{\text{linearity}} = g(x) - g(x) = 0$$

$$\underbrace{L(y-y_P)}_{\text{unknown fun}^c \text{ or sol}^n} = 0 \longrightarrow y - y_P \text{ is sol}^n \text{ to the associated homogenous version of our ODE}$$

$$y_c(x) = y(x) - y_P(x) \longrightarrow \boxed{y(x) = y_c(x) + y_P(x)} \leftarrow \text{finding } y_P(x)$$

↑ complimentary sol^{n} (sol^{n} to $L(y)=0$)

variation of parameters

Particular Solution Notes

$$y_P(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2 \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1 & f(x) \end{vmatrix}$$

↑ finding u_1' ↑ finging u_2'

Week 5: Lecture 1 Sep 30, 2024

For y_P in $L(y) = y'' + P(x)y' + Q(x)y = f(x)$

1) Determine 2 lin. indep. sol^{n}s y_1 and y_2 of $L(y) = 0$

2) Solve for $u_1'(x)$ and $u_2'(x)$ from

$$\boxed{\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= f(x) \end{aligned}}$$, which gives

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2 f}{W(y_1 y_2)}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 f}{W(y_1 y_2)}$$

$$W(y_1 y_2) = y_1 y_2' - y_1' y_2$$

3) Integrate results from 2) to get $u_1(x)$ and $u_2(x)$

$$u_1(x) = \int u_1'(x)dx = -\int \frac{y_2 f}{W(y_1 y_2)} dx$$

$$u_2(x) = \int u_2'(x)dx = \int \frac{y_1 f}{W(y_1 y_2)} dx$$

4) Compute $y_P = u_1 y_1 + u_2 y_2$

Ex: $y'' - 2y' + y = \frac{e^t}{1+t^2}$ $y(0) = 2$ $y'(0) = 0$

Q: what do I know?

A: constant coefficients

- $m^2 - 2m + 1 = 0 \implies m_1 = 1 \quad m_2 = 1$

$$\underline{y_c(t) = C_1 e^t + C_2 t e^t = (C_1 + C_2 t) e^t}$$

- $f(t)$ cont. on $(-\infty, \infty)$

1) $y_1(t) = C_1 e^t$ $y_2(t) = C_2 t e^t$

2) Find $y_p(t) \implies y_p(t) = u_1 y_1 + u_2 y_2$

$$y_p(t) = \left[\left(-\int \frac{y_2 f(t)}{W(y_1 y_2)} dt\right) y_1 + \left(\int \frac{y_1 f(t)}{W(y_1 y_2)} dt\right) y_2\right]$$

$$W(y_1 y_2) = \begin{vmatrix} e^t & te^t \\ e^t & \end{vmatrix}$$

$$y_p(t) = \underbrace{-\frac{1}{2} \ln|1+t^2| e^t}_{u_1(t)} + \underbrace{\tan^{-1}(t)}_{u_2(t)} t e^t$$

3) $y(t) = y_p(t) + y_c(t)$

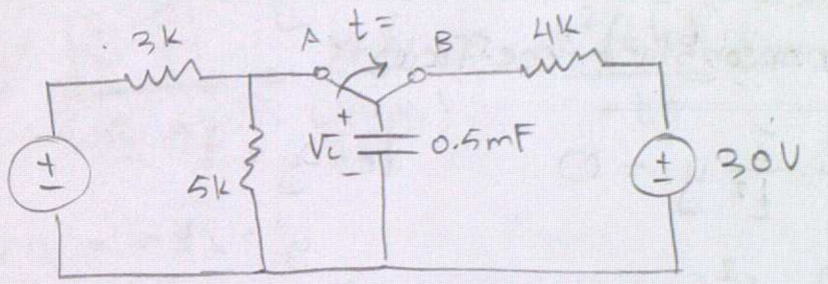
4) Solve IVP $\implies C_1 = 2 \quad C_2 = -2$

Ex: $t^2 y'' - 2ty' + 2y = t \ln t$ with $t > 0$

Q: what do I know?

A: Non-constant coefficients

Ex: step-response of RC ckt



Q: Find $v_c(t)$ for $t \geq 0$

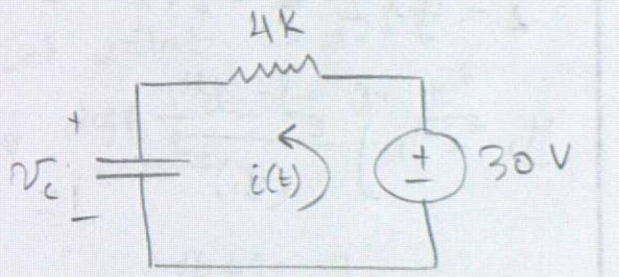
$v_c(t^-) = 24\left(\frac{5}{8}\right)$

$= 15$ V

For $t \geq 0$

KUL: $iR + v_c(t) = 30$

but $i = C v_c'$



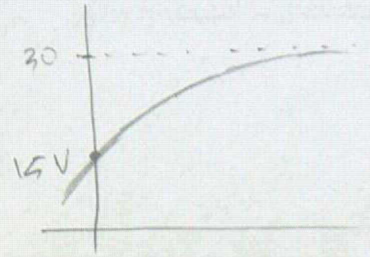
$$\boxed{v_c' + \frac{1}{RC} v_c = \frac{30}{RC}}$$

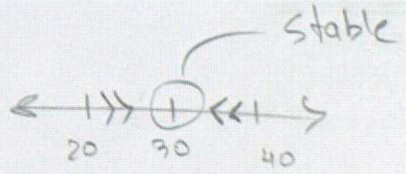
constant coeffient ODE (auxiliary equ.)

nonhomogenous (variation of paratmeters)

autonomous (critical ponits, phase plots)

$v_c' = \frac{30}{RC} - \frac{1}{RC} v_c \leftarrow$ critical point at $v_c = 30$ [V]





Ex 2: $t^2y'' - 2ty' + 2y = t\ln t$, $t > 0$

$$y'' - \frac{2t}{t^2}y' + \frac{2}{t^2}y = \frac{t\ln t}{t^2}$$

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = \ln t$$

nonhomogenous second order linear ordinary differential equation with nonconstant coefficients

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 0$$

Let $y = t^2$

$y' = 2t$

$y'' = 2$

$$2 - \frac{2}{\cancel{t}}\cdot 2\cancel{t} + \frac{2}{\cancel{t^2}}\cdot \cancel{t^2}$$

$$2 - 4 + 2 = -2 + 2 = 0$$

$\therefore y_1(t) = C_1t^2$ is sol^n

$$y_2 = y_1\int \frac{e^{-\int -\frac{2}{t}dt}}{(y_1)^2}\,dt$$ per reduction of order

$$= t^2\int \frac{e^{2\int \frac{1}{t}dt}}{(t^2)^2}\,dt$$

$$= t^2\int \frac{e^{2\ln t}}{t^4}\,dt = t^2\int \frac{t^2}{t^4}\,dt = t^2\int t^{-2}\,dt$$

$$= t^2(-t^{-1}) = \frac{t^2}{-t} = -t$$

$\therefore y_2(t) = C_2 t$

Now we can use our y_1 and y_2 to find particular solution

$$W(y_1, y_2) = \begin{vmatrix} t^2 & t \\ 2t & 1 \end{vmatrix} = t^2 - t(2t) = t^2 - 2t^2 = -t^2$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2 \end{vmatrix}}{W(y_1,y_2)} = \frac{-y_2f}{W(y_1,y_2)} = \frac{-t\,(t\ln t)}{-t^2} = \frac{-t^2\ln t}{-t^2} = \ln t$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1 & f \end{vmatrix}}{W(y_1,y_2)} = \frac{y_1f}{W(y_1,y_2)} = \frac{t^2(t\ln t)}{-t^2} = -t\ln t$$

$\Rightarrow u_1' = \ln t$

$$\int u_1'\,dt = u_1 = \int \ln t\,dt = t\ln t - t$$

$\Rightarrow u_2' = -t\ln t$

Let $u = \ln t$ $\quad dv = t$

$du = \frac{1}{t}dt$ $\quad v = \frac{t^2}{2}$

$$\int u_2'\,dt = -\int t\ln t\,dt$$

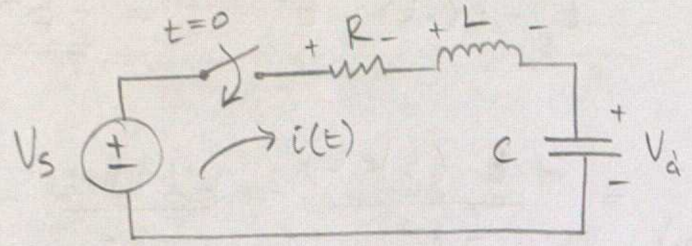
$$= -\left(\frac{t^2\ln t}{2} - \int \frac{t^2}{2}\cdot\frac{1}{t}\,dt\right)$$

$$= \frac{-t^2\ln t}{2} + \frac{1}{2}\int t\,dt = \frac{-t^2\ln t}{2} + \frac{1}{2}\cdot\frac{t^2}{2} = \frac{t^2 - 2t^2\ln t}{4}$$

$$\therefore y_p = u_1y_1 + u_2y_2 = (t\ln t - t)(t^2) + \left(\frac{t^2 - 2t^2\ln t}{4}\right)(t)$$

$$y_p = t^3\ln t - t^3 + \frac{t^3 - 2t^3\ln t}{4}$$

Switched RLC ckt w/h Sinusoidal Input



$i(t) = q'(t)$

$V_R = iR$ $\quad i_c = C V_c'(t)$

$V_L = L i_L'(t)$

For the ckt, it is known that $L = 500\,mH$, $C = 50\,\mu F$, and src given by $V_s(t) = 5\sin 400t$ [V]

a) Determine what values of R would lead to overdamped, underdamped, and critically damped behaviour.

b) Solve for the current $i(t)$ if $R = 520\,\Omega$

---

a) Do KVL: $V_s = V_R + V_L + V_c$

$\int i = q$

$i = q'$

$i' = q''$

$$V_s = iR + Li' + \frac{1}{C}\int i$$

$$V_s = Rq' + Lq'' + \frac{1}{C}q$$

Standard form: $Lq'' + Rq' + \frac{1}{C}q = V_s$

$$q'' + \frac{R}{L}q' + \frac{1}{LC}q = \frac{V_s}{L}$$

homogenous equation: $q'' + \frac{R}{L}q' + \frac{1}{LC}q = 0$

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0$$

$$m = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - 4(1)\left(\frac{1}{LC}\right)}}{2}$$

$$m = \frac{-R}{2L} \pm$$

Week 5: Lecture 2 Oct. 2, 2024

Intro to Laplace Transforms

ODEs: $y''' - 2y' + 3y = 5\cos x$

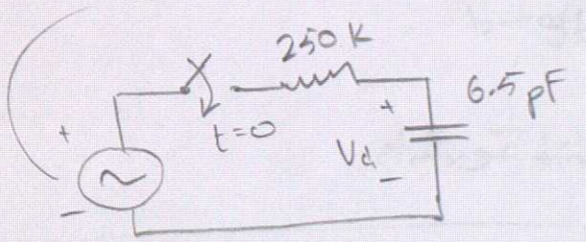
$y' - xy = 0$

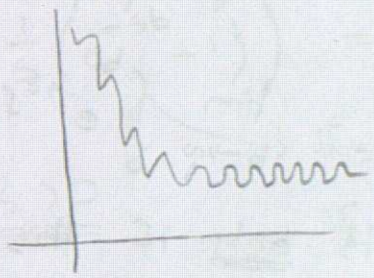
Some limitations:

1) conts. coefficient
2) Integrating factors, variation of parameters
3) forcing functions limited

→ Laplace transforms help us get rid of some issues.

$V_0(t) = 5\cos(\omega t), \omega = 2\pi(10)^6$





$$V_c' + \frac{1}{RC} V_c = \frac{5}{RC}\cos(\omega t) \Rightarrow V_c(t) = \underbrace{0.5\cos(\omega t - 84^\circ)}_{\text{steady state}} + \underbrace{4.95e^{-t/\tau}}_{\text{transient}}$$

3 ways to analyze

1) Time-domain analysis : ODE techiques
2) Phasor-domain : steady state only
3) s-domain : uses only algebra, Laplace Transform

$$V_c(s) = \frac{sV_s/RC}{(s^2+\omega^2)(s+\frac{1}{RC})} + \frac{V_0}{s+\frac{1}{RC}} \xrightarrow[\text{Laplace transform}]{\text{inverse}} \boxed{\quad}$$

phasor transform

$V(t) = V_0 \cos(\omega t + \theta) \xrightarrow{\text{phasor transform}} \overline{V} = V_0 \angle \theta$

$V(t) = Re($

Laplace Transform

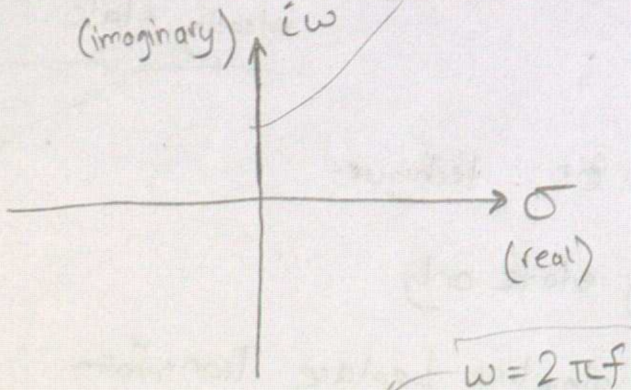
$$\boxed{L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt}$$

- Since this is an improper integral, we have in reality:

$$L\{f(t)\} = \lim_{b \to \infty} \int_0^b e^{-st} f(t)\,dt$$

- F(s) exists <u>only</u> if this limit exists

S - Domain



phasor domain

$$e^{-st} = e^{-(\sigma + i\omega)t}$$

$$= e^{-\sigma t} e^{-i\omega t}$$

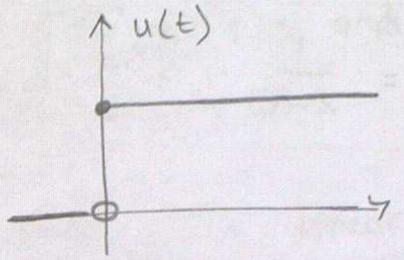
$$e^{-i\omega t} = \cos\omega t - i \sin\omega t$$

usually, $\boxed{s = \sigma + i\omega}$

$\boxed{\omega = 2\pi f}$

$\frac{2\pi}{T} =$

Ex: Step function (Heaviside f^n)



$$f(t) = u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

u(t) for $t \geq 0$

$$L\{f(t)\} = L\{u(t)\} = \lim_{b \to \infty} \int_0^b e^{-st}(1)\,dt$$

$$= \lim_{b \to \infty} \left[\frac{-e^{-st}}{s} \right]_0^b$$

$e^{-s(0)}$

$$= \lim_{b \to \infty} \left(-\frac{1}{s}\left(e^{-sb} - 1\right) \right) = F(s)$$

this is a problem if $s \leq 0$ or $\sigma \leq 0$

$$= \frac{1}{s}, \quad \underline{\underline{s > 0}}$$

$$\boxed{\therefore L\{u(t)\} = \frac{1}{s}, \ s > 0}$$

Existence of Laplace Transforms

if 1) ✓ and 2) ✓, then $\boxed{\lim_{s \to \infty} F(s) = 0}$

Sufficient Conditions

1) f(t) is piecewise cont. on $[0, \infty)$ (finite # of discontinuities)

2) f(t) is of exponential order, c $|f(t)| \leq Me^{ct}, \ t \geq T$

① take function, divide by e^{ct}

② take limit as $t \to \infty$ to see if limit exists

ex: $f(t) = t^2$ ← <u>is</u> of exp. order ex: $f(t) = e^{t^3}$ ← <u>not</u> of exp. order

Ex: exponential signal $f(t)$

$$f(t) = e^{at} u(t)$$

$$\boxed{\therefore F(s) = \frac{1}{s-a} \quad , s > a}$$

$$= \begin{cases} 0 & t < 0 \\ e^{at} & t \geq 0 \end{cases}$$

$$L\{f(t)\} = F(s) = \lim_{b \to \infty} \int_0^b e^{-st} f(t) dt$$

$$= \lim_{b \to \infty} \int_0^b e^{-st} e^{at} dt$$

$$= \lim_{b \to \infty} \int_0^b e^{(a-s)t} dt$$

$$= \lim_{b \to \infty} \left[\frac{1}{a-s} e^{(a-s)t} \right]_0^b$$

$$= \lim_{b \to \infty} \left[\frac{1}{a-s} e^{(a-s)b} \right] - \left[\frac{1}{a-s} \cancelto{1}{e^{(a-s)(0)}} \right]$$

$$= \lim_{b \to \infty} \frac{1}{a-s} e^{(a-s)b} - \frac{1}{a-s}$$

$$= \underbrace{\lim_{b \to \infty} \frac{1}{a-s} \left(e^{(a-s)b} - 1 \right)} = \frac{1}{a-s}$$

exists if: $\hookrightarrow a - s < 0$

$a < s$

$\underline{\underline{s > a}}$ — or really, $\sigma > a$

Quiz 3 (3.1 3.2 3.3 Review) Oct 2, 2024

General Linear DE's : Conditions for existence and uniqueness

1) all $a_n(x)$'s and $g(x)$ are continuous

2) $a_n(x) \neq 0$

Homogenous Linear DE's : General solutions (fundamental set)

1) n^{th} order ODE has n linearly independent solutions

2) general solution of ODE is linear combination with arbitrary constants C_n of linearly independent solutions

Linearly Independent Solutions :

Set of solutions $y_1(x), y_2(x), \cdots, y_n(x)$ to homogenous DE are linearly independent if $W(y_1, y_2 \cdots y_n) \neq 0$ on int. I

Reduction of Order

Given DE $y'' + P(x)y' + Q(x)y = 0$ and a solution $y_1(x)$,

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2} dx$$

and the general solution is:

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

Constant Coefficients

- distinct real roots : $m_1, m_2, \cdots, m_k \longrightarrow y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots + C_m e^{mx}$
- real repeated roots : $m_1 = m_2 = \cdots = m_k \Rightarrow y = C_1 e^{mx} + C_2 x^1 e^{mx} + C_3 x^2 e^{mx} + \cdots$
- imaginary roots : $m = \alpha + \beta i$ $\hookrightarrow y = e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x]$

Week 5 : Lecture 2 Oct 3, 2024

Laplace Transform Linearity

→ $$L\{\alpha f(t) + \beta g(t)\} = \alpha L\{f(t)\} + \beta L\{g(t)\}$$

Ex: Find $L\{\sin(\omega t)u(t)\}$ ← $u(t)$ is step function, basically means we start after 0

Remember $\sin(\omega t) = \dfrac{e^{i\omega t} - e^{-i\omega t}}{2i}$

$$L\{\sin(\omega t)u(t)\} = \frac{1}{2i}L\{e^{i\omega t}\} - \frac{1}{2i}L\{e^{-i\omega t}\}$$

$$= \frac{1}{2i}\frac{1}{s-i\omega} - \frac{1}{2i}\frac{1}{s+i\omega}$$

remember $L\{e^{at}\} = \dfrac{1}{s-a}$

$$= \frac{1}{2i}\frac{s+i\omega - s + i\omega}{(s-i\omega)(s+i\omega)}$$

$$= \frac{\omega}{s^2+\omega^2}$$

$$\boxed{\therefore L\{\sin(\omega t)u(t)\} = \frac{\omega}{s^2+\omega^2}}$$

$$\boxed{L\{\cos(\omega t)u(t)\} = \frac{s}{s^2+\omega^2}}$$

Inverse Laplace Transforms and IVPs (Sec. 4.2)

Tranforms of Time Derivatives

$$\boxed{y''} - \boxed{6y'} + \boxed{3y} = \boxed{\sin(t)} \quad y(0) = 1 \quad y'(0) = 2 \quad t \geq 0$$

$y'' \to ?$ $\quad 6y' \to ?$ $\quad 3y \to 3Y(s)$ $\quad \sin(t) \to \dfrac{1}{s^2+1^2}$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$L\left\{\frac{dy}{dt}\right\} = L\{y'\} = \int_0^\infty \underbrace{e^{-st}}_{u}\underbrace{\left(\frac{dy}{dt}\right)dt}_{dv}$$

$$= \underbrace{e^{-st}y(t)\Big|_0^\infty}_{-y(0)} - \int_0^\infty y(t)(-se^{-st})\,dt$$

$$= -y(0) + s\underbrace{\int_0^\infty e^{-st}y(t)\,dt}_{Y(s)}, \quad (s>0)$$

$$\boxed{\therefore L\{y'\} = sY(s) - y(0)}$$

In general: $$\boxed{L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)}$$

Ex: $\underbrace{y''+y}_{Y(s)} = \underbrace{f(t)}_{F(s)}$, $y(0)=0$, $y'(0)=0$

s-domain version

$$\boxed{F(s) = s^2Y(s) + Y(s)} \rightarrow \boxed{Y(s) = \frac{F(s)}{s^2+1}}$$

$$L\{y''\} = s^2Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} = s^2Y(s)$$

Inverse Laplace Transform

$F(s) \longrightarrow$ what is $f(t) = L^{-1}\{F(s)\}$

$$\boxed{f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\, ds}$$

Linearity

$$L^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha L^{-1}\{F(s)\} + \beta L^{-1}\{G(s)\} = \alpha f(t) + \beta g(t)$$

Ex: $F(s) = \dfrac{6s^3 + 25s^2 + 156s + 600}{s(s+6)(s^2+25)}$ } proper rational func^r, order num < order denom.

$\therefore \lim_{s \to \infty} F(s) = 0$

our 2 condi. are true then
1) $f(t)$ cont.
2) $f(t)$ exp order.

$$F(s) = \frac{4}{s} + \frac{2}{s+6} + \frac{1}{s^2+25}$$

$$\underline{\underline{f(t) = 4u(t) + 2e^{-6t}u(t) + \frac{1}{5}\sin(5t)u(t)}}$$

we got inverse by applying linearity property of inverse Laplace transforms

Partial Fractions Expasion

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{100(s+3)}{(s+6)(s+3-4i)(s+3+4i)}$$

$$F(s) = \frac{A}{s-P_1} + \frac{B}{s-P_2} + \frac{C}{s-P_3}$$

$P_1 = -6$
$P_2 = -3 + 4i$
$P_3 = -3 - 4i$
} poles

$F(s) = \dfrac{A}{s-a}$

$f(t) = Ae^{-at}$

$$F(s) = \frac{A}{s-P_1} + \frac{Bs + C}{(s-P_2)(s-P_3)}$$ OR

$$f(t) = \Big[-12e^{-6t} + \underbrace{(6-8i)e^{(-3+4i)t} + (6+8i)e^{(-3-4i)t}}\Big]u(t)$$

for $p = -\alpha \pm i\beta \rightarrow B = |B|\angle\theta$, $C = \overline{B} = |B|\angle{-\theta}$

this becomes

$$2|B|e^{-\alpha t}\cos(\beta t + \theta)$$

$$f(t) = \Big[-12e^{-6t} + 20e^{-\alpha t}\cos(4t - 53.1^\circ)\Big]u(t)$$

used cover-up method

$$B = (s+3-4i)F(s)\Big|_{s=-3+4i} = 6 - 8i = 10\angle{-53.1^\circ}$$

$C = \overline{B} = 6 + 8i = 10\angle{53.1^\circ}$ since P_2 and P_3 conjugates poles

Week 6: Lecture 1 Oct. 7, 2024

Translation Theorems

$y'' + 2y' + y = 0$, $y(0) = 1$, $y'(0) = 1$, $t \geq 0$

Transform ODE: $y'' = s^2 Y(s) - sy(0) - y(0)$

$= s^2 Y(s) - s(1) - 1 = \underline{s^2 Y(s) - s - 1}$

$y' = sY(s) - y(0) = \underline{sY(s) - 1}$

$y = \underline{Y(s)}$

After transform: $s^2 Y(s) - s + 2(sY(s) - 1) + Y(s) - 1 = 0$

$s^2 Y(s) - s + 2sY(s) - 2 + Y(s) - 1 = 0$

$s^2 Y(s) + 2sY(s) + Y(s) = s + 2 + 1$

$(s^2 + 2s + 1) Y(s) = s + 2 + 1$

$Y(s) = \frac{s+3}{s^2 + 2s + 1} = \frac{s+3}{(s+1)^2}$

$\frac{s+3}{(s+1)^2} = \frac{A_1}{(s+1)^2} + \frac{A_2}{(s+1)} = \frac{A_1 + A_2(s+1)}{(s+1)^2} = \frac{A_1 + A_2 s + A_2}{(s+1)^2}$

s: $1 = A_2$

constant: $3 = A_1 + A_2$ → $A_1 = 3 - A_2 = 3 - 1 = 2$

$\therefore \underline{A_1 = 2}$ $\underline{A_2 = 1}$

$\therefore Y(s) = \frac{2}{(s+1)^2} + \frac{1}{s+1}$

→ Laplace Trasform Property 3: s-shifling

$G(s) = F(s-a) = \int_0^\infty e^{-(s-a)t} f(t) = \int_0^\infty e^{-st} \cdot \underbrace{e^{at} \cdot f(t)}_{g(t)} dt$

. take $f(t)$, multiply be e^{at}

$$\boxed{\therefore L\{e^{at} f(t)\} = F(s-a)}$$

if a function in the s-domain is shifted by an amount a, this results in the time-domain fun^c being multiplied by e^{at}

Ex: $L^{-1}\left\{\frac{1}{s+1} + \frac{2}{(s+1)^2}\right\}$

Pecall $L^{-1}\left\{\frac{1}{s}\right\} = u(t)$, replace s with s+1, so we have

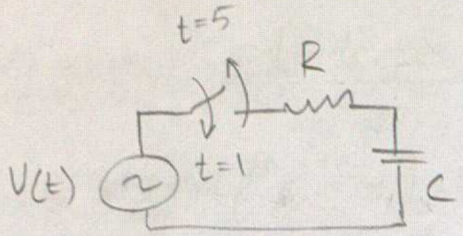
$F(s) \to f(t)$

$L^{-1}\left\{\frac{1}{s+1}\right\} = \underbrace{e^{-t}}_{e^{at}} \underbrace{u(t)}_{f(t)}$

Recall $L^{-1}\left\{\frac{2}{s^2}\right\} = tu(t)$, replace s with s+1, so we have

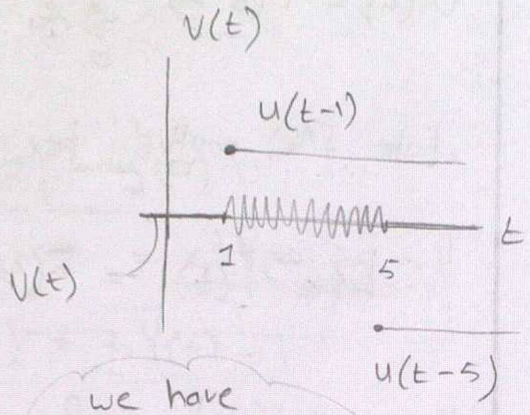
$L^{-1}\left\{\frac{2}{(s+1)^2}\right\} = \underbrace{e^{-t}}_{e^{at}} \underbrace{tu(t)}_{f(t)}$

Ex:



ON: $t = 1s$

OFF: $t = 5s$



$$V(t) = 5\cos(\omega t)\left[u(t-1) - u(t-5)\right]$$

we have to subtract this from u(t-1) to "turn of"

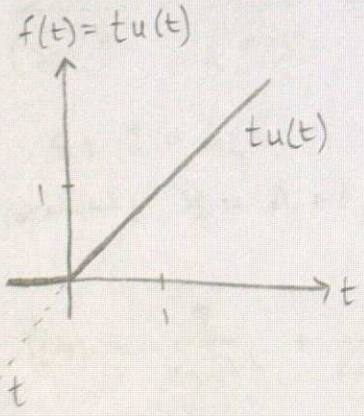
Property 4: t-Shifting

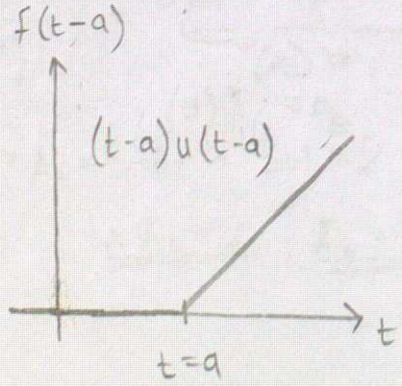
$$\boxed{L\{f(t-a)u(t-a)\} = e^{-as}F(s)}$$

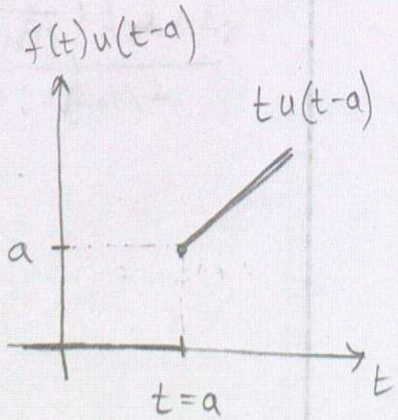
if a signal in the time domain is shifted by an amount a, this results in the s-domain signal being multiplied by e^{-as}.

$$\boxed{e^{-as}L\{g(t+a)\} = L\{g(t)u(t-a)\}}$$

Forcing Functions and Switches







Week 6: Lecture 2

Oct. 8, 2024

Property #5: Derivatives of Transforms

$$L\{y'\} \longrightarrow sY(s) - (\text{initial conditions})$$

→ take derivative of s-domain function:

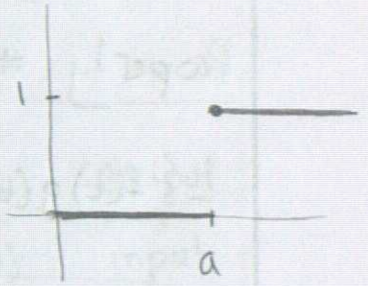
$$G(s) = \frac{dF(s)}{ds} = \frac{d}{ds}\underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{F(s)}$$

$$= \int_0^\infty \frac{\partial e^{-st}}{\partial s} f(t)\,dt$$

$$= \underbrace{\int_0^\infty e^{-st}\underbrace{\left[-t f(t)\right]}_{g(t)}dt}_{G(s)}$$

Unit Step Func^{n}

$$u(t-a) = \begin{cases} 0, & t<a \\ 1, & t \geqslant a \end{cases}$$



In general:

$$\boxed{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)}$$

- find F(s) of hard func^{n}
- find

time diff. ⟶ multiplication by s

s - diff. ⟶ multiplication by t

Eg $g(t) = t^2u(t) = t^2 f(t)$ $\qquad \frac{d}{ds} s^{-1} = -s^{-2}$

$n=2$, $L\{t^2u(t)\} = (-1)^2 \frac{d^2}{ds^2} F(s)$ $\qquad \frac{d}{ds} -s^{-2} = 2s^{-3}$

$= \frac{d^2}{ds^2}\left[\frac{1}{s}\right]$ $\qquad L\{u(t)\} = F(s)$

$F(s) = \frac{1}{s}$

$$\boxed{G(s) = \frac{2}{s^3}}$$

Property #6 : Convolution Theorem

$L\{f(t)g(t)\} \neq F(s)G(s)$

$$\boxed{L^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)g(t-\tau)\,d\tau = f(t)*g(t)}$$

convolution operator

$F(s) = \frac{1}{s} \leftarrow u(t)$ $\qquad G(s) = \frac{1}{s+1} \leftarrow e^{-t}u(t)$

$F(s)G(s) = \frac{1}{s(s+1)} \longrightarrow f(t)*g(t) = \int_0^t \underbrace{u(\tau)}_{f(t)=u(t)} \underbrace{e^{-(t-\tau)}}_{g(t)=e^{-t}u(t)} d\tau$

1) $f(t)*g(t) = g(t)*f(t)$

Ckt Convolution Theorem

LC ckt

DE: $V_c'' + \frac{1}{LC}V_c = \frac{V_{in}}{LC}$

$\omega_0 = \frac{1}{\sqrt{LC}}$ $\qquad \Rightarrow V_c'' + \omega_0^2 V_c = \omega_0^2 V_{in}(t)$

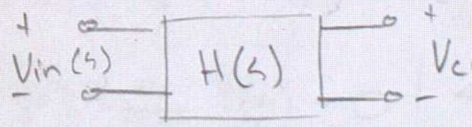
$\omega_0^2 = \frac{1}{LC}$ $\qquad$ convert to s-domain

$V_c(0) = V_c'(0) = 0$ $\qquad s^2V_c(s) + \omega_0^2 V_c(s) = \omega_0^2 V_{in}(s)$

solving for $V_c(s)$

$$V_c(s) = \underbrace{\left(\frac{\omega_0^2}{s^2+\omega_0^2}\right)}_{H(s)\,\rightarrow\,\text{transfer function}} V_{in}(s) = H(s)V_{in}(s)$$

↳ $H(s) = \frac{V_c(s)}{V_{in}(s)} = \frac{\text{output}}{\text{input}}$



H(s) completely characterizes

- finding the inverse of $V_c(t) = L^{-1}\{H(s)V_{in}(s)\} = \int_0^t h(\tau)V_{in}(t-\tau)\,d\tau$

$h(t) = L^{-1}\left\{\frac{\omega_0^2}{s^2+\omega_0^2}\right\} = \omega_0 \sin(\omega_0 t)$

$h(t) = L^{-1}\{H(s)\}$

↳ impulse response

Now I can apply any input:

1) $V_{in}(t) = V_0 \cos(\omega_0 t) \longrightarrow \omega_0 = \frac{1}{\sqrt{LC}}$

Using our identity from earlier, we know $V_C(s)$ is s-dom

$$V_C(s) = H(s) V_{in}(s) = \left(\frac{\omega_0^2}{s^2+\omega_0^2}\right)\left(V_0 \cdot \frac{s}{s^2+\omega_0^2}\right)$$

to find $V_C(s)$ in time domain, we apply the following

$$V_C(t) = L^{-1}\left\{\left(\frac{\omega_0^2}{s^2+\omega_0^2}\right)\left(\frac{V_0 s}{s^2+\omega_0^2}\right)\right\}$$

$$= L^{-1}\left\{\frac{V_0\omega_0^2 s}{(s^2+\omega_0^2)^2}\right\}$$

observe that

$$(-1)\frac{d}{ds}\underbrace{\left(\frac{V_0\omega_0^2}{s^2+\omega_0^2}\right)}_{} = (-1)\left(\frac{-2V_0\omega_0^2 s}{(s^2+\omega_0^2)^2}\right)$$

$F(s) \to V_0\omega_0 \sin(\omega_0 t) = f(t)$

$$= t\,\frac{f(t)}{2}$$

$$= \frac{\omega_0 V_0}{2} t \sin(\omega_0 t) = h(t) * V_{in}(t) = \int_0^t h(\tau) f(t-\tau)\, d\tau$$

Week 6: Lecture 3

Oct 10, 2024

$\underline{ty'' + (t-3)y' + y = 0}$ $\quad y(0)=0,\ y'(0)=0$

↑ solution not guaranteed (maybe not unique) since at $t=0$, we might have some problelms

① Trasform into s-domain

$$L\{t\underbrace{y''}_{}\} = (-1)\frac{d}{ds}F(s) = (-1)\frac{d}{ds}[s^2 Y(s)] = -s^2 Y'(s) - 2sY(s)$$

$f(t) \to F(s) = s^2 Y(s)$

$$L\{(t-3)y'\} = L\{t\underbrace{y'}_{}\} - 3L\{\underbrace{y'}_{}\} = (-1)\frac{d}{ds}G(s) - 3G(s)$$

$g(t) \to G(s) = sY(s)$

$$= -\frac{d}{ds}[sY(s)] - 3sY(s) = -Y(s) - sY'(s) - 3sY(s)$$

$$\therefore -s^2Y'(s) - 2sY(s) - Y(s) - sY'(s) - 3sY(s) + Y(s) = 0$$

$$\boxed{Y'(s) + \frac{s}{s+1}Y(s) - 0}$$ ← ODE in s-domain
1st order linear, homogenous

$$\mu(s) = e^{\int p(s)ds} = e^{\int \frac{s}{s+1}ds} = e^{5\ln(s+1)} = (s+1)^5$$ ← integrating factor

$$\therefore Y(s) = \frac{c}{(s+1)^5}$$ ← actually a particular sol^n since Laplace takes care of I.C.'s

$$Y(s) = \underbrace{\frac{c}{(s+1)^5}} \Rightarrow y(t) = L^{-1}\left\{\frac{c}{(s+1)^5}\right\}$$

think of this as $B(s) = \frac{1}{s^5} \Rightarrow b(t) = \frac{t^4}{4!}u(t)$

but our function has an s-shift, so $y(t)$ is:

$$y(t) \Rightarrow e^{-t} t^4 u(t)$$

We have a constant, so our final answer is

$$y(t) = L^{-1}\left\{\frac{c}{(s+1)^5}\right\} = \boxed{c^* e^{-t} t^4 u(t)}$$

takes care of the $\frac{1}{4!}$ from $b(t) \ldots y(t)$

We have confirmed that infite solutions go through our initial conditions (since we violated the hypothesis of the unique $sol^{\underline{n}}$ theorem)

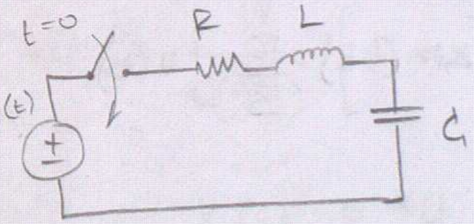
Property #7: Transform of a Time Integral

What is $L\left\{\int_0^t f(\tau)d\tau\right\}$?

↳ use convolution prop: $L\left\{\int_0^t\right.$

$$\boxed{L\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}}$$

Ex: $R=1$, $L=500\,mH$, $C=200\,mF$, $V_s(t)=5e^{-2t}$ ... Find $i(t)$



from ckt analysis

$$\text{KVL: } \boxed{iR + Li' + \frac{1}{C}\int_0^t i\,d\tau = 5e^{-2t}}$$

① Transform to s-domain: initially uncharged, no I.C.'s terms

$$R\,I(s) + Ls\,I(s) + \frac{1}{sC}I(s) = \frac{V_0}{s+2} \leftarrow 5$$

② Solve for $I(s)$:

$$I(s) = \underbrace{\left(\frac{5}{s+2}\right)}_{V_s(s)}\underbrace{\left(\frac{\frac{5}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}}\right)}_{H(s):\text{ transfer function}}$$

relates input function to output

$$I(s) = \left(\frac{10}{s+2}\right)\left(\frac{s}{s^2+2s+10}\right)$$

$$= \left(\frac{10}{s+2}\right)\left(\frac{s}{(s+1)^2+9}\right)$$

③ Find inverse transform, $i(t)$

$$\sin kt = \frac{k}{s^2+k^2}$$

$$I(s) = \left(\frac{A}{s+2}\right) + \left(\frac{Bs+C}{(s+1)^2+9}\right)$$

mistake

partial fractions, $A = -1$, $B = 1$, $C = 10$

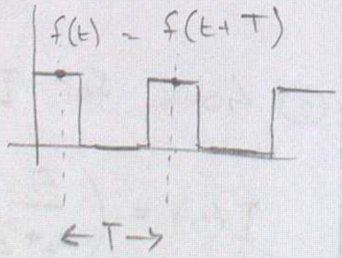
$$\therefore i(t) = \left[-e^{-2t} + e^{-t}\cos(3t) + e^{-t}\sin(3t)\cdot\frac{10}{3}\right]u(t)$$

$(s+1)$ shift

Property #8 : Transform of Periodic Function

- consider a f^n where $f(t+T) = f(t)$

$$\boxed{F(s) = L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt}$$



Week 7: Lecture 1

Oct 15, 2024

Ex: $x' + x = f(t)$, $x(0) = 0$

s-domain: $sX(s) + X(s) = F(s)$

$$X(s) = \frac{F(s)}{s+1} \leftarrow \text{from a prev. example}$$

$$\therefore X(s) = \frac{1}{s(s+1)(1+e^{-s})} = \underbrace{\underbrace{\frac{1}{s}\left(\frac{1}{s+1}\right)}_{G(s)}\underbrace{\left(\frac{1}{1+e^{-s}}\right)}_{H(s)}}_{\text{integral of the result } g(t)*h(t)}$$

input as a pulse

observe that $H(s) = \frac{1}{1+e^{-s}} = 1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots$

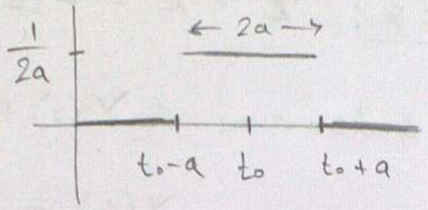
$$X(s) = \underbrace{\frac{1}{s(s+1)}}\left[1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots\right] \rightarrow L\{e^{-sa}k(s)\}$$

$$k(s) \Rightarrow k(t) = (1-e^{-t})u(t) \qquad = k(t-a)u(t-a)$$

$$\therefore x(t) = L^{-1}\{X(s)\} = (1-e^{-t})u(t) - \left[1-e^{-(t-1)}\right]u(t-1)$$

$$+ \left[1-e^{-(t-2}\right]u(t-2) - \cdots$$

Dirac Delta Function



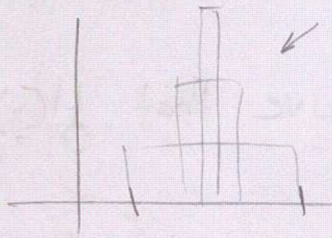
- a very short burst of energy
- function $\delta_a(t-t_o)$

$$\delta_a(t-t_o) = \begin{cases} 0, & 0 \le t \le t_o - a \\ \frac{1}{2a}, & t_o - a \le t \le t_o + a \\ 0, & t \geqslant t_o + a \end{cases}$$

Let's have $a \to a$

$$\delta_a(t-t_o) = \lim_{a \to 0} \delta_a(t-t_o)$$

$$= \begin{cases} \infty, & t = t_o \\ 0, & t = t_o \end{cases}$$



squeezin it but keepind area same

↳ area same

↳ value of $\delta \; r \to \infty$

As well, $\int_0^\infty \delta_a(t-t_o)\,dt = 1$ ← unit area

- more properly defined by sifting property:

$$\boxed{\int_0^\infty f(t)\,\delta(t-t_o)\,dt = f(t_o)}$$

if $f(t) = 1$

$$\int_0^\infty \delta(t-t_o)\,dt = 1$$

$$\boxed{L\{\delta(t-t_o)\} = \int_0^\infty e^{-st}\,\delta(t-t_o)\,dt = e^{-st_o}}$$

↑ shifted δ function

$$\boxed{L\{\delta(t)\} = 1}$$

unshifted δ function

Ex: $V''_c + \frac{1}{LC} V_c = \frac{V_{in}(t)}{LC}$ ← Let's have $V_{in}(t) = \delta(t)$

$$V''_c + \omega_o^2 V_c = \omega_o^2\,\delta(t) \qquad \omega_o = \frac{1}{\sqrt{LC}}$$

Transform: $s^2 V_c(s) + \omega_o^2 V_c(s) = \omega_o^2 (1)$ ← I am seeing the response of the system to all frequencies at one

$$V_c(s) = \frac{\omega_o^2}{s^2 + \omega_o^2}$$ ← H(s), our transfer function

$$V_c(t) = \omega_o \sin(\omega_o t)\,u(t) = h(t)$$ ← impulse response

- Input δ gives transfer function as output (impulse response)
- I can now get output given ~~any~~ ou input (I have transfer func)

Ex: $y'' + 2y' = \delta(t-1) \quad y(0) = 0,\ y'(0) = 1$

$$s^2 Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) = e^{-s}$$ ← a time shift

$$(s^2 + 2s)\,Y(s) = 1 + e^{-s} \Rightarrow Y(s) = \frac{1+e^{-s}}{s(s+2)} = \left(\frac{A}{s} + \frac{B}{s+2}\right)(1+e^{-s})$$

$$A = \frac{1}{2},\ B = -\frac{1}{2}$$

a=1, we have time shift

$$Y(s) = \frac{1/2}{s} - \frac{1/2}{s+2} + e^{-s}\left(\frac{1/2}{s} - \frac{1/2}{s+2}\right)$$

$$\boxed{y(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{2}u(t-1) - \frac{1}{2}e^{-2(t-1)}u(t-1)}$$

17.3/17.4 Sets in Complex Plane

Week 7: Lecture 2 Oct 17, 2027

Review : Complex Numbers

$z = x + iy$ or $z = |z|(\cos\theta + i\sin\theta)$

$= |z| e^{i\theta}$ where $|z| = \sqrt{x^2+y^2}$

$= |z| \angle\theta$

principal argument : $\mathrm{Arg}(z) = \theta_o = \tan^{-1}\left(\frac{y}{x}\right)$

$\downarrow$

$-\pi \leq \theta_o \leq \pi$

general argument : $\arg(z) = \theta_o + 2\pi k$, $k = 0, \pm 1, \pm 2, \cdots$

multiplication / division

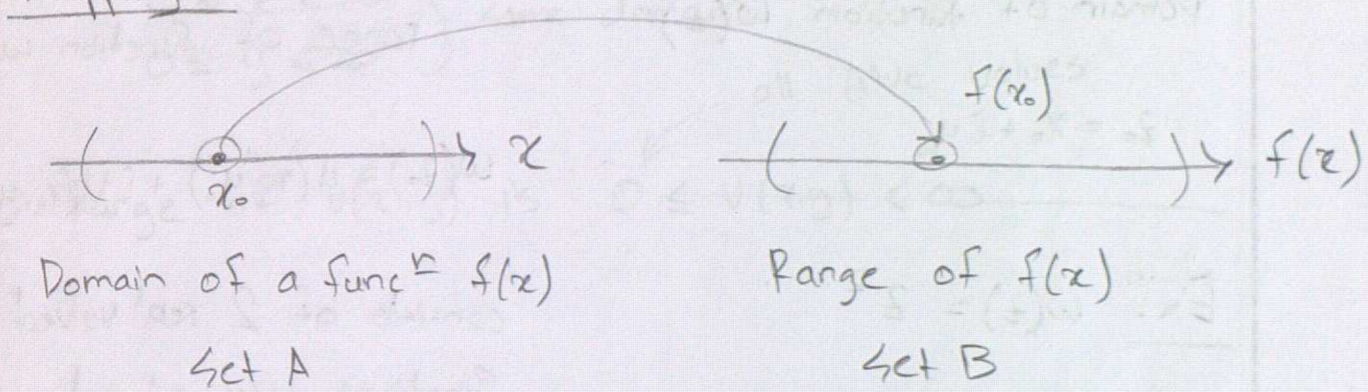
$z_1 z_2 = r_1 r_2 \angle \theta_1 + \theta_2$

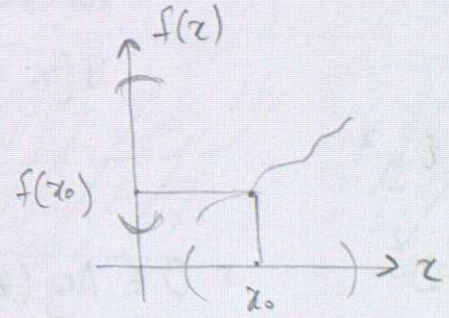
$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle \theta_1 - \theta_2$

Complex Functions

- Laplace Transforms, $F(s)$, $s = \sigma + i\omega$ $F(s) = \frac{41}{s^2+3s+7}$
- Transfer Functions, $H(j\omega)$, $H(j\omega) = \frac{1}{1-\omega^2 LC + j\omega RC}$ ← RLC ckt
- Electromagnetic Waves, $E(x, y, z, j\omega)$ and $B(x, y, z, j\omega)$

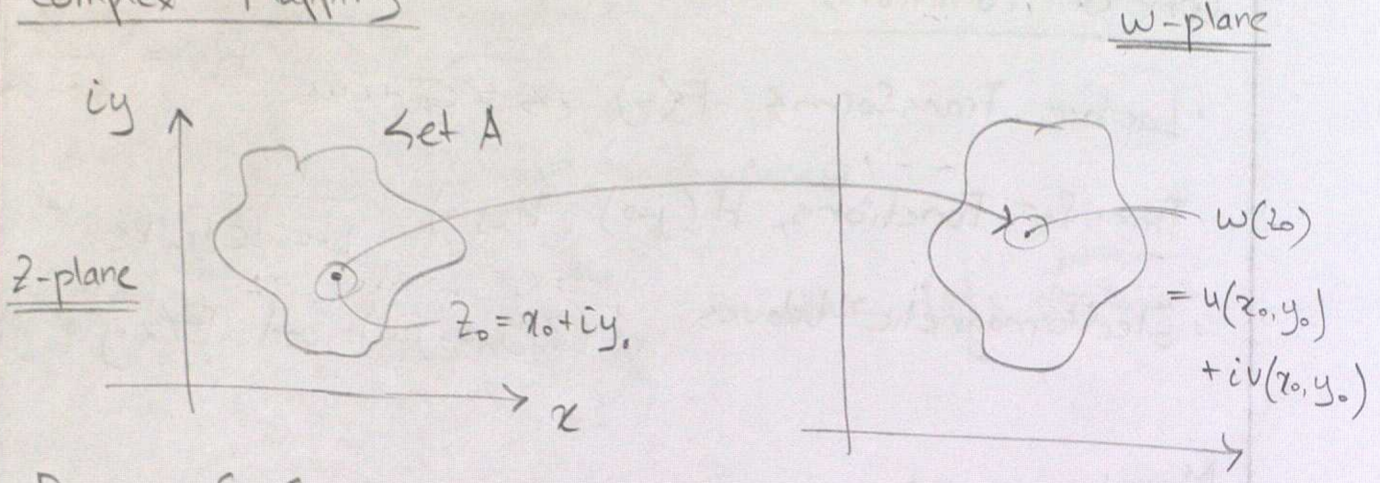
Mapping





a one-to-one mapping from set A (dom) to set B (range)

Complex Mapping



Domain of function $w(z)$

$z_0 = x_0 + iy_0$

range of function $w(z)$

$w(z_0) = u(x_0,y_0) + iV(x_0,y_0)$

consists of 2 real valued functions $u(x_0,y_0)$ and $V(x_0,y_0)$

Ex: $w(z) = z^2$

$$w(z) = (x+iy)^2$$

$$= x^2 + 2xiy + i^2y^2$$

$$= x^2 + 2xyi - y^2$$

$$= \underbrace{x^2 - y^2}_{\text{real}} + i\underbrace{2xy}_{\text{im}}$$

$\therefore u(x,y) = x^2 - y^2$

$V(x,y) = 2xy$

$$\underbrace{0 \leq \text{Arg}(z) \leq \frac{\pi}{2}}$$

basically means domain is 1st Quadrant

$0 \leq x < \infty$

$0 \leq y < \infty$

$u(x,y) = x^2 - y^2$

$$\left.\begin{array}{l} 0 \leq x < \infty \\ 0 \leq y < \infty \end{array}\right\} \text{domain of } w(z) \text{ set A}$$

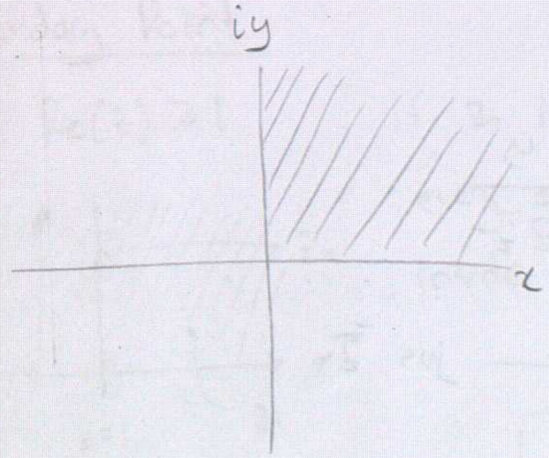
$\therefore$ Range of $u(x,y)$ is $-\infty < u(x,y) < \infty$

$V(x,y) = 2xy$

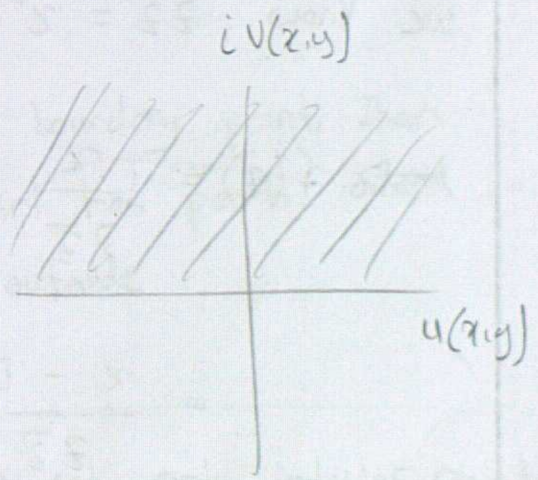
$$\left.\begin{array}{l} 0 \leq x < \infty \\ 0 \leq y < \infty \end{array}\right\} \text{same domain}$$

all (+)ve values

$\therefore$ Range of $V(x,y)$ is $0 \leq V(x,y) < \infty$



domain of $w(z)$



range of $w(z)$

Alternate Representations

$$\underbrace{\text{Let } f(z) = f(x+iy) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}}_{\text{express in terms of } z}$$

observe $\boxed{z = x+iy \text{ ①}}$ $\boxed{\bar{z} = x-iy \text{ ②}}$

① + ② $\boxed{x = \frac{z+\bar{z}}{2}}$ ③ ① − ② $\boxed{y = \frac{z-\bar{z}}{2i}}$ ④

take ③ and ④ to plug into function

we know $z\bar{z} = x^2+y^2$

$$\Rightarrow f(z) = \frac{x}{z\bar{z}} - i\frac{y}{z\bar{z}}$$

$$= \frac{x-iy}{z\bar{z}} \leftarrow \text{jus } \bar{z}$$

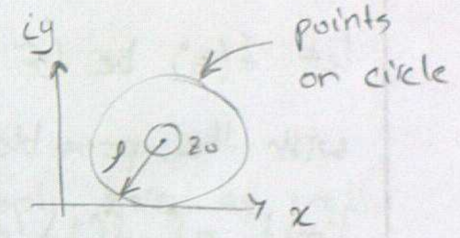
$$= \frac{\bar{z}}{z\bar{z}}$$

$$f(z) = \frac{1}{z}$$

Sets in the Complex Plane

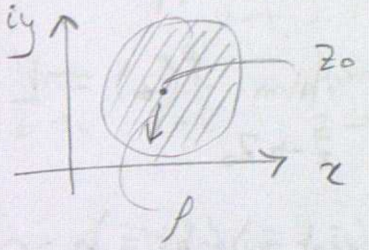
- "set" is a collection of points

ex: $|z-z_0| = \rho \leftarrow$ real number, a circle



ex: $|z-z_0| \leq \rho$

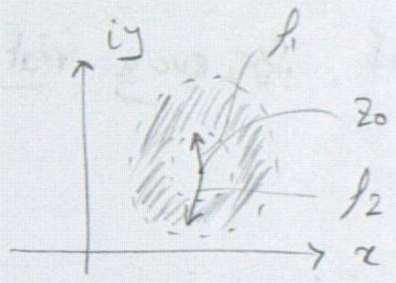
closed set



ex:

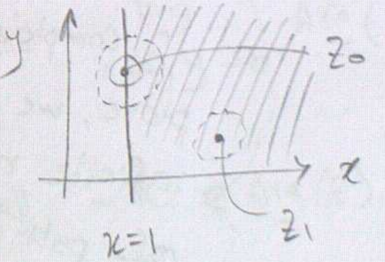
$\rho_1 < |z-z_0| < \rho_2$

open set



Boundary Point

ex: $Re(z) \geqslant 1$



if z_0 is a boundary point, then every open set has points around z_0 inside and outside

Interior Points

if z_1 is an interior point, then there is an open set around z_1 ~~is~~ with points inside S disk

Open set: only interior points

Closed set: both interior and boundary points

connected set: any 2 pnts can be connected by finite straight line segments

domain: open connected set

Week 8: Lecture 1 (17.4)   Oct 21, 2024

Limits of Complex Functions

→ "open disk" $|z - z_0| < \rho$

Let $f(z)$ be a func^n defined in $\overbrace{\text{neighbourhood}}$ of z_0 with the possible exception of z_0 itself. We say limit of $f(z)$ as z approaches z_0 is the number L and write

$$\lim_{z \to z_0} f(z) = L$$

if, for every real number $\rho > 0$

<u>Ex:</u> $\lim_{z \to i} (z+i) = 2i$

$$|f(z) - L| = |z+i-2i| < \varepsilon \quad ①$$

$$0 < |z-i| < \delta \quad ②$$

$$① \to |z+i-2i| = |z-i| < \varepsilon$$

Let $\varepsilon = \frac{\delta}{2}$ $\Longrightarrow$ $|z-i| < \frac{\delta}{2}$

In complex plane, we have infinite number of paths we can take as $z \to z_0$

Properties

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} u(x,y) + i \lim_{z \to z_0} v(x,y)$$

- basic rules/properties of real limits apply here as well

<u>Ex:</u> $\lim_{z \to i} \frac{z^2+1}{z-i} = \lim_{z \to i} \frac{z^2}{z-i} + \lim_{z \to i} \frac{1}{z-i}$

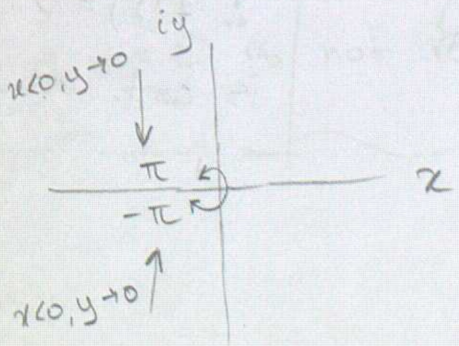
1) Factoring $\frac{z^2+1}{z-i} = \frac{(z+i)(z-i)}{z-i} = z+i$

$$\lim_{z \to i} z+i = i+i = \boxed{2i}$$

2) L'Hopital's Rule: $\lim_{z \to i} \frac{2z}{1} = \boxed{2i}$

<u>Ex:</u> $\lim_{x<0,\, y \to 0} \text{Arg}(z) = ?$

Recall, $-\pi < \text{Arg}(z) \leq \pi$



$\boxed{\therefore \text{ limit DNE}}$

since from top, we get π and from the bottom, we get closer to -π but never get there → 2 path test proves limit DNE

Continuity of Complex Function

open disk $|z-z_0| < \rho$

Function $f(z)$ defined in a $\underbrace{\text{neighborhood}}$ of z_0, then

a) $f(z_0)$ is defined

b) $\lim_{z \to z_0} f(z) = f(z_0)$

$\Bigg\}$ $f(z)$ is cont. at z_0

→ all polynomials, expontials, ant sinusoidal func^n cont.

Ex: is $f(z) = \bar{z}$ cont. for all z?

complex conjugaten func^n

$$f(z) = \bar{z} = \overline{x+iy} = x - iy$$

$u(x,y) = x$ $\qquad$ $v(x,y) = -y$

a) $f(z)$ is defined for all (x,y)

b) Does $\lim_{z \to z_0} f(z) = f(z_0)$ everywhere?

$$\lim_{z \to z_0} \bar{z} = \lim_{\substack{z \to z_0 \\ (x \to x_0 \\ y \to y_0)}} x - i \lim_{\substack{z \to z_0 \\ (x \to x_0 \\ y \to y_0)}} y$$

$$= x_0 - i y_0$$

$$= \overline{z_0}$$

$\boxed{\therefore f(z) = \bar{z} \text{ is cont. everywhere}}$

Discontiuities

Ex: $f(z) = \dfrac{z^2+4}{z(z-2i)}$

$f(z)$ is undefined at these points

Observe that $f(z)$ is discont. at $z=0$, $z=2i$

However, for $z \neq 2i$ and $z \neq 0$, we can write:

$$f(z) = \frac{(z+2i)(z-2i)}{z(z-2i)} = \frac{z+2i}{z}$$

$$\lim_{z \to 2i} \left(\frac{z+2i}{z}\right) = \frac{2i+2i}{2i} = 2$$

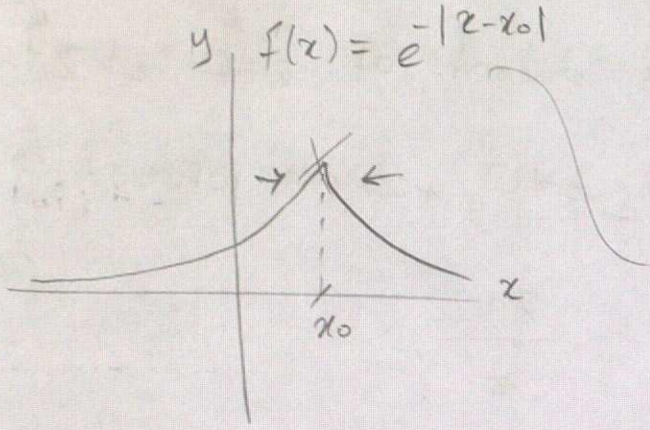
Now we "adjust" of func^n $f(z)$ as:

$$\boxed{f(z) = \begin{cases} \dfrac{z^2+4}{z(z-2i)} & z \neq 2i \\ 2 & z = 2i \end{cases}}$$

← continuous $f(z)$ for all z except $z=0$

- This is example of removable discontinuity (or hole)
- $z=0$ is not removable discontinuity

Derivative of Complex Func^n



$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$f'(x_0)$ DNE

If differen. then also cont.

Complex Derivative:

let $\Delta z = \Delta x + i\Delta y$

$$\boxed{f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}}$$

limit must exist for all possible paths for $f'(z_0)$ to exist

Ex: What is $f'(z)$ for $f(z) = z^2$

$$f'(z) = \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z) - f(z)}{\Delta z}\right] = \lim_{\Delta z \to 0}\left[\frac{(z+\Delta z)^2 - z^2}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0}\left[\frac{\cancel{z^2} + 2z\cancel{\Delta z} + \Delta z^{\cancel{2}} - \cancel{z^2}}{\cancel{\Delta z}}\right]$$

$$= \lim_{\Delta z \to 0}\left[2z + \Delta z\right] = \boxed{2z}$$

Week 8: Lecture 2 (17.5)

Oct 22, 2024

Derivatives of Complex Func^n

$f(z) = \bar{z}$

$$f'(z_0) = \lim_{\Delta z \to 0}\left[\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}\right]$$

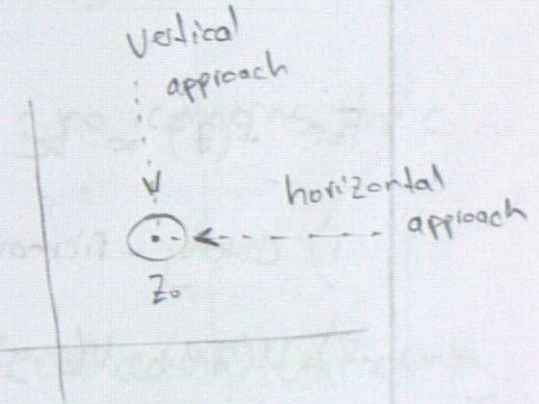
$$= \lim_{\Delta z \to 0}\left[\frac{\overline{z_0+\Delta z} - \bar{z}_0}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0}\left[\frac{\bar{z}_0 + \overline{\Delta z} - \bar{z}_0}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0}\left[\frac{\overline{\Delta z}}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0}\left[\frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}\right]$$

$$\boxed{\therefore f'(z_0) = \frac{d}{dz}(\bar{z}) \text{ does not exist}}$$



Path #1: horizontal approach ($\Delta z = \Delta x$, $\Delta x \to 0$, $\Delta y = 0$)

$$\therefore f'(z_0) = \lim_{\Delta x \to 0}\left[\frac{\Delta x}{\Delta x}\right] = \textcircled{1}$$

different limit

Path #2: vertical approach ($\Delta z = \Delta y$, $\Delta y \to 0$, $\Delta x = 0$)

$$\therefore f'(z_0) = \lim_{\Delta y \to 0}\left[\frac{-i\Delta y}{i\Delta y}\right] = \textcircled{-1}$$

Derivative of Complex Func$^{'r}$: $f(z) = u(x,y) + i\,v(x,y)$

Path #1 : horizon: $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

Path #2 : vertical: $f'(z) = \dfrac{\partial u}{\partial y} - i\dfrac{\partial v}{\partial y}$

Cauchy Riemann Equa^{ns}

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

For $f(z)$ to be diff. at z_0, then

1) cauchy-riemann equa^{r}s must be satisfied

2) $u(x,y)$ $v(x,y)$ and partials are continuous throughout some interval of z_0

$$\boxed{f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} = u_x + iv_x = v_y - iu_y$$

$$= u_x - iu_y = v_y + iv_x$$

Ex: find derivative of $f(z) = \sin z$

① Find u, v for $f(z) = \sin z = \underbrace{\sin x \cosh y}_{u} + i\underbrace{\cos x \sinh y}_{v}$

1) CR equa^{r}s: $\dfrac{\partial u}{\partial x} = \cos x \cosh y \overset{?}{\underset{\checkmark}{=}} \dfrac{\partial v}{\partial y} = \cos x \cosh y$

$\dfrac{\partial u}{\partial y} = \sin x \sinh y \overset{?}{\underset{\checkmark}{=}} -\dfrac{\partial v}{\partial x} = \sin x \sinh y$

the CR Equa^{r}s true for all x and all y $\Rightarrow$ ∴ for all z

2) Continuity of u, v and their partials : no concerns here (all are cont. for all z)

$$\therefore \frac{d}{dz}[\sin z] = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \cos x \cosh y - i \sin x \sinh y$$

$$= \cos z$$

True for all z

Ex: $f(z) = x^2 + iy^2$, $u(x,y) = x^2$ $v(x,y) = y^2$

1) $\frac{\partial u}{\partial x} = 2x \overset{?}{\underset{\times}{=}} \frac{\partial v}{\partial y} = 2y$ ⟵ note that these are equal if $x = y$

$\frac{\partial u}{\partial y} = 0 \overset{?}{\underset{\checkmark}{=}} -\frac{\partial v}{\partial x} = -0$

$\boxed{\therefore \text{ differentiable on } x = y}$

2) cont. everywhere

⟹ for $f'(z)$ to exist, the first CREqua^n must be satisfied, and that is the case when $x = y$

$\boxed{\therefore \text{ not analytic anywhere}}$

Analytic Functions

- more restrictive version of differentiability
- $f'(z)$ exists not only at z_0, but everywhere in neighborhood around z_0

→ if function is analytic, it can be expanded as a convergent power series

→ if $f(z)$ is analytic for all z, it is an entire function

→ if $f(z)$ is not analytic at z_0, then it is singular there

Ex: where is $f(z) = e^z$ analytic ?

$f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

⟹ $u(x,y) = e^x \cos y$ $v(x,y) = e^x \sin y$

1) do the CREqua^{n}s : $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x} = e^x \cos y \overset{?}{\underset{\checkmark}{=}} \frac{\partial v}{\partial y} = e^x \cos y$

$\frac{\partial u}{\partial y} = -e^x \sin y \overset{?}{\underset{\checkmark}{=}} -\frac{\partial v}{\partial x} = -e^x \sin y$

2) u, v, and all partial's cont. for all x, y so ∴ all z

$\boxed{\text{Since the above hypotheses are verified for all } z, \therefore f(z) \text{ is analytic everywhere}}$

e^z → differentiable everywhere → analytic everywhere

$\boxed{\therefore e^z \text{ is entire function}}$

Example top left: $f(z) = x^2 + iy^2$ is analytic nowhere

Beautiful Idea #1 : Harmonic Functions

Definition: A real-valued function is harmonic if it has second partial derivatives and satisfies

Laplace's Equation: $\nabla^2 f = 0 = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$

Theorem: If $f(z) = u(x,y) + iv(x,y)$ is analytic on a domain D, then the functions u and v are harmonic. You can say v is conjugate harm. fnc. to u

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Week 8: Lecture 3

Oct 24, 2024

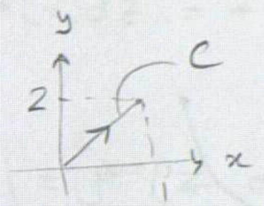
Contour Integration in Complex Plane

Real Domain Contour

$$\int_0^{\pi/2} \cos x \, dx \quad \text{and} \quad \iint x e^y \, dx dy$$

$\Rightarrow$ what if we have path C?

$\int_C xy^2 d\ell$ where C is line



parameterize C wrt t, let $0 \le t \le 1$

$$x(t) = t \quad y(t) = 2t \quad \vec{\ell}(t) = t\hat{i} + 2t\hat{j}$$

$$\Rightarrow \int_C xy^2 d\ell = \int_0^1 x(t)[y(t)]^2 \underbrace{d\ell} \qquad \rightarrow d\ell = |\dot{\vec{\ell}}(t)| dt$$

$$= \sqrt{5} dt$$

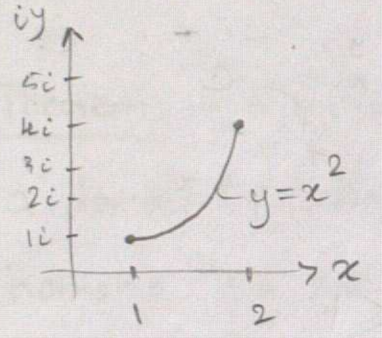
$$= \int_0^1 t(2t)^2 \sqrt{5} dt$$

$$= \sqrt{5}$$

Complex Domain Contour

$$\int_C z^2 dz, \quad C: y = x^2, \quad 1 \le x \le 2$$

1.1) Draw path in xy-plane



1.2) Parametrize

$1 \le t \le 2, \quad x(t) = t$
$\qquad\qquad y(t) = t^2$

$$z(t) = x + iy$$

① $z(t) = t + it^2$

(ii) $1 \le t \le 4: \; z(t) = \sqrt{t} + it$

2.) Integral Evaluation

$$\boxed{\int_C f(z)dz = \int_C f[z(t)]\frac{dz}{dt}dt = \int_C f[z(t)]\, z'(t)dt}$$

① $1 \le t \le 2: \; \int_C z^2 dz = \int_1^2 \underbrace{(t+it^2)^2}_{f[z(t)]}\underbrace{(1+2ti)}_{z'(t)}dt$

$$= -\frac{86}{3} - 6i$$

(ii) $1 \le t \le 4: \; \int_C z^2 dz = \int_1^4 (\sqrt{t}+it)^2\left(\frac{1}{2\sqrt{t}}+i\right)dt$

$$= -\frac{86}{3} - 6i$$

Properties of Contour Integrals

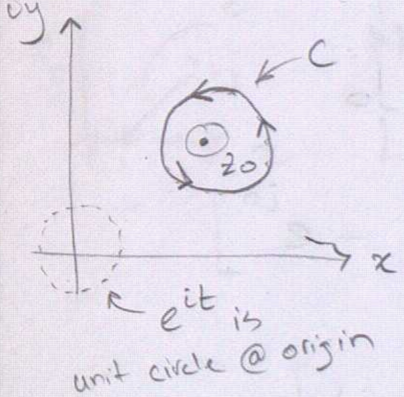
1) Linearity: $\int_C [\alpha f(z) + \beta g(z)]dz = \alpha\int_C f(z)\overset{dz}{{}_{\wedge}} + \beta\int_C g(z)dz$

2) Path Decomposition: $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$

3) Reversal of path orientation: $\int_{-C} f(z)dz = -\int_C f(z)dz$

Ex: $\int_C (z-z_0)^n$ with $C: \underbrace{|z-z_0| = \rho}$ ← orient CCW

an open circle of radius ρ



$$C: \; z(t) = z_0 + \rho e^{it}$$
$$0 \le t \le 2\pi$$

e^{it} scaled by ρ, moved up to z_0

$$f[z(t)] = (\overset{z(t)}{z} - z_0)^n = (\cancel{z_0} + \rho e^{it} - \cancel{z_0})^n = (\rho e^{it})^n = \rho^n e^{int}$$

$$z'(t) = i\rho e^{it}$$

$$\oint_C f(z)dz = \int_0^{2\pi}\underbrace{(\rho^n e^{int})}_{f[z(t)]}\underbrace{(i\rho e^{it})}_{z'(t)}dt = i\rho^{n+1}\int_0^{2\pi} e^{i(n+1)t}dt$$

For $n=-1$

$$\circlearrowleft \oint_C f(z)\,dz = \oint_C \frac{dz}{(z-z_0)} = i\overbrace{\rho^{(-1+1)}}^{\rho^0 = 1} \int_0^{2\pi} e^{\overbrace{i(-1+1)t}^{e^0=1}}\,dt$$

$$= i\int_0^{2\pi} = \boxed{2\pi i}$$

For $n \neq -1$

$$\circlearrowleft \oint_C (z-z_0)^n\,dz = i\rho^{(n+1)} \int_0^{2\pi} e^{i(n+1)t}\,dt$$

$$= \cancel{i}\rho^{(n+1)} \left[\frac{e^{i(n+1)t}}{i(n+1)} \right]_0^{2\pi}$$

$$= \frac{i\rho^{(n+1)}}{n+1} \left[e^{i(n+1)(2\pi)} - e^{i0} \right]$$

$$= \frac{i\rho^{(n+1)}}{n+1} \left[\underbrace{e^{i2\pi(n+1)}}_{\underbrace{\cos[2\pi(n+1)]}_{1} + i\underbrace{\sin[2\pi(n+1)]}_{0}} - 1 \right]$$

$$= \frac{\rho^{(n+1)}}{n+1} [1-1] = \boxed{0}$$

$$\boxed{\circlearrowleft \oint_C (z-z_0)^n \quad C: |z-z_0| = \rho \quad = \begin{cases} 2\pi i, & n=-1 \\ 0, & n \neq -1 \end{cases}}$$

← very important integral

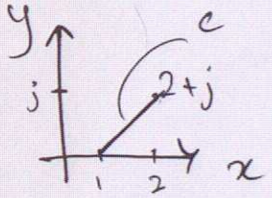
A Bound for a Contour Integral

If f is cont. on smooth curve C and if $|f(z)| \leq M$ for all z on C, then

$$\boxed{\left| \int_C f(z)\,dz \right| \leq ML}$$

← "L" is the length of C

Ex: Find upper bound of $\left| \int_C \frac{1}{z}\,dz \right|$, C is line that connects $z=1$ to $z=2+j$



$$M = \max_{\text{on } C} \left(\frac{1}{z}\right) = 1 \leftarrow \text{at } \underline{\underline{z=1}}$$

$$L = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\boxed{\therefore \left| \int_C \frac{1}{z}\,dz \right| \leq \sqrt{2}}$$

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32, 33, 36, 38, 40, 41, 44, 45, 47ac

Section 17.6 : ~~1~~, 6, 11, 12, ~~15~~, 16, 17, 18, 20, 24, 28

✓1. $e^{\frac{\pi}{6}i} = e^{1}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$$\boxed{= \frac{\sqrt{3}}{2} + i\frac{1}{2}}$$

• 6. $e^{\left(-\pi + \frac{3\pi}{2}i\right)} = e^{-\pi}\left(\cancelto{0}{\cos\frac{3\pi}{2}} + i\cancelto{-1}{\sin\frac{3\pi}{2}}\right)$

$$\boxed{= -ie^{-\pi}}$$

R 11. $\dfrac{e^{1+\frac{5\pi i}{4}}}{e^{-1-\frac{\pi}{3}i}} = e^{2+\frac{19\pi}{4}} = e^{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$1 + \frac{5\pi i}{4} + 1 + \frac{\pi}{3}i = 2 + i\left(\frac{15\pi}{4} + \frac{4\pi}{4}\right) = 2 + \frac{19\pi}{4}$

$\frac{19\pi}{4} \Rightarrow \frac{3\pi}{4}$ $\quad = e^{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \boxed{-\frac{e^{2}}{\sqrt{2}} + i\frac{e^{2}}{\sqrt{2}}}$

• 12. $\dfrac{e^{2+3\pi i}}{e^{-3+\frac{\pi}{2}i}}$

$2 + 3\pi i + 3 - \frac{\pi}{2}i$

$= 5 + i\left(3\pi - \frac{\pi}{2}\right)$

$= 5 + i\left(\frac{6\pi - \pi}{2}\right) = 5 + \frac{5\pi}{2}i$

$\frac{5\pi}{2} \Rightarrow \frac{\pi}{2}$

$e^{5+\frac{5\pi}{2}i} = e^{5}\left(\cancelto{0}{\cos\frac{\pi}{2}} + i\cancelto{1}{\sin\frac{\pi}{2}}\right) = \boxed{ie^{5}}$

✓15. $f(z) = e^{z^2} = e^{(x+iy)^2} = e^{x^2+2xyi+i^2y^2} = e^{x^2+2ixy-y^2} = e^{x^2-y^2+2xyi}$

$= e^{x^2-y^2}\left(\cos 2xy + i\sin 2xy\right)$

$f(z) = \underbrace{e^{x^2-y^2}\cdot\cos 2xy}_{u} + i\,\underbrace{e^{x^2-y^2}\cdot\sin 2xy}_{v}$

P · 16. $f(z) = e^{\frac{1}{z}} = e^{\frac{1}{x+iy}}$

$$\frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y^2}{x^2+y^2}$$

$$f(z) = e^{\frac{x}{x^2+y^2} - i\frac{y^2}{x^2+y^2}} = \frac{e^{\frac{x}{x^2+y^2}}}{e^{i\frac{y^2}{x^2+y^2}}}$$

· 17. $|e^z| = e^x$

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = \underbrace{e^x}_{|e^x| = e^x} \underbrace{(\cos y + i\sin y)}_{|\cos y + i\sin y| = 1}$$

$$\boxed{\therefore |e^z| = e^x}$$

· 18. $\frac{e^{z_1}}{e^{z_2}} = \frac{e^{x_1+iy_1}}{e^{x_2+iy_2}} = e^{x_1-x_2+i(y_1-y_2)} = e^{z_1-z_2}$

· 20. $(e^z)^n = (e^{x+iy})^n = e^{xn+iyn} = e^{n(x+iy)} = e^{nz}$

· 24. $\ln(z) = \ln(-ei) = \log_e(e) + \text{Arg}(-ei) + 2\pi n$

$$= 1 + -\frac{\pi}{2}i + 2\pi n \quad (n = 0, \pm 1, \pm 2, \cdots)$$

· 28. $\ln(-\sqrt{3}+i) = \ln(2) + i\left[\frac{5\pi}{6} + 2\pi n\right] \quad (n = 0, \pm 1, \pm 2, \cdots)$

$$r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$

$$\theta = \frac{5\pi}{6}$$

32. $z = 3 - 4i$, $|z| = 5$, $\text{Arg}(z) = -53.13^\circ$

$$\text{Ln}(z) = 5 + i[-53.13 + 2\pi n] \quad n = 0$$

$$= \log_e(5) - 53.13i$$

☆ 33. $z = (1 + i\sqrt{3})^5 \qquad |z| = 2, \ \text{Arg}(z) = \frac{\pi}{3}$

$$\text{Ln}((1+i\sqrt{3})^5) = 5\,\text{Ln}(1+i\sqrt{3}) = 5\left(\ln 2 + \frac{\pi}{3}i\right) = \cancel{3.466 + \frac{5\pi}{3}i}$$

$$z^5 = 2^5\left(\cos 5\frac{\pi}{3} + i\sin\frac{5\pi}{3}\right) = 2^5\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 16 - i16\sqrt{3}$$

$$\text{Ln}(16 - i16\sqrt{3}) \qquad r = 32, \ \theta = -\frac{\pi}{3}$$

$$= \log_e(32) + i\left[-\frac{\pi}{3}\right] = 3.466 - \frac{\pi}{3}i$$

36. $e^{\frac{1}{z}} = -1 \qquad r = 1, \ \theta = \pi$

$$\frac{1}{z} = \ln(-1) = \ln(1) + i(\pi + 2\pi n), \ n = 0, \pm 1, \pm 2, \cdots$$

$$\boxed{\therefore z = \frac{1}{i(\pi + 2n\pi)}, \ n = 0, \pm 1, \pm 2, \cdots}$$

38. $e^{2z} + e^z + 1 = 0 \implies (e^z)^2 + (e^z) + 1 = 0 \qquad \text{Let } e^z = x$

$$x^2 + x + 1 = 0 \qquad x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \sqrt{(-1)}\sqrt{(3)}$$

$$x = -\frac{1}{2} \pm i\sqrt{3}$$

$$r = \frac{\sqrt{13}}{2}, \ \theta = 106^\circ$$

① $e^z = -\frac{1}{2} + i\sqrt{3}$

$$\boxed{\begin{aligned} z &= \ln\left(-\tfrac{1}{2} + i\sqrt{3}\right) = \ln\left(\tfrac{\sqrt{13}}{2}\right) + i(106^\circ + 360^\circ n) \quad n = 0, \pm 1, \pm 2, \cdots \\ z &= \ln\left(-\tfrac{1}{2} - i\sqrt{3}\right) = \ln\left(\tfrac{\sqrt{13}}{2}\right) + i(-106^\circ + 360^\circ n) \quad n = 0, \pm 1, \pm 2, \cdots \end{aligned}}$$

40. $3^{\frac{i}{\pi}} = e^{\frac{i}{\pi}\ln 3}$

$\ln(3) = \ln(3) + i(0^{\circ} + 2\pi n)$

$3^{\frac{i}{\pi}} = e^{\frac{i}{\pi}(\ln 3 + 2\pi i n)} = \boxed{\dfrac{e^{i\frac{\ln 3}{\pi}}}{e^{2n}}}$ $\quad n = 0, \pm 1, \pm 2, \cdots$

$\frac{i}{\pi}\ln 3 + \frac{2\cancel{\pi} i^2 n}{\cancel{\pi}}$

$= \frac{i\ln 3}{\pi} - \frac{2\pi}{\pi} n =$

☆ 41. $(1+i)^{(1+i)} = e^{(1+i)\ln(1+i)}$

$\ln(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi n\right)$

$(1+i)\left(\ln\sqrt{2} + \frac{\pi}{4}i + 2\pi i n\right) = \ln\sqrt{2} + \frac{\pi}{4}i + 2\pi i n + i\ln\sqrt{2} + \frac{\pi i^2}{4} + 2\pi i^2 n$

$= \ln\sqrt{2} + \frac{\pi}{4}i + 2\pi i n + i\ln\sqrt{2} - \frac{\pi}{4} - 2\pi n$

$= \left(\ln\sqrt{2} - \frac{\pi}{4} - 2\pi n\right) + i\left(\frac{\pi}{4} + 2\pi n + \ln\sqrt{2}\right)$

$(1+i)^{(1+i)} = e^{\left(\ln\sqrt{2} - \frac{\pi}{4} - 2\pi n\right)} \cdot e^{i\left(\ln\sqrt{2} + \frac{\pi}{4} + 2\pi n\right)}$ $\quad n = 0, \pm 1, \pm 2, \cdots$

$=$

44. $(1-i)^{2i} = e^{2i\,\mathrm{Ln}(1-i)}$ $\quad r = \sqrt{2},\ \theta = -\frac{\pi}{4}$

$\mathrm{Ln}(1-i) = \ln\sqrt{2} + i\left(-\frac{\pi}{4}\right) = \ln\sqrt{2} - i\frac{\pi}{4}$

$(2i)\left(\ln\sqrt{2} - i\frac{\pi}{4}\right) = 2i\ln\sqrt{2} - 2i^2\frac{\pi}{4} = 2i\ln\sqrt{2} + \frac{\pi}{2}$

$(1-i)^{2i} = e^{\frac{\pi}{2}}\left(\cos(2\ln\sqrt{2}) + i\sin(2\ln\sqrt{2})\right)$

$= \boxed{e^{\frac{\pi}{2}}\cos(2\ln\sqrt{2}) + i e^{\frac{\pi}{2}}\sin(2\ln\sqrt{2})}$

45. $z_1 = i,\ r = 1,\ \theta = \frac{\pi}{2}$ $\quad z_1 z_2 = \sqrt{2}\,\angle\frac{\pi}{2} + \frac{3\pi}{4} = \sqrt{2}\,\angle\frac{5\pi}{4}$

$z_2 = -1+i,\ r = \sqrt{2},\ \theta = \frac{3\pi}{4}$

$\mathrm{Ln}(z_1 z_2) = \log_e(\sqrt{2}) + i\left(-\frac{3\pi}{4}\right) = \ln\sqrt{2} - i\frac{3\pi}{4}$

$\mathrm{Ln}(z_1) = \ln(1) + i\left(\frac{\pi}{2}\right) = \ln 1 + i\frac{\pi}{2}$

$\mathrm{Ln}(z_2) = \ln(\sqrt{2}) + i\left(\frac{3\pi}{4}\right) = \ln\sqrt{2} + i\frac{3\pi}{4}$

$\mathrm{Ln}(z_1) + \mathrm{Ln}(z_2) = (\ln 1 + \ln\sqrt{2}) + i\left(\frac{\pi}{2} + \frac{3\pi}{4}\right) \neq \mathrm{Ln}(z_1 z_2)$ //

47.a) Just showing that power rule for logs don't work

48. $(z^{\alpha})^n = (z^{n\alpha})$ does not hold if n is complex

Proofs

✓ ✓ ✓ ✓ ✓ ✓

Section 17.7 : 2, 3, 4, 12, 14, 15, 19, 21, 23, 27, 28, 31, 32

2. $\sin(-2i) = \dfrac{e^{+i(-2i)} - e^{-i(-2i)}}{2i}$ ← too hard

$-2i = 0 - 2i \Rightarrow x = 0,\ y = -2$

$\sin(-2i) = \cancel{\sin(0)}\overset{\to 0}{\cosh(-2)} + i\cos(0)\sinh(-2)$

$= i\sinh(-2)$

3. $\sin\left(\frac{\pi}{4} + i\right) \Rightarrow x = \frac{\pi}{4},\ y = 1$

$= \sin\left(\frac{\pi}{4}\right)\cosh(1) + i\cos\left(\frac{\pi}{4}\right)\sinh(1)$

$= \dfrac{\cosh(1)}{\sqrt{2}} + i\,\dfrac{\sinh(1)}{\sqrt{2}}$

4. $\cos(2-4i) \Rightarrow x = 2,\ y = -4$

$\cos(2-4i) = \boxed{\cos(2)\cosh(-4) - i\sin(2)\sinh(-4)}$

12. $\cosh(2+3i) = \cos(i(2+3i))$

$i(2+3i) = 2i + 3i^2 = -3 + 2i \Rightarrow x = -3,\ y = 2$

$\cos(-3+2i) = \boxed{\cos(-3)\cosh(2) - i\sin(-3)\sinh(2)}$

14. $\cos\left(\frac{\pi}{2} + i\ln(2)\right) = -\frac{3}{4}i$

$x = \pi/4$

$y = \ln 2 = \ln 2 + 0 = \ln 2$

$= \cos\left(\frac{\pi}{4}\right)\cosh(\ln 2) - i\sin\left(\frac{\pi}{4}\right)\sinh(\ln 2)$

$= \dfrac{\cosh(\ln 2)}{\sqrt{2}} - i\,\dfrac{\sinh(\ln 2)}{\sqrt{2}} \; \therefore\Rightarrow$

$\cosh(\ln 2) = \dfrac{e^{\ln 2} + e^{-\ln 2}}{2} = \dfrac{2 + \frac{1}{2}}{2} = 1 + \frac{1}{4} = \frac{5}{4}$

$\sinh(\ln 2) = \dfrac{e^{\ln 2} - e^{-\ln 2}}{2} = \dfrac{2 - \frac{1}{2}}{2} = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore\Rightarrow \dfrac{5}{4\sqrt{2}} - i\,\dfrac{3}{4\sqrt{2}} = \dfrac{1}{4\sqrt{2}}(5 - 3i)$

$\cos\left(\frac{\pi}{2} + i\ln 2\right) = \dfrac{e^{i(\frac{\pi}{2} + i\ln 2)} + e^{-i(\frac{\pi}{2} + i\ln 2)}}{2}$

$i\left(\frac{\pi}{2} + i\ln 2\right) = \frac{\pi}{2}i - \ln 2$

$-i\left(\frac{\pi}{2} + i\ln 2\right) = -\frac{\pi}{2}i - i^2\ln 2 = -\frac{\pi}{2}i + \ln 2$

$= \dfrac{e^{-\ln 2 + \frac{\pi}{2}i} + e^{\ln 2 - \frac{\pi}{2}i}}{2} = \dfrac{e^{-\ln 2}e^{\frac{\pi}{2}i} + \frac{e^{\ln 2}}{e^{\pi/2 i}}}{2} = \dfrac{\frac{e^{\frac{\pi}{2}i}}{2} + \frac{2}{e^{\pi/2 i}}}{2}$

$= \dfrac{e^{\frac{\pi}{2}i}}{4} + \dfrac{1}{e^{\frac{\pi}{2}i}} = \dfrac{\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}}{4} + \dfrac{1}{\cos\frac{\pi}{4} + i\sin\frac{\pi}{2}}$

$= \dfrac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}{4} + \dfrac{1}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i} = \dfrac{1}{4\sqrt{2}} + \dfrac{1}{4\sqrt{2}}i +$

$= \cos\left(\frac{\pi}{12}\right)\cosh(\ln 2)$ // I used the wrong angle

$\boxed{\therefore = -\frac{3}{4}i}$

15. P

$\sin z = 2$

$$2 = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$4i = e^{ix-y} - e^{-ix+y}$$

$$4i = \frac{e^{ix}}{e^{y}} - \frac{e^{y}}{e^{ix}}$$

$$2 = \frac{e^{iz} - e^{-iz}}{2i}$$

$$e^{iz}(4i) = (e^{iz} - e^{-iz})e^{iz}$$

$$4ie^{iz} = e^{2iz} - 1$$

$$-e^{2iz} + 4ie^{iz} + 1 = 0$$

Let $a = e^{iz}$

$$-(e^{iz})^2 + 4i(e^{iz}) + 1 = 0$$

$$\sqrt{-12} = \sqrt{(-1)}\sqrt{(12} = \sqrt{12}\,i$$

$$-a^2 + 4ia + 1 = 0$$

$$a = \frac{-4i \pm \sqrt{(4i)^2 - 4(-1)(1)}}{-2} = \frac{-4i \pm \sqrt{-16+4}}{-2} = \frac{-4i \pm \sqrt{-12}}{-2}$$

$$= \frac{-4i \pm \sqrt{12}\,i}{-2} = 2i \pm \frac{\sqrt{12}}{-2}i \left\langle \begin{matrix} -2i + \frac{\sqrt{12}}{2}i \\ -2i - \frac{\sqrt{12}}{2}i \end{matrix} \right.$$

$$e^{iz} = i\left(2 \pm \frac{\sqrt{12}}{2}\right)$$

$$r = \sqrt{(2)^2 + \left(\frac{\sqrt{12}}{2}\right)^2} = \sqrt{7}$$

$$iz = \ln\left(i\left(2 \pm \frac{\sqrt{12}}{2}\right)\right)$$

$$\theta \left\langle \begin{matrix} 2 + \frac{\sqrt{12}}{2} \rightarrow > 0 \\ 2 - \frac{\sqrt{12}}{2} \rightarrow > 0 \end{matrix} \right\} 0^\circ$$

$$z = \frac{1}{i} \cdot \ln\left[\left(2 \pm \frac{\sqrt{12}}{2}\right)i\right]$$

$$= \frac{1}{i} \cdot \left(\log_e(\sqrt{7}) + i(0 + 2\pi n)\right)$$

$$= \frac{\log_e(\sqrt{7})}{i} + \left(-\frac{\pi}{2} + 2\pi n\right), \quad n = 0, \pm1, \pm2, \cdots$$

19.

$\cos z = \sin z$

$$\frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iz} - e^{-iz}}{2i}$$

$$i(e^{iz} + e^{-iz})e^{iz} = e^{iz}(e^{iz} - e^{-iz})$$

$$i(e^{2iz} + 1) = e^{2iz} - 1$$

$$ie^{2iz} + i = e^{2iz} - 1$$

$$ie^{2iz} - e^{2iz} + i + 1 = 0$$

$$e^{2iz}(i-1) = -i-1$$

$$e^{2iz} = \frac{-i-1}{i-1} \cdot \frac{i+1}{i+1} = \frac{-i^2-i-i-1}{-1-1}$$

$$e^{2iz} = \frac{-2i}{-2} = i$$

$$2iz = \ln(i) = \ln(1) + i\left(\frac{\pi}{2} + 2\pi n\right)$$

$$z = \frac{\underset{0}{\log_e(1)} + \cancel{i}\left(\frac{\pi}{2} + 2\pi n\right)}{2\cancel{i}}$$

$$\boxed{z = \frac{\pi}{4} + \pi n}$$

$n = 0, \pm1, \pm2, \cdots$

21. P

$\cos z = \cosh(2)$

$$\frac{e^{iz} + e^{-iz}}{2} = \frac{e^{2} + e^{-2}}{2}$$

$$e^{iz} + e^{-iz} = e^{2} + e^{-2}$$

$$e^{2iz} + 1 = e^{2+iz} + e^{-2+iz}$$

$$e^{2iz} - e^{2+iz} - e^{-2+iz} = -1$$

$$e^{2iz} - e^{2}e^{iz} - e^{-2}e^{iz} = -1$$

$$e^{2iz} - e^{iz}(e^{2} + e^{-2}) = -1$$

$$e^{2iz} - (e^{2} + e^{-2})e^{iz} + 1 = 0$$

$$(e^2 + e^{-2})^2$$

$$(e^2)^2 + 2(e^2)(e^{-2}) + (e^{-2})^2$$

$$= e^4 + 2 + e^{-4}$$

Let $a = e^{iz} \Longrightarrow a^2 - (e^2 + e^{-2})a + 1 = 0$

$$a = \frac{(e^2 + e^{-2}) \pm \sqrt{(e^2 + e^{-2})^2 - 4(1)(1)}}{2} = \frac{(e^2 + e^{-2}) \pm \sqrt{e^4 + e^{-4} - 2}}{2}$$

$$= \frac{e^2}{2} + \frac{e^{-2}}{2} \pm \frac{\sqrt{e^4 + e^{-4} - 2}}{2}$$

$$z = \frac{1}{i} \cdot \ln\left(\frac{e^2}{2} + \frac{e^{-2}}{2} \pm \frac{\sqrt{e^4 + e^{-4} - 2}}{2}\right)$$

Review (pg. 862) : 3, 9, 10, 15, 18

✓ 3. $z = 3+4i, \; Re\left(\frac{z_1}{\bar{z}_1}\right) = ?$

$$\boxed{\therefore Re(z_1/\bar{z}_1) = -7/25}$$

$$\frac{3+4i}{3-4i} \cdot \frac{3+4i}{3+4i} = \frac{9+2(3)(4i)\overset{-16}{+16i^2}}{9+16} = \frac{-7+24i}{25} = -\frac{7}{25} + \frac{24i}{25}$$

✓ 9. $e^z = 2i \Rightarrow z = \ln(2i), \; r=2, \theta = \frac{\pi}{2}$

$$\boxed{z = \log_e(2) + i\left(\frac{\pi}{2} + 2\pi n\right), \; n = 0, \pm1, \pm2, \dots}$$

10. $e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x \underbrace{(\cos y + i\sin y)}_{\text{magnitude} = 1}$

$$\boxed{\therefore \text{False, counter is } z = 0}$$

$\therefore |e^z| = |e^x|$

$|e^x| = 1 \Rightarrow x = 0 \leftarrow Re(z)$ must $= 0$, but $Im(z)$ can take any value, making z purely imaginary

✓15. $Ln(-ie^3), \; |z| = e^3, \; \theta = -\frac{\pi}{2}$

$\therefore Ln(-ie^3) = \log_e(e^3) + i\left(-\frac{\pi}{2}\right)$

$$\boxed{= 3 - i\frac{\pi}{2}}$$

18. $$\frac{3-i}{2+3i} + \frac{2-2i}{1+5i} = \frac{(3-i)(1+5i) + (2-2i)(2+3i)}{2+10i+3i+15i^2} = \frac{18+16i}{-13+13i} \cdot \frac{-13-13i}{-13-13i}$$

$3+15i-i\overset{+5}{-i^2 5} = 8+14i$

$4+6i-4i\overset{+6}{-i^2 6} = 10+2i$

$\} + = 18+16i$

$$\boxed{\frac{-26-442i}{338}}$$

$-13+13i \; \{ -234-234i-208i \overset{+208}{-208i^2}$

$-26-442$

Week 9: Lecture 1
Sec 18.1, MAT291, ECE221

Line Integrals, Circulation, Net flux — Nov 4, 2024

Line Integration with Vector Fields

Work $= \int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (P(x,y), Q(x,y)) = \vec{F}(x,y)$

and C given by $\vec{r}(t)$

$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} = \langle x(t), y(t) \rangle$

from $a \le t \le b$

$\vec{r}\,'(t) = \langle x'(t), y'(t) \rangle$

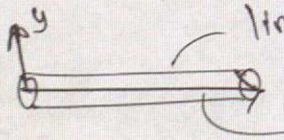
$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$\downarrow$$

$$= \int_a^b P(x,y)\,x'(t)\,dt + \int_a^b Q(x,y)\,y'(t)\,dt$$

$$= \int_C \underbrace{P\,dx}_{dx = x'(t)dt} + Q\,\underbrace{dy}_{dy = y'(t)dt}$$

Line Integration with Scalar Fields



linear density $= \rho_0$ [=] Ω/m

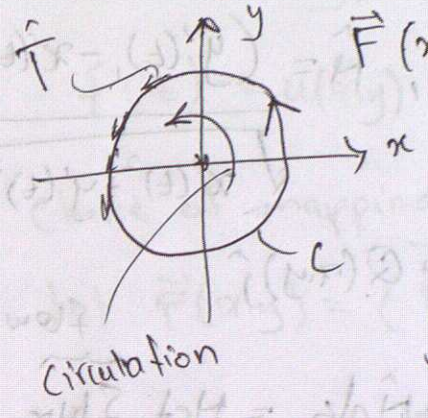
$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

Total Resistance $= R = \int \rho_0 \overset{\frown}{ds}$ $\quad |\vec{r}\,'(t)|dt = \sqrt{x'(t)^2 + y'(t)^2}$

$$= \int_a^b \rho_0 \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_a^b \rho_0 |\vec{r}\,'(t)|\,dt$$

Circulation

- line integral of the tangential component over a closed curve



$\vec{F}(x,y) = \langle -y, 0 \rangle = P(x,y)\hat{i} + Q(x,y)\hat{j}$

"curl" is the circulation per area

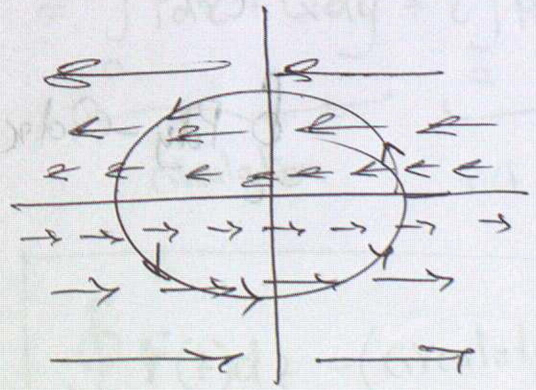
$$\oint_C \vec{F} \cdot d\vec{r} = \text{circulation} = \underbrace{\int_a^b \vec{F}(r(t)) \cdot \vec{r}\,'(t)\,dt}$$

$\hat{T} = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$ ← unit tangent vector

$\vec{r}\,'(t) = \hat{T}\,|\vec{r}\,'(t)|$

$$= \int_a^b \vec{F}(r(t)) \cdot T\,\underbrace{|\vec{r}\,'(t)|\,dt}_{ds}$$

$$\therefore \text{Circulation} = \oint_C \vec{F} \cdot \hat{T}\,ds \; \boxed{= \oint_C P\,dx + Q\,dy}$$



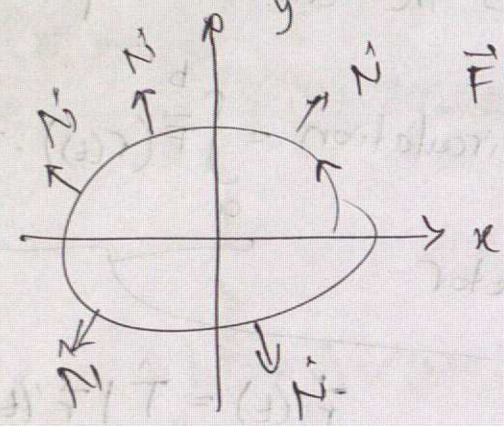
circulation $= \oint_C \vec{F} \cdot \hat{T} ds \boxed{> 0}$

$\vec{\nabla} \times \vec{F}$

Net Flux

- the line integral of the Normal Component over a closed curve

$\hat{T}\cdot\hat{N} = 0$, $\hat{T} = \frac{(x'(t), y'(t))}{\sqrt{x'(t)^2 + y'(t)^2}}$, $\hat{N} = \frac{(y'(t), -x'(t))}{\sqrt{x'(t)^2 + y'(t)^2}}$

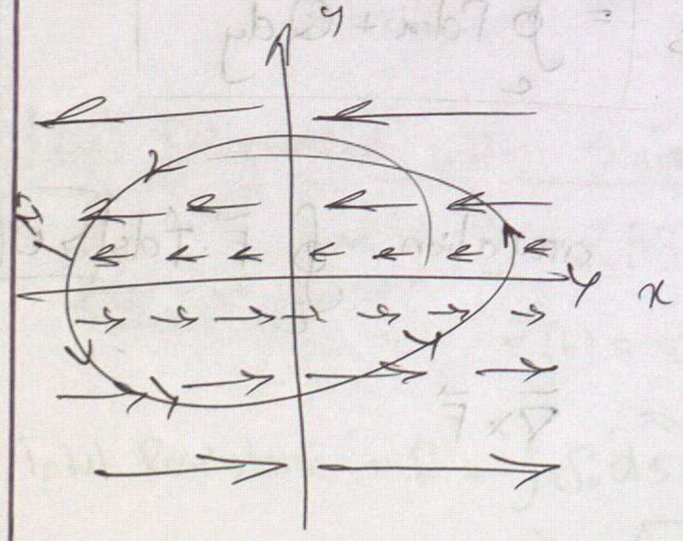


$\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$

$$\oint_C \vec{F}\cdot\hat{N}\,ds = \text{Net } \overbrace{\text{flux}}^{\text{flow}}$$

outward

$$\oint_C \vec{F}\cdot\hat{N}\,ds = \oint_C P\,dy - Q\,dx$$



Net outward flux $= \oint_C \vec{F}\cdot\hat{N}\,ds = 0$

$$= \oint_C P\,dy - Q\,dx \boxed{= 0}$$

Relantionship with Complex Contour Integrals

Let ~~P~~ $\vec{F}(x,y) = (P, Q) = (u(x,y), v(x,y))$

$f(z) = u(x,y) + i\,v(x,y)$

Choice of mapping s.t. $P \to u$, $Q \to -v$

Let $\vec{F}(x,y) = (P, Q) = (u(x,y), -v(x,y))$

$$\oint_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt$$

$$= \int_a^b \underbrace{\left(u(x,y) + i\,\overbrace{v(x,y)}^{v(x(t),y(t))}\right)}_{F[z(t)]} \underbrace{\left(x'(t) + i y'(t)\right)}_{z'(t)} dt$$

$(i^2 = -1)$

$$= \int_a^b u x'\,dt - v y'\,dt + i\int_a^b u y'\,dt + v x'\,dt$$

$$= \underbrace{\int_a^b P\,dx + Q\,dy}_{\text{circulation}} + i\underbrace{\int_a^b P\,dy - Q\,dx}_{\text{net outward flux}}$$

using $P = u$ and $Q = -v$

textbook: $P = u$, $Q = v$
circulation: $\mathrm{Re}\left\{\oint F(z)\,dz\right\}$

$$\boxed{\oint_C F(z)\,dz = (\text{circulation}) + i\,(\text{net outward flux})}$$

Ex: $\vec{F} = (-y, 0) = (P, Q)$

$P = -y = u$ $\quad f(z) = P - iQ = -y$

$Q = 0 = -v$

$\oint_C f(z)dz$ $\quad$ C parametrized by $z(t) = e^{it}$

$z(t) = (\cos t, \sin t)$ $\quad 0 \leq t \leq 2\pi$

$$\oint_C = f(z)dz = \int_0^{2\pi} f[z(t)]\, z'(t)dt$$

$$= \int_0^{2\pi} (-\sin t)(-\sin t + i\cos t)dt$$

$$= -i(0) + \pi$$

flux is zero (imaginary part of this integral)

circulation (real part of this integral)

Week 9: Lecture 2

Nov 5, 2024

Ex: $D = \frac{\rho_e}{2\pi}\left(\frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j}\right) = (P, Q) = \underbrace{(u, -v)}_{f(z)}$

find net flux and circulation of unit ~~sphere~~ circle

$$f(z) = (P, -Q) = \frac{\rho_e}{2\pi}\left(\frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}\right)$$

$$\oint_C f(z)dz = \int f[z(t)]\, z'(t)dt$$

sphere parametrization: $|z + i| = 1$ a circle e^{it}

$z + i = e^{it}$

$Z(t) = -i + e^{it}$

$z(t) = e^{it} = x(t) + iy(t)$ $\quad \rightarrow z' = ie^{it}$

$= \cos(t) + i\sin(t)$

$$= \int_0^{2\pi} \frac{\rho_e}{2\pi}\left(\frac{\cos t}{\cos^2 t + \sin^2 t} - i\frac{\sin t}{\cos^2 t + \sin^2 t}\right)(ie^{it})dt$$

$$= \int_0^{2\pi} i\frac{\rho_e}{2\pi}\left(\cos t\, e^{it} - i\sin t\, e^{it}\right)dt$$

$$= \underbrace{i\rho_e}_{\text{this it is the flux}} + 0$$

there is no circulation

$$\boxed{\iiint_{Vol} \vec{\nabla}\cdot\vec{D}\, dV = \oiint_{Area} \vec{D}\cdot d\vec{A}} \rightarrow = Q_{enclosed}$$

Divergence theorem

Vector Fields with Analytic Functions

If $f(z)$ is analytic $\longrightarrow \vec{F}(u,-v) = $ (P,Q)

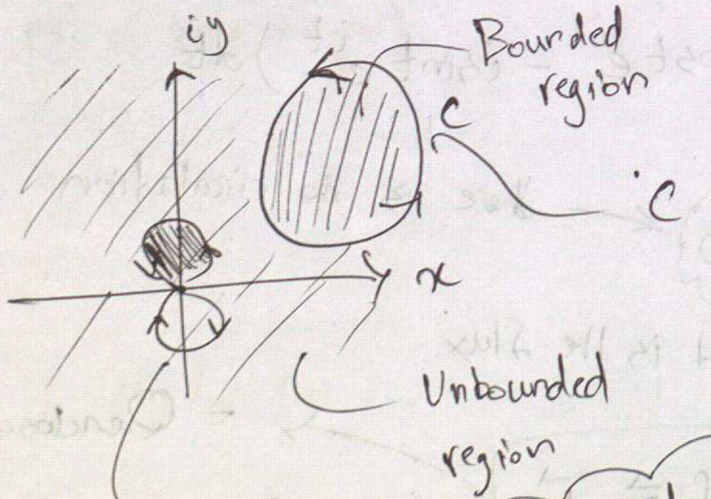
$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

if we know $f(z)$ is analytic, then C-RE's are satisfied

Analytic functions have zero curl and divergence in real word

Analaticity means a curl-free and divergence-less vector fields $\vec{F}(u,-v)$ in real world

Simple Closed Contour



bounded (interior) region is always on your L.H.S

ccw is (+)ve orientation

C is a simple closed contour

as I walk around, my interior is not a bounded region ∴ not simple closed contour

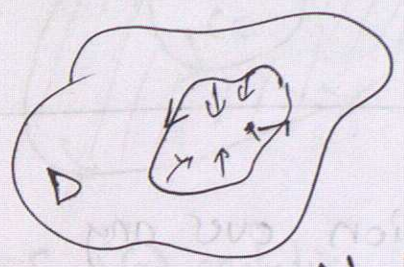
Cauchy - Goursal Integral Theorem and deformation of Contours

Section 18.2

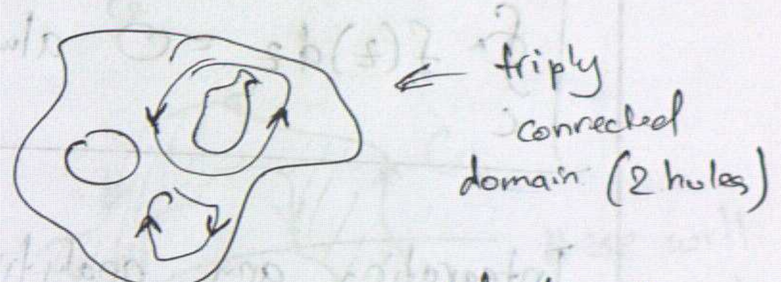
Connected Domains

a simply connected domain D is one where every simple closed contour can be shrunk to a point without leaving D

AKA $\Longrightarrow$ NO HOLES

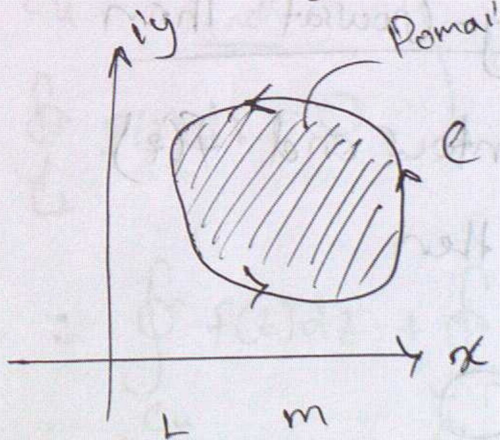


simply connected domain



multiply connected domain

Contour Integration Over Simple Closed Curve



$f(z) = u(x,y) + i\,v(x,y)$

$dz = dx + i\,dy$

$$\oint_C f(z)\,dz = \oint_C \left[u(x,y) + i v(x,y)\right](dx + i dy)$$

$$\Rightarrow \oint_C \overset{L}{u\,dx} - \overset{m}{v\,dy} + i\oint_C \overset{m}{u\,dy} + \overset{L}{v\,dx}$$

Green's Theorem

$$\oint_C L\,dx + m\,dy$$

$$= \iint_D \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy \xrightarrow{} 0 \;-\; i\iint_D \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)dx\,dy \xrightarrow{} 0 \;=\; \iint_D \left(\frac{\partial m}{\partial x} - \frac{\partial L}{\partial y}\right)dx\,dy$$

if function analytic

∴ $\oint_C f(z)dz = 0$ if $f(z)$ is analytic on and within D and C is a simple closed contour, then

$\oint_C f(z)dz = 0$ always

Integrating any analytic function over any simple closed contour will always give you 0

Beautiful Idea #2: Cauchy-Goursat Thrm

If C is simple closed contour and $f(z)$ is analytic in and on C, then

$$\oint_C f(z)dz = 0$$

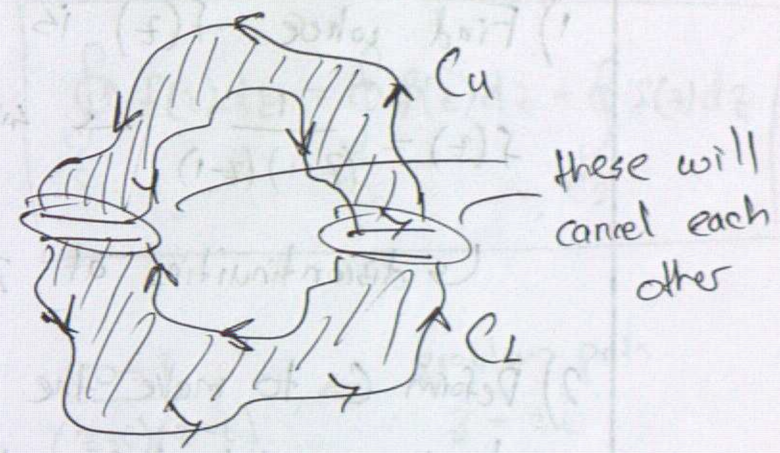
Ex: $\oint_C e^z dz = 0$ or $\oint_C \sin z\, dz = 0$ or $\oint_C \cos z\, dz = 0$

or $\oint_C z^2 dz = 0$ or $\oint_C (\text{any polynomial})\, dz = 0$

if C is simple closed contour } they're all entire functions

Deformation of Contours

How do $\oint_{C_1} f(z)dz$ and $\oint_{C_2} f(z)dz$ relate?



If $f(z)$ analytic

$$\left.\begin{aligned}&\oint_{C_u} f(z)dz = 0\\ &\text{and}\\ &\oint_{C_L} f(z)dz = 0\end{aligned}\right\} \text{CG thrm}$$

$$\therefore \oint_{C_u} f(z)dz + \oint_{C_L} f(z)dz = 0 = \text{what?}$$

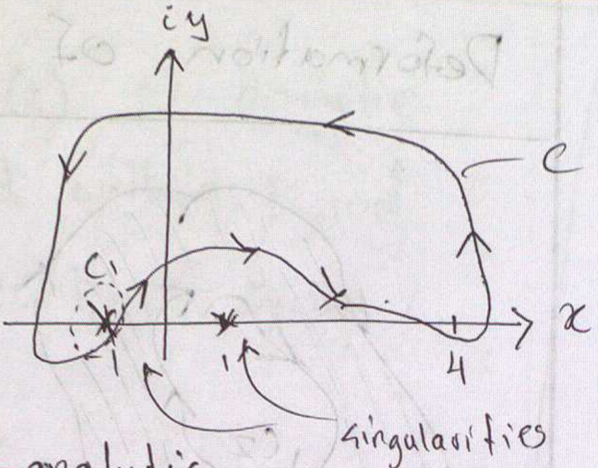
$$\oint_{C_u} f(z)dz + \oint_{C_L} f(z)dz = 0 = \oint_{C_1} f(z)dz - \oint_{C_2} f(z)dz = 0$$

$$\boxed{\therefore \oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz}$$

(As long as I don't pass thru an non-analytic region

Week 9: Lecture 3 Nov 7, 2024

<u>Ex:</u> $\oint_C \frac{1}{z^2-1}\,dz$, C:



the contour is tough to parametrize

1) Find where f(z) is non-analytic

$f(z) = \frac{1}{(z+1)(z-1)} \rightarrow$ singularities at $z = \pm 1$

↳ discontinuities at $z = \pm 1$

2) Deform C to make the integral easier to solve

↳ C_1 is valid deformation of C → C_1 is a circle of radius $\underbrace{\rho < 2}$ centered at $z = -1$

so I avoid z=1 problem point

$$\oint_C \frac{1}{z^2-1}\,dz = \oint_{C_1} \frac{1}{z^2-1}\,dz = \underbrace{\oint_{C_1} \frac{\frac{1}{2}}{z-1}\,dz - \oint_{C_1} \frac{\frac{1}{2}}{z+1}\,dz}$$

partial fractions

$\frac{1/2}{z-1}$ is fully analytic within C_1

$\oint_{C_1} \frac{1/2}{z-1}\,dz = 0$ by CG thrm

very important integral

$$\oint_C \frac{1}{(z-z_0)^n}\,dz = \begin{cases} 0, & n \neq 1 \\ 2\pi, & n = 1 \end{cases}$$

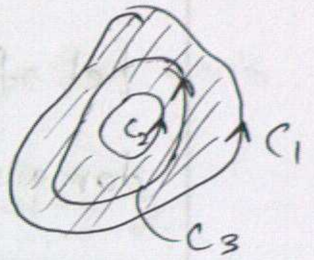
↳ $|z-z_0| = \rho$

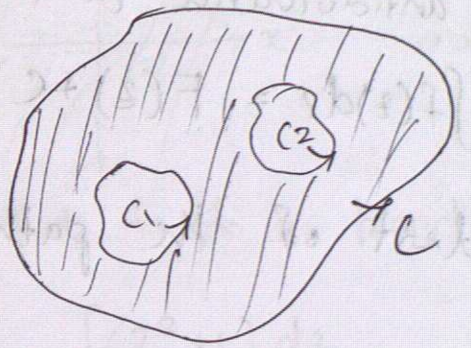
$$0 - \left(\tfrac{1}{2}\right)(2\pi i) = 0 - \pi i = \boxed{-\pi i}$$

Cauchy-Goursat Thm for Multiply Connected Domain

- doubly → 1 hole

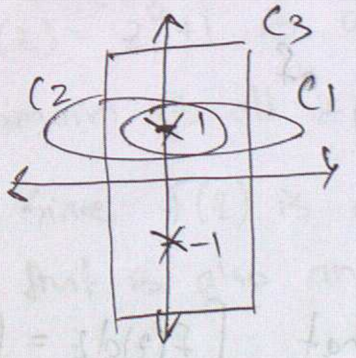
$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz = \oint_{C_3} f(z)\,dz$$





$$\boxed{\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz}$$

<u>Ex:</u> $f(z) = \frac{\cos z}{z^2+1} = \frac{\cos z}{(z+i)(z-i)}$ ← problem pnts $z = \pm i$



$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz \checkmark$$

C_3 cannot shrink to C_1 or C_2 because it'll have to cross the problematic point $z = -i$

Independence of Path and Analaticity of Ln(z)

Sec 18.3, part of 17.6

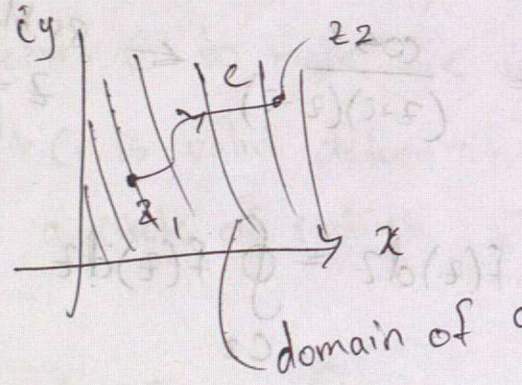
Fundamental Thm of Complex Integral Calculus

Let $f(t)$ be analytic in a simply connected domain D (no holes), then

1) $f(z)$ has an analytic antiderivative on in D

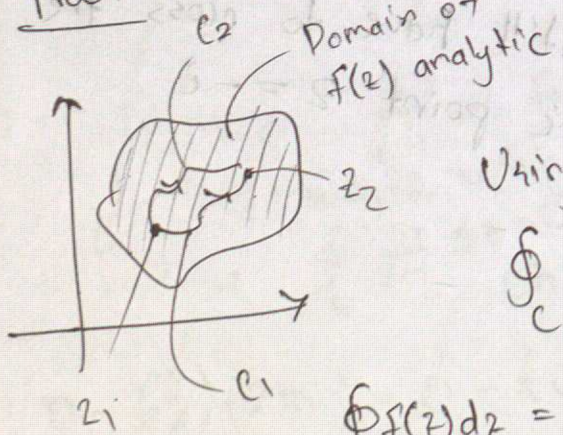
$F'(z) = f(z)$ and $\int f(z)\,dz = F(z) + C$

2) $\int_C f(z)\,dz$ is independent of the path of C



$$\int_C f(z)\,dz = F(z_2) - F(z_1)$$

Proof



Prove that $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$

Using CG thm,

$\oint_{C} f(z)\,dz = 0$ ← combination } of C_1 and $-C_2$

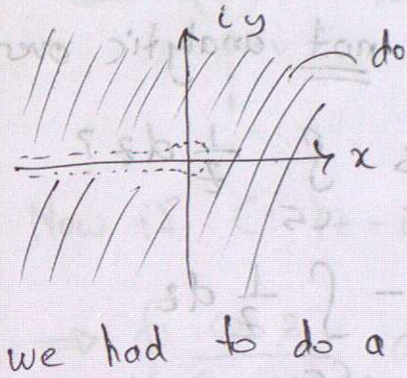
$$\oint_C f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\boxed{\int_{C_1} f(z)\,dz = \int_{C_2} f(t)\,dz = F(z_2) - F(z_1)}$$

f(z) has to be analytic on C_1 and C_2

Ln(z) ← discontinuous at $z=0$ and along negative real axis

- Ln(z) is analytic if $z=0$ and z not (-)ve real axis



∴ In domain D, Ln(z) is analytic and $\boxed{\frac{d}{dz}[\text{Ln}(z)] = \frac{1}{z}}$

we had to do a branch cut to make f(z) analytic

Ex: $\int_C (z^2+1)\,dz$

a) suggest a simply connected where integral is path independent

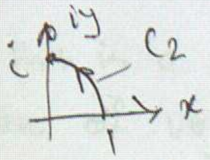
$f(z) = z^2+1$ is analytic for all z, so our simply connected domain is all z, and ∴ $\int f(z)\,dz$ will be path independent

b) Since f(z) is analytic on D, it has an antiderivative that is also analytic on D.

$$\therefore \int (z^2+1)\,dz = \frac{z^3}{3} + z + C - F(z)$$

Ex: $\int_{C_2} \frac{1}{(z+1)^2}\,dz$ ← contours thru $z=-1$ can't be determined by independence of path

b) what if $C_2: |z|=1,\ 0 \le \text{Arg}(z) \le \frac{\pi}{2}$

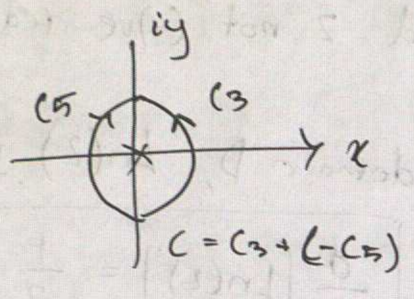


$$\int_{C_2} \frac{1}{(z+1)^2}\,dz = \left.\frac{-1}{(z+1)}\right|_1^i = F(i) - F(1) = \frac{-1}{i+1} - \left(-\frac{1}{2}\right)$$

$$= \frac{1}{2}i$$

Week 10: Lecture 1

Ex: $\int_c \frac{1}{z}\,dz$ for the following contours



- while $\frac{1}{z}$ is analytic on both C_5 and C_3, its antiderivative $\therefore$ $Ln(z)$ is <u>not</u> analytic over C_5
- so, what is $\int_{C_5} \frac{1}{z}\,dz$?

Note that
$$\underbrace{\oint_c \frac{1}{z}\,dz}_{\text{very important integral}: 2\pi i} = \underbrace{\int_{C_3} \frac{1}{z}\,dz}_{Ln(i) - Ln(-i) = \pi i} - \int_{C_5} \frac{1}{z}\,dz$$

$$\therefore \int_{C5} \frac{1}{z}\,dz = \int_{C_3} \frac{1}{z}\,dz - \oint_c \frac{1}{z}\,dz = \pi i - 2\pi i = \boxed{-i\pi}$$

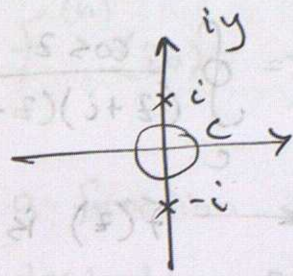
Cauchy's Integral Formula (Sec 18.4)

1) Contour integration thru parametization: $\int_c f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt$

2) Integration using antiderivatives: $\int_c f(z)\,dz = F(z_2) - F(z_1)$

3) Using Cauchy-Goursat Thm and deformation of contours

Ex: $\oint_c \frac{\cos z}{z^2+1}\,dz$, $C: |z| = \frac{1}{2}$



Since $f(z) = \frac{\cos z}{z^2+1}$ is analytic on and within C, by Cauchy-Goursat we know $\oint_c f(z)\,dz = \boxed{0}$

Now if $C: |z - i| = 1$, we have circle radius 1 @ (0,1)

$\Rightarrow \oint_c \frac{\cos z}{(z+i)(z-i)}\,dz = ?$ "We don't have tools yet"

Beautiful Idea #3: Cauchy Integral Formula

Let f(z) be analytic on and within C, then

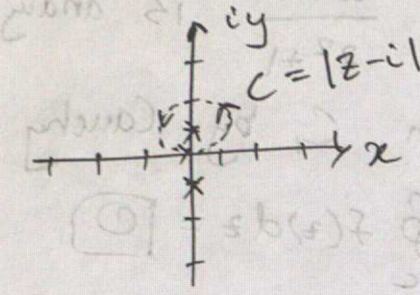
$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0}\,dz}$$

1) By knowing all the values of f(z) <u>on</u> the contour, we can find <u>all</u> the values of f(z) <u>inside</u> the contour (i.e., $f(z_0)$)

2) We can use this formula to evaluate integrals of this type $\boxed{\oint_c \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0)}$

- this is a special case of very important integral

Ex: $\oint_C \frac{\cos z}{z^2+1}\,dz$, $C: |z-i|=1$



$$\oint_C \frac{\cos z}{z^2+1}\,dz = \oint_C \frac{\cos z}{(z+i)(z-i)}\,dz$$

$$= \oint_C \frac{\frac{\cos z}{(z+i)}}{z-i}\,dz$$ ← f(z) is analytic

$z_0 = i$

$$= 2\pi i\left(\frac{\cos z}{z+i}\right)\Big|_{z=i} = 2\pi i f(z_0)$$

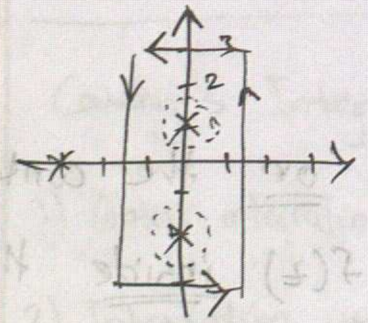
$$= 2\pi i\left(\frac{\cos i}{i+i}\right) = \pi \cos i = \underline{\underline{\pi \cosh(1)}}$$

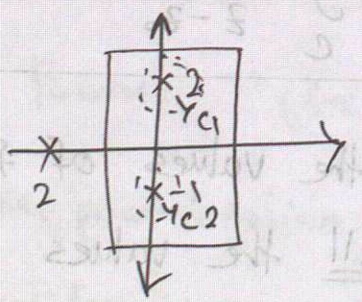
since $\cos z = \cos x \cosh y - i \sin x \sinh y$

Ex: $I = \oint_C \frac{\cos z}{(z+2)(z+i)(z-2i)}\,dz$

$z \neq 2$
$z \neq -i$
$z \neq 2i$

a error in diagram





$$\Rightarrow \oint_{C_1} \frac{\frac{\cos z}{(z+2)(z+i)}}{z-2i}\,dz + \oint_{C_2} \frac{\frac{\cos z}{(z+2)(z-2i)}}{z+i}\,dz$$

$f_1(z)$ — $f_2(z)$

$$= 2\pi i f_1(2i) + 2\pi i f_2(-i) = \underline{0.677 + 2.67i}$$

Beautiful Idea 3b : Exteded Version

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

- if a function is analytic, then it will have all orders of derivatives if they have one derivative
- to solve "higher order" singularities,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i f^{(n)}(z_0)}{n!}$$

Week 10: Lecture 2 Nov 12, 2024

Fundamental Thm of Algebra

Every non-constant polynomial of order n with real or complex coefficients has at least one complex root.

$P(z) = a_n z^n + \cdots + a_1 z^1 + a_0 \longrightarrow$ non-constant poly

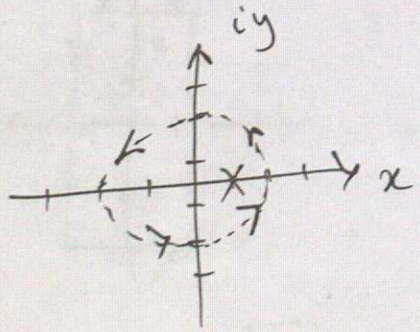
$\hookrightarrow$ At least one root $P(z) = 0$

$P(z)$ w/n degree n $\longrightarrow$ n roots (real, complex)

Ex: $\oint_C \frac{z^3+2z+1}{(z-1)^3} dz$, $C = |z| = 2$

$z_0 = 1,\ n+1 = 3 \Longrightarrow n = 2$, $f(z) = z^3+2z+1$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$



$$\oint_C \frac{z^3+2z+1}{(z-1)^3} dz = \frac{2\pi i}{2!} f''(z_0=1)$$

$$= \frac{2\pi i}{2}(6z|_{z=1})$$

$$= \frac{12\pi i}{2}$$

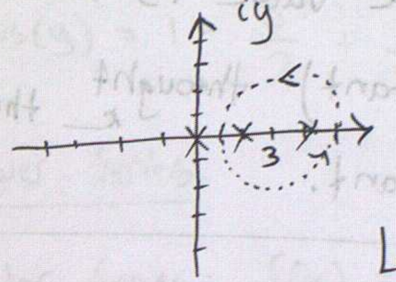
$$\underline{\underline{= 6\pi i}}$$

$f(z) = z^3+2z+1$

$f'(z) = 3z^2+2$

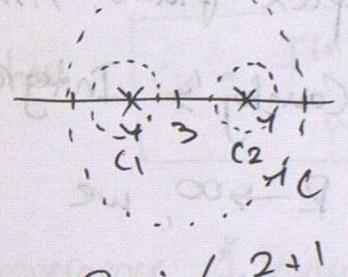
$f''(z) = 6z$

Ex: $\oint_C \frac{z+1}{z(z-2)(z-4)^3} dz$, $C: |z-3| = 2$



$$\frac{z+1}{z(z-2)(z-4)^3} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{(z-4)^{\cancel{3}}}$$

$$\text{Let } f(z) = \frac{z+1}{z} \qquad + \frac{D}{(z-4)^2} + \frac{E}{(z-4)^3}$$



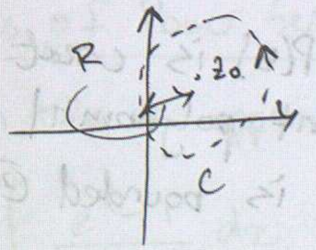
$$\Longrightarrow \oint_{C_1} \frac{z+1}{z(z-4)^3} \cdot \frac{1}{(z-2)} dz + \oint_{C_2} \frac{z+1}{z(z-2)} \cdot \frac{1}{(z-4)^3} dz$$

$$= 2\pi i \left(\frac{2+1}{2(2-4)^3}\right) + \frac{2\pi i}{2!}\left(\frac{d^2}{dz^2}\left[\frac{z+1}{z(z-2)}\right]\Big|_{z=4}\right) = -\frac{\pi}{64} i$$

Cauchy's Inequality

Let $f(z)$ be analytic on and within $|z-z_0| = R$ and

$$\underline{\underline{|f(z)| \leq M_0 \text{ on } C}}$$



$$|f^{(n)}(z_0)| = \left|\frac{n!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{n+1}} dz\right| \leq \frac{n!}{2\pi} ML$$

$$M = \left|\frac{f(z)}{(z-z_0)^{n+1}}\right| = \frac{|f(z)|}{|(z-z_0)^{n+1}|} = \frac{M_0}{R^{n+1}}$$

$L = 2\pi R$

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi}\left(\frac{M_0}{R^{n+1}}\right) 2\pi R = \frac{n!\, M_0}{R^n}$$

Liouville's Theorem

An entire function whose absolute value is bounde (i.e., does not exceed some constant) throught ← throughout the complex plane is a constant.

Proof: Let z_0 be any point in complex plane. Assume $|f(z)| \leq M_0$ everywhere, then from Cauchy's Integral Inequality, $|f'(z_0)| \leq \frac{M_0}{R}$, so letting $R \rightarrow \infty$, we have $|f'(z_0)| = 0 \Longrightarrow \therefore f(z_0) = \text{constant!}$

Prooving Fundamental Thm of Algebra

Suppose $P(z)$ is a non-constant polynomial, and that $P(z) \neq 0$ for all z. Prove by contradiction

Let $f(z) = \frac{1}{P(z)} \rightarrow$ since $P(z) \neq 0$, $f(z)$ is entire ①

Let $|z| \rightarrow \infty$, so $\lim_{|z| \rightarrow \infty} f(z) = 0$ since $P(z)$ is cnot constant polynomial

Since $\lim_{|z| \rightarrow \infty} f(z)$ doesn't "blow" up, it is bounded ②

By Liouville's Thm, we can say $f(z)$ is a constant since it's an entire function ① and is bounded ②.

Since $f(z) = \text{constant} \Longrightarrow P(z) = \frac{1}{f(z)} = \frac{1}{\text{constant}} = \text{constant}.$

↑ thus, our contradiction

Sequences and Series

Week 10: Lecture 3

Nov 14, 2024

Real-Valued Taylor Series of cos(y)

$$\cos(y) = 1 - \frac{y^2}{2} + \frac{y^4}{24} - \frac{y^6}{720} + \frac{y^8}{40320}$$

Power Series

Taylor Series: $f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 \dots$

where $\boxed{c_n = \frac{f^{(n)}(x_0)}{n!}}$

ex// $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

ex// $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \rightarrow \frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{(x+1)^n}{2^{n+1}}$ ↑ about $x = -1$

Convergence & Divergence

$f(x) = \frac{1}{1-x}$

$= 1 + x + x^2 + x^3 + \cdots$

power series representations usually only valid over an interval of convergence

↳ interval of convergence is $(-1, 1)$

Uses of Power Series

- can help us approximate hard functions

$\int \frac{e^x - 1}{x} dx$ has no elementary antiderivative, so → Let $e^x \approx 1 + x + \frac{x^2}{2}$

$$\int \frac{e^x - 1}{x} dx \approx \int \frac{1 + x + \frac{x^2}{2} - 1}{x} dx = \int \left(1 + \frac{x}{2}\right) dx = \underline{\underline{x + \frac{1}{4}x^2 + C}}$$

Complex Valued Series

1) How do we know if a series converges or diverges?

$$\sum_{n=0}^{\infty} x^n, \quad |x|<1$$

2) What is the region of convergence? $\sum_{n=0}^{\infty} C_n(z-z_0)^n$

3) What are other types of power series? $\longrightarrow \sum_{n=-\infty}^{\infty} C_n(z-z_0)^n$

Convergence of Complex Sequences

$z_n = 1+i^n \longrightarrow \{z_n\}$

$z_1 = 1+i, \; z_2 = 0, \; z_3 = 1-i, \; z_4 = \overset{2}{1+i} \; \cdots$

> A sequence $\{z_n\}$ converges to a complex number L I.A.O.I $Re(z_n)$ converges to $Re(L)$ and $Im(z_n)$ converges to $Im(L)$.

Ex: $\{z_n\}, \; z_n = 1 - \frac{1}{n^2} + i\left(2+\frac{4}{n}\right)$

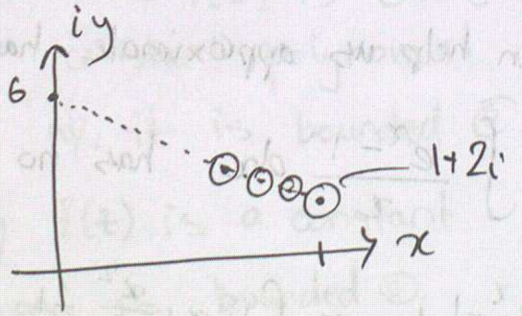
$z_1 = 1 + \frac{1}{1} + i\left(2+\frac{4}{1}\right) = 6i$

$z_2 = \frac{3}{4} + 4i$

$z_3 = \frac{8}{9} + \frac{10}{3}i$

$\vdots$

$z_\infty = 1+2i$ ← converges to this



Convergence of Complex Series

N'th term test:

> If $\sum_{k=1}^{\infty} z_k$ converges, then $\lim_{n\to\infty} |z_n| = 0$,
>
> Otherwise, series diverges

ex// $\sum_{k=1}^{\infty} \frac{3+2i}{(1+i)^k}$ converge?

$$z_k = \frac{3+2i}{(1+i)^k} \longrightarrow |z_k| = \frac{|3+2i|}{|(1+i)^k|} = \frac{|3+2i|}{|1+i|^k} \Rightarrow \lim_{n\to\infty} |z_n| = 0 \quad ✓$$

Since $\lim_{n\to\infty} |z_n| = 0$, series converges absolutely → does not rely on "adding and subtracting" to convergence

$$\text{Sum} = S = \sum_{k=1}^{\infty} \frac{3+2i}{(1+i)^k}$$

$$= \frac{3+2i}{1+i} \sum_{k=1}^{\infty} \underbrace{\frac{1}{(1+i)^{k-1}}}_{r=\frac{1}{1+i}} = \left(\frac{3+2i}{1+i}\right)\left(\frac{1}{1-\frac{1}{1+i}}\right)$$

$$= 2-3i$$

Geometric Series

$$\sum_{k=1}^{\infty} r^{k-1} = 1 + r + r^2 + \cdots = \frac{1}{1-r}$$

if $|r|<1$

Ratio Test

Consider series $\sum_{k=1}^{\infty} z_k$ s.t. $L = \lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|$

a) series absolutely converges if $L<1$

b) series divergent if $L>1$

c) inconclusive if $L=1$

Ex: $\sum_{k=1}^{\infty} \frac{3+2i}{(1+i)^k}$

$L =$

Convergence of Power Series

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots = \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

Use ratio test $L = \lim_{n\to\infty} \left| \frac{a_{n+1}(z-z_0)^{\cancel{n}+1}}{a_n \cancel{(z-z_0)^n}}\right|$

$= \lim_{n\to\infty} \left|\frac{a_{n-1}}{a_n}\right| |z-z_0|$ ← radius of circle is R

for absolute convergence, $L<1$ $\quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| R < 1$ ← series convergence absolutely

↓ $\therefore R = \frac{1}{\lim_{n\to\infty}}$

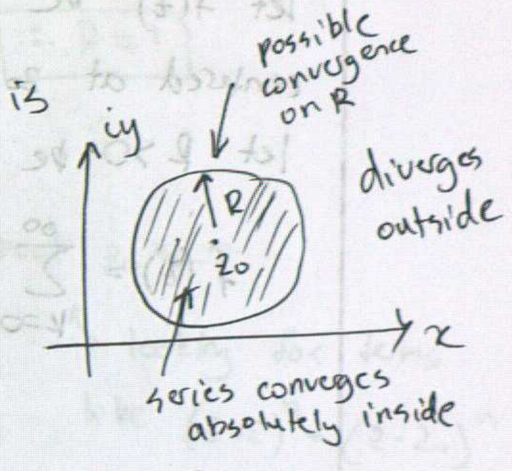
$\boxed{\therefore R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|}$ ← radius of convergence

Radius of Convergence

Consider the series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ and define $\tilde{L} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

a) If $\tilde{L}\neq 0$, then radius of convergence is

$$R = \frac{1}{\tilde{L}} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$



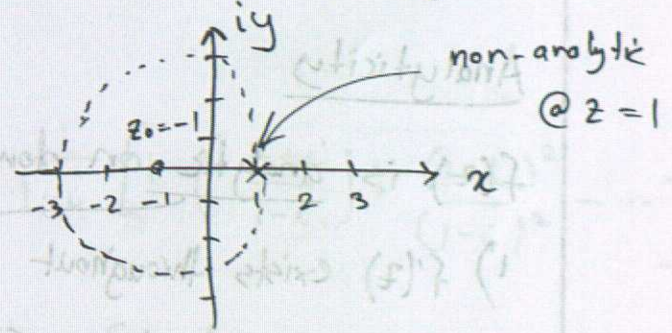
b) If $\tilde{L}=0$, then $R=\infty$, so that the power series converges for all z

c) If $\tilde{L}=\infty$, $R=0$ and series converges only @ centre $z=z_0$

Ex: $f(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} \frac{(z+1)^k}{2^{k+1}} \Longrightarrow a_k = \frac{1}{2^{k+1}}$

$R = \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{2^{n+1}}}{\frac{1}{2^{n+2}}}\right|$ Taylor series about $z_0=-1$

$= \lim_{n\to\infty}|2| = 2$



For $R<2$ → this series converges

$|z+1|<2$

- radius of convergence is from where series is centered to nearest singularity of function you're approximating

Section 19.2

Week 11: Lecture 1 Nov 18, 2024

Taylor's Theorem

let $f(z)$ be analytic at z_0. Let C be the largest circle centered at z_0, inside which $f(z)$ is analytic everywhere, and let $R > 0$ be radius of C, then we have pwr series:

$$f(z) = \sum_{k=0}^{\infty} \underbrace{\frac{f^{(k)}(z_0)}{k!}}_{a_k\text{, our coefficients in front of power terms}} (z - z_0)^k \qquad |z - z_0| < R$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!} \leftarrow R = \infty$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{k=0}^{\infty} z^k \leftarrow R = 1$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \leftarrow R = \infty$$

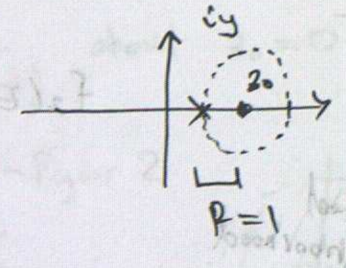
Analyticity

$f(z)$ is analytic on domain $\mathbb{D}$ if any are true

1) $f'(z)$ exists throughout D
2) $f(z)$ has derivatives of all orders through D
3) $f(z)$ has a Taylor series expansion valid in a neighborhood of each point in D

Ex: radius of convergence of $f(z) = \frac{z}{z^2 - 2z + 1}$ centered at $z = 2$

$$f(z) = \frac{z}{(z-1)^2}$$



$$\boxed{\therefore R = 1}$$

Ex: Find Taylor series for $f(z) = \frac{1}{1+z}$ about $z_0 = -i$

↖ looking for terms like $(z+i)^n = (z - z_0)^n$

$$f(z) = \frac{1}{1+z} = \frac{1}{\underbrace{1-i}+z+i} = \underbrace{\frac{1}{1-i}\left(\frac{1}{1+\frac{z+i}{1-i}}\right)}_{\text{looks like geometric series}}$$

$$= \frac{\left.\frac{1}{1-i}\right]a}{1+\underbrace{\left(-\frac{z+i}{1-i}\right)}_{r}} = \frac{a}{1-r}$$

geometric series

$$\frac{a}{1-r} = a\sum_{k=0}^{\infty} r^k, \quad |r| < 1$$

observe $|r| < 1$

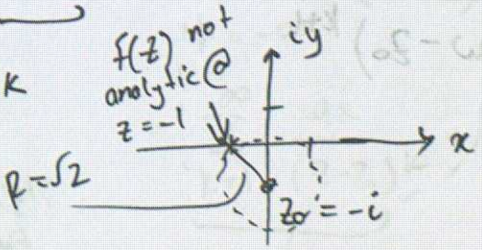
$$\left|-\frac{z+i}{1-i}\right| < 1$$

$$\therefore f(z) = \frac{1}{1-i}\sum_{k=0}^{\infty}\left(-\frac{z+i}{1-i}\right)^k = \frac{1}{1-i}\left[1 + \left(-\frac{z+i}{1-i}\right) + \frac{(z+i)^2}{(1-i)^2} - \cdots\right]$$

$$= \sum_{k=0}^{\infty} \underbrace{\frac{(-1)^k}{(1-i)^{k+1}}}_{a_k} (z+i)^k \quad \text{for } |z+i| < R$$

↓

$R = \sqrt{2}$

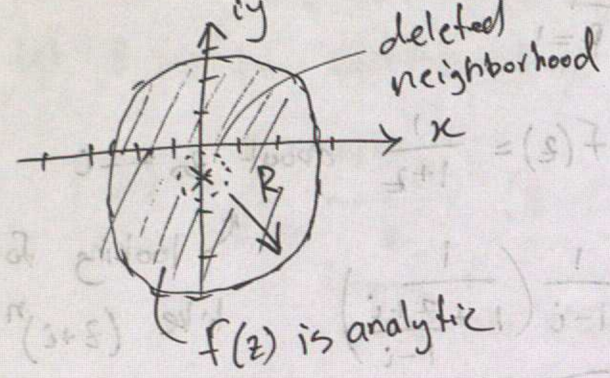


Laurent Series (19.3)

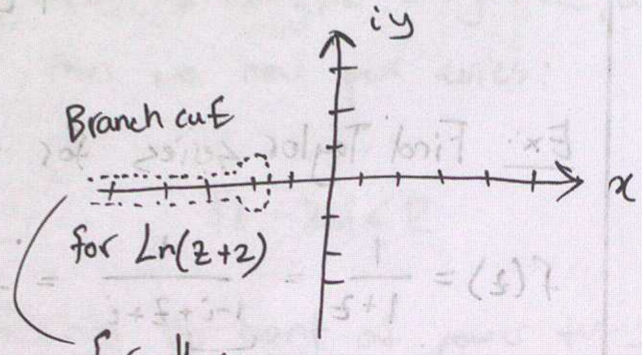
Week 11: Lecture 2 Nov 19, 2024

Isolated Singularities

Ex: $f_1(z) = \frac{1}{z+i}$ $f_2(z) = Ln(z+2)$



in fact even if $R \to \infty$



for these singularities of $Ln(z+2)$, we cannot isolate these with a deleted neighborhood

If we can delete a neighborhood (open annulus) to make ~~z~~ f(z) analytic everywhere else, then that was an isolated singularity $\to 0 < |z - z_0| < R$

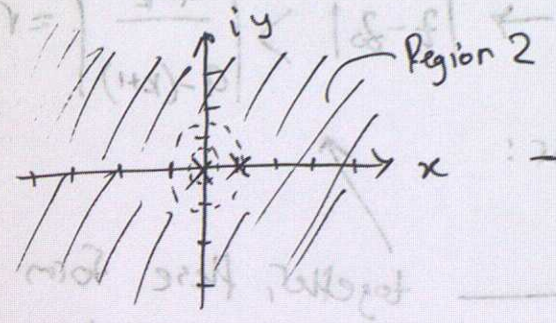
Laurent Series

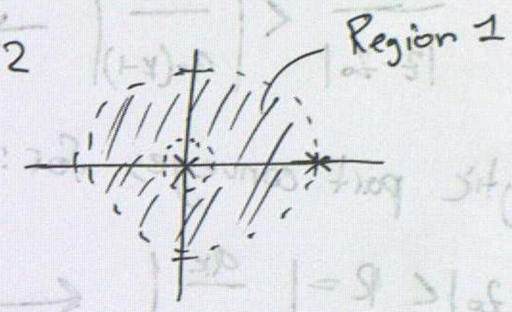
$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$ when f(z) analytic inside annular domain D

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z_0)^{k+1}} \, dw \qquad k = 0, \pm 1, \pm 2$$

Region of Convergence

Ex: $f(z) = \frac{1}{z(z-1)}$ about $z_0 = 0$





For region 1: $0 < |z| < 1$

unique

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} - 1 - z - z^2 - \dots$$

For region 2: $1 < |z| < \infty$

unique

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \dots$$

- We can now have multiple expansions of f(z) about $z_0 \to$ one for each different region
- region of convergence is an open annulus ($r < |z - z_0| < R$)

Principle and Analytic Parts

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k = \sum_{k=-\infty}^{-1} a_k (z - z_0)^k + \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

$$= \underbrace{\sum_{k=1}^{\infty} \frac{a_k}{(z - z_0)^k}}_{\text{principle part}} + \underbrace{\sum_{k=0}^{\infty} a_k (z - z_0)^k}_{\text{analytic part}}$$

The principle part converges for:

$$a_{-}\quad \frac{1}{|z-z_0|} < \left|\frac{a_{-k}}{a_{-(k-1)}}\right| \rightarrow |z-z_0| > \left|\frac{a_{-k}}{a_{-(k+1)}}\right| = r$$

the analytic part converges for:

$$|z-z_0| < R = \left|\frac{a_k}{a_{k+1}}\right|$$

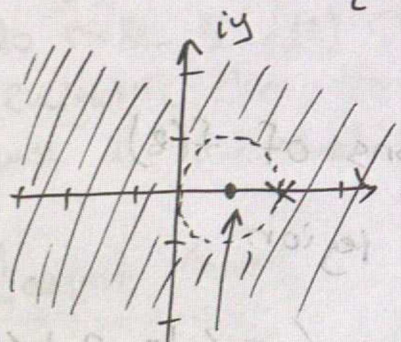
together, these form the open annulus

$$r < |z-z_0| < R$$

<u>Ex</u>: Find Laurent series for

$$f(z) = \frac{z^2-2z+3}{\cancel{z-2}}\ \text{in region}\ |z-1|>1$$

$z-2$



$z_0 = 1$

1) Rearrange $f(z)$ to express in terms of $(z-1)$

$$\frac{1}{z-2} = \frac{1}{z-1-1} = \frac{-1}{1-(\cancel{z-1})}$$

but $|z-1|>1$ in our region

∴ we can't use this "r"

$$\frac{1}{z-1}\left(\frac{1}{1-\underbrace{\frac{1}{z-1}}}\right)$$

$$r = \frac{1}{z-1} \rightarrow |r|<1$$

For $|z-1|$ ~~1~~ >1, $\frac{1}{z-1}\sum_{k=0}^{\infty}\frac{1}{(z-1)^k}$

what about z^2-2z+3?

↓

need this in powers of $(z-1)$

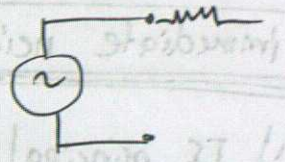
Zeroes and Singularities
Section 19.4

Week 11: Lecture 3

Nov 21, 2024

Importane of zeroes and poles

$$H(s) = \frac{s/RC}{s^2 + s/RC + \frac{1}{LC}} = \frac{s/RC}{(s-p_1)(s-p_2)}$$

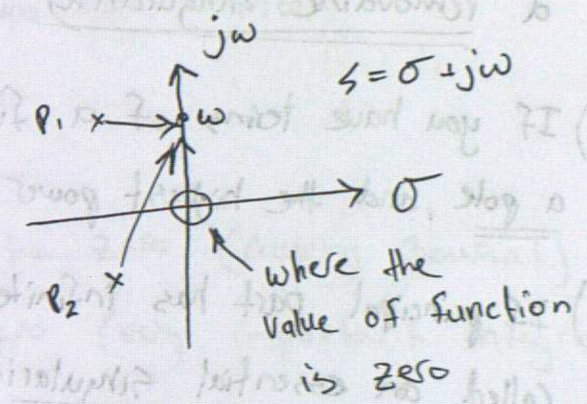


$$p_1, p_2 = -\frac{1}{2RC} \pm j\sqrt{\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2}$$

$$= \sigma + j\omega$$

growth or decay

oscillation



$$V_{in} = V_0 \cos(\omega t)$$

$$|V_{out}| = |V_{in}||H(s)| \propto \frac{|s|}{|s-p_1||s-p_2|}$$

Zeroes:

$f(z) = (z+2)^3 \longrightarrow$ zero @ $z=-2$ $\rightarrow$ of what order?

$\therefore f(z=-2) = (-2+2)^3 = 0$, $f'(z=-2) = 3(-2+2)^2 = 0$

$f''(z=-2) = 6(-2+2) = 0$

$f'''(z=-2) = 6 \neq 0$

zero of order 3 since $f'''(z) \neq 0$

zero of order 3 since $f'''(z_0) \neq 0$ //

Note: The Taylor series for $f(z)$ about $z=-2$ would have the first 3 terms missing ($a_0 = a_1 = a_2 = 0$ since those 3 derivatives are zero).

☆ **Singularities** (based on Laurent Series)

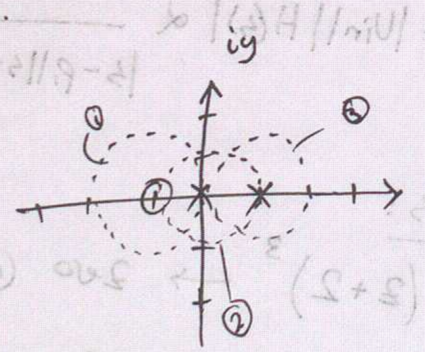
- use principal part of Laurent Series (expand in the $\boxed{\text{immediate neighborhood}}$) of the singularity

1) If principal part is zero, then $z = z_0$ is called a removable singularity

2) If you have terms of a finite number, $z = z_0$ is called a pole, and the highest power is the order of the pole

3) If principal part has infinite # of terms, $z = z_0$ is called an essential singularity.

Ex: $f(z) = \dfrac{z+1}{z^2(z-1)^3}$



for $0 < |z+1| < 1$ ①

$f(z) = -\frac{1}{8}(z+1) - \cdots$ ← just the first term ($a_0 = 0$) is missing → zero of order 1

for $0 < |z| < 1$ ②

$f(z) = \underbrace{-\frac{1}{z^2} - \frac{4}{z}}, + 9 - 16z - \cdots$ ← 2 terms, so pole of oreder 2

for $0 < |z-1| < 1$ ③

$f(z) = \underbrace{\frac{2}{(z-1)^3} - \frac{3}{(z-1)^2} + \frac{4}{(z-1)}}, - 5 + 6(z-1) \cdots$ ← 3 terms, pole of order 3

Week 12: Lecture 1

Nov 25

Zeroes and Poles

- for rational functions: the only singularities are removable singularities or poles.

$$f(z) = \frac{(z-z_1)(z-z_2)\cdots(z-z_m)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

what if $z_2 = p_2$, the two can cancel

Section 19.5 : Residues

- all the analytic part goes to zero (cauchy Goursat)
- most principal terms to zero (very important integral)
- the only term in the Laurent series that survives is the $\underline{k=-1}$ term!

residue of singularity

Cauchy's Residue Thm

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k)$$

$$\oint_C f(z)\,dz = 2\pi i \left[\text{Res}(f(z), z_1) + \text{Res}(f(z), z_2) + \cdots + \text{Res}(f(z), z_n)\right]$$

"$2\pi i$ times sum of all residues"

Residue: pole of order n (singularity of $f(z)$ @ $z = z_0$ is a pole of order n)

$$a_{-1} = \text{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z) \right]$$

Residue: pole of order 1

$$a_{-1} = \text{Res}(f(z), z_0) = \lim_{z \to z_0} \left[(z-z_0) f(z) \right]$$

$$a_{-1} = \text{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)} \quad \text{if } f(z) = \frac{g(z)}{h(z)}$$

analytic at $z = z_0$

Ex: $f(z) = \dfrac{e^z}{(z^2+1)z^2}$ → does numerator have a zero at my pole? → no here

$g(z) = \dfrac{e^z}{z^2(z-i)}$

$h(z) = (z+i)$

$$f(z) = \frac{e^z}{z^2(z+i)(z-i)}$$

$z = 0$, pole order 2

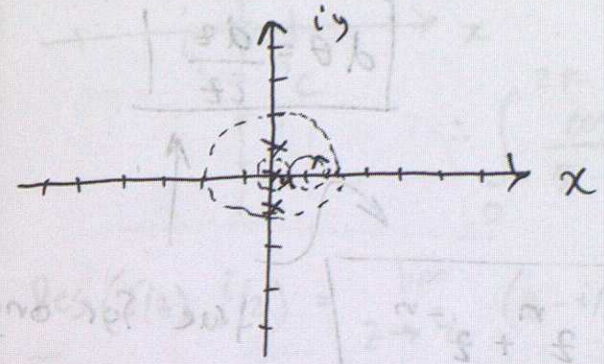
$z = \pm i$, poles order 1

$$\text{Res}(f(z), z=i) = \lim_{z \to i} (z-i) f(z) = \lim_{z \to i} \frac{e^z}{z^2(z+i)} = i\frac{e^i}{2} \,//$$

$$\text{Res}(f(z), z=-i) = \frac{g(-i)}{h'(-i)} = \frac{e^{-i}}{\underset{-i-i}{(-2i)}\underset{i^2}{(-1)}\underset{h'(-i)}{(1)}} = -i\frac{e^{-i}}{2} \,//$$

$$\text{Res}(f(z), z=0) = \frac{1}{(2-1)!} \lim_{z \to 0} \frac{d}{dz}\left(z^2 f(z)\right) = \lim_{z \to 0}\left[\frac{e^z(z^2+1) - 2ze^z}{(z^2+1)^2}\right] = 1 \,//$$

a) $|z-1| = \frac{1}{2}$ C_1 b) $|z| = \frac{1}{2}$ C_2 c) $|z| = 2$ C_3



a) $\displaystyle\oint_{C_1} f(z)\,dz \boxed{= 0}$ Cauchy Goursat

b) $\displaystyle\oint_{C_2} f(z)\,dz = 2\pi i\, \text{Res}(f(z), z=0)$

$= 2\pi i(1)$

$\boxed{= 2\pi i}$

c) $$\oint_{C_3} f(z)\,dz = 2\pi i\left[\text{Res}(f(z), z=0) + \text{Res}(f(z), z=i) + \text{Res}(f(z), z=-i)\right]$$

$$= 2\pi i\left[1 + \underbrace{\frac{e^{-i}}{2i} - \frac{e^{i}}{2i}}_{\sin(1)}\right] = \boxed{2\pi i(1 - \sin(1))}$$

Ex: $\displaystyle\int \frac{e^{z}-1}{\sin^3 z}\,dz$ over $|z| = 1$

19.6: Real Integrals Part 1

Week 12: Lecture ~~8~~ 2

Nov 28, 2024

Form #1: $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\boxed{d\theta = \frac{dz}{iz}}$$

$$\boxed{\begin{aligned} \cos n\theta &= \frac{e^{in\theta} + e^{-in\theta}}{2} = \frac{z^n + z^{-n}}{2} \\ \sin n\theta \ \cancel{\cos} n\theta &= \frac{e^{in\theta} - e^{-in\theta}}{2i} = \frac{z^n - z^{-n}}{2i} \end{aligned}}$$

true for only parametrization $z = e^{i\theta}$

Ex: $\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\sin\theta}\, d\theta$

1) convert into complex domain: $\cos(2\theta) = \frac{z^2 + z^{-2}}{2}$, $d\theta = \frac{dz}{iz}$, $\sin\theta = \frac{z - z^{-1}}{2i}$

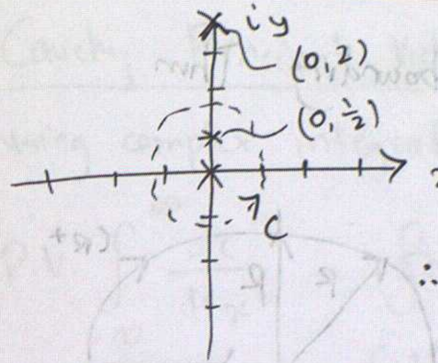
↳ these rely on the parametrization above

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\sin\theta} = \oint_{|z|=1} \frac{\frac{z^2 + z^{-2}}{2}}{5 - 4\left(\frac{z - z^{-1}}{2i}\right)} \left(\frac{dz}{iz}\right) \cdot \left(\frac{z^2}{z^2}\right)$$

gets rid of (-)ve powers

$$= -\frac{1}{4} \oint_{|z|=1} \frac{z^4 + 1}{z^3\left(z^2 - \frac{5i}{2}z - 1\right)}\, dz = -\frac{1}{4} \oint_{|z|=1} \underbrace{\frac{z^4 + 1}{z^3(z - i/2)(z - 2i)}}\, dz$$

$\oint_c f(z)\, dz$



$$\Rightarrow -\frac{1}{4}(2\pi i)\left[\operatorname{Res}(f(z), 0) + \operatorname{Res}(f(z), i/2)\right]$$

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\sin\theta}\, d\theta = -\frac{1}{4}(2\pi i)\left(-\frac{17}{6}i - \frac{5i}{6}\right)$$

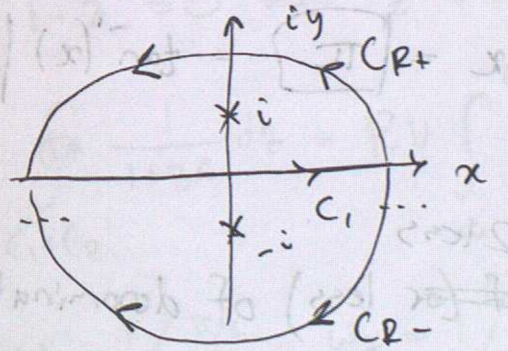
$$\downarrow$$

$$\boxed{= -\frac{\pi}{6}}$$

$$\operatorname{Res}(f(z), i/2) = \lim_{z \to i/2} (z - i/2) f(z) = -\frac{17}{6}i$$

$$\operatorname{Res}(f(z), 0) = \lim_{z \to 0} \frac{d}{dz}\left(f(z) z^2\right) = -\frac{5}{6}i$$

Form #2: $\int_{-\infty}^{\infty} f(x)\, dx$, ex: $\int_{-\infty}^{\infty} \frac{1}{x^2 + 1}\, dx$



Observe that:

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + 1}\, dx = \int_{C_1} \frac{1}{z^2 + 1}\, dz$$

↳ along C_1, $z = x$

$$\oint_{C_1 + C_{R+}} \frac{1}{z^2 + 1}\, dz = \int_{-\infty}^{\infty} \frac{1}{x^2 + 1}\, dx + \int_{C_{R+}} \frac{1}{z^2 + 1}\, dx = 2\pi i\left(\operatorname{Res}\left(\frac{1}{z^2 + 1}, z = i\right)\right)$$

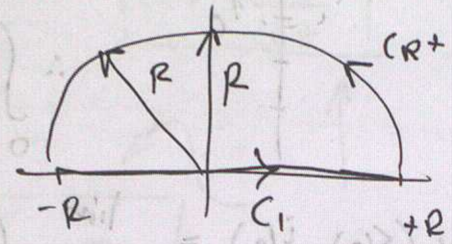
$$= \frac{1}{2i}(2\pi) = \boxed{\pi}$$

↑ simple poles at $z = \pm i$

↑ only the pole at $z = i$ matters

↑ result of closed integral, not our answer what about $\int_{CR+} \frac{1}{z^2+1} dz \to$ ML bounding Thm

$$\left| \int_{CR+} \frac{1}{z^2+1} dz \right| \leq ML$$



$M = \frac{1}{z^2+1} \Big|_{\max \text{ on } C_{R+}} \approx \frac{1}{R^2}$, $L = \pi R$, $ML = \frac{\pi}{R}$

$\therefore \left| \int_{CR+} \frac{1}{z^2+1} \right| \leq \frac{\pi}{R}$, so as $R \to \infty$, this contribution is zero

$$\therefore \oint_{C_1+C_{R+}} \frac{1}{z^2+1} dz = \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \boxed{\pi} = \tan^{-1}(x) \Big|_{-\infty}^{\infty}$$

- power of numerator must be ~~half~~ 2less (or less) of denominator
- pour of den. 2 or mor powr of numerator

$$\boxed{f(z) = \frac{P(z)}{Q(z)}, \text{ if degree } Q \geq 2 + \text{degree } P}$$

then $\int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum \text{Res}$

Week 12: Lecture 3

Nov 28, 2024

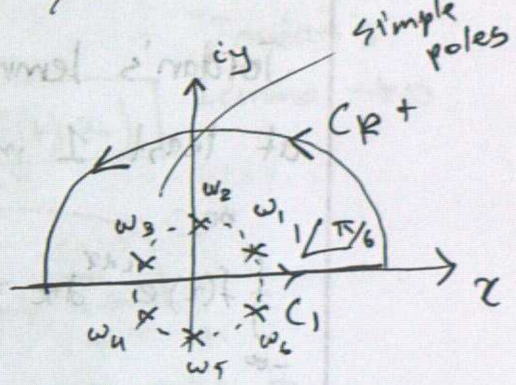
Cauchy Principal Values

- using complex integration $\int_{-\infty}^{\infty} \frac{dx}{1+x^6}$

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{1+x^6} = \oint_{C_1+C_R^+} \frac{dz}{1+z^6} = \int_{C_1} \frac{dx}{1+x^6} + \cancel{\int_{CR^+} \frac{dz}{1+z^6}}$$

0 since order of $Q(z) \geq 2+$ order of $f(z)$

$$= 2\pi i \sum \text{Res}\left(\frac{1}{1+z^6}, z_k\right)$$



For $\oint_{C_1+C_R^+} \frac{1}{1+z^6} dz \to$ what are our singularities?

$1+z^6 = 0 \Rightarrow z^6 = -1 = 1\angle\pi$ (find 6th roots of -1)

$$\oint_{C_1+C_R^+} \frac{1}{1+z^6} dz = \text{P.V.} \int_{-\infty}^{\infty} \frac{1}{1+x^6} dx = 2\pi i \sum_{k=1}^{3} \text{Res}\left(\frac{1}{1+z^6}, \omega_k\right)$$

$\omega_1 = 1\angle\pi/6$, $\omega_2 = 1\angle\pi/3$, $\omega_3 = 1\angle 5\pi/6$

$$\text{Res}\left(\frac{1}{1+z^6}, 1\angle\pi/6\right) = (z-\omega_1)\left(\frac{1}{1+z^6}\right)\Big|_{z=\omega_1} = \frac{g(\omega_1)}{h'(\omega_1)}$$

$\to g(z) = 1$

$\to h(z) = 1+z^6$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^6} dx = 2\pi i \left[\frac{1}{6\angle 5\pi/6} + \frac{1}{6\angle 5\pi/2} + \frac{1}{6\angle 25\pi/6} \right] = \frac{2\pi}{3} //$$

19.6 Real Integrals Part 2

Form #3: $\int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$

$$\Rightarrow \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx = \int_{-\infty}^{\infty} f(x)\cos(\alpha x) dx + i\int_{-\infty}^{\infty} f(x)\sin(\alpha x) dx$$

Jordan's lemma: degree of denominator must be at least 1 more than degree of the numerator

$$\int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx = \text{P.V.}\int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx = \lim_{R\to\infty}\int_{-R}^{R} f(x) e^{i\alpha x} dx$$

$$= \begin{cases} \int_{C_1} f(x) e^{i\alpha x} dx + \int_{C_R^+} f(x) e^{i\alpha x} dx = 2\pi i \sum_{k=1}^{k} \text{Res}(f(x), z_k) \\ \int_{C_1} f(x) e^{-i|\alpha| x} dx + \int_{C_R^-} f(x) e^{-i|\alpha| x} dx = -2\pi i \sum_{k=1}^{k} \text{Res}(f(x), z_k) \end{cases}$$

upper half singularities

lower half singularities

$\alpha > 0$ → choose upper half

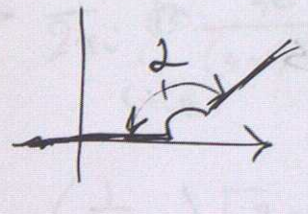
$\alpha < 0$ → choose lower half

Week 13: Lecture 1

Dec 2, 2024

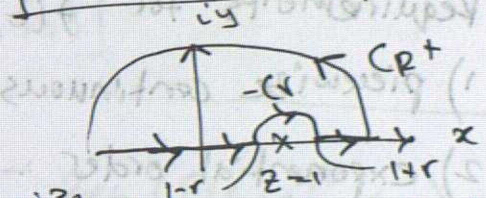
Indentation of Contours Simple Poles

$$\oint_C f(z) dz = \int_{-R}^{c-r} f(x) dx + \int_{-C_r} f(z) dz + \int_{c+r}^{R} f(x) dx + \underbrace{\int_{C_R^+} f(z) dz}_{\text{Jordan's Lemma} \to 0}$$



just include some fraction of the residue → $\boxed{-\pi i \text{Res}(f(z), z_0)}$

Ex: $\text{P.V.}\int_{-\infty}^{\infty} \frac{\cos 3x}{x-1} dx$



1) convert to $f(z)$: $f(z) = \frac{e^{i3z}}{z-1}$ ← real part to isolate $\cos 3x$

2) Singularities of $f(z)$ on contour: $z = 1$

3) Incorporate indentation of contours

~~$\oint \frac{e^{i3z}}{z-1} dz = \int_{-R}^{1-r} \frac{e^{i3x}}{x-1} dx + \int_{-C_r} \frac{1}{e^{i}}$~~

Week 13: Lecture 2 Dec 3, 2024

Laplace Transform

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt \quad \text{with } s = \sigma + i\omega$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)e^{st}\,ds$$

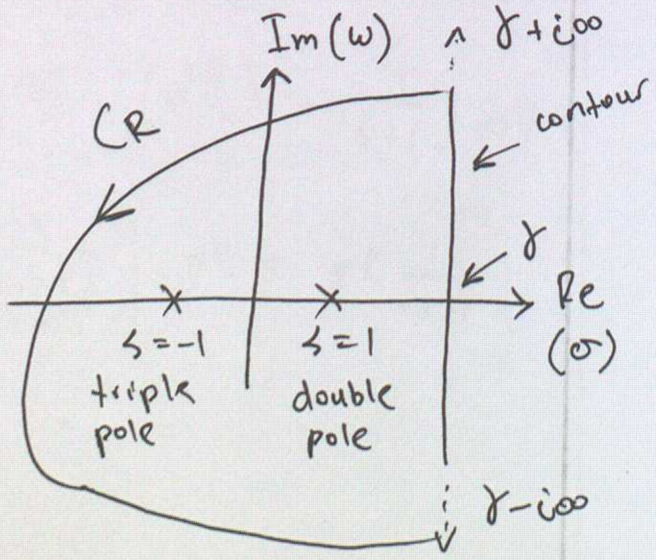
- Requirements for $f(t)$
  1) piecewize continuous
  2) exponential order : $\lim_{s\to\infty} |F(s)| = 0$

Ex: Find $\mathcal{L}^{-1}\{F(s)\}$ for $F(s) = \frac{s}{(s+1)^3(s-1)^2}$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)\,e^{st}\,ds$$

- think of s as z where $s = \sigma + i\omega$ ← $\omega = 2\pi f$

$$\Rightarrow \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{se^{st}}{(s+1)^3(s-1)^2}\,ds$$



- what is γ ? What Contour?

  ↳ Rule: Place this contour to the right of all the poles of F(s)

- To use residue theory, we can evaluate this
  → close up contour with C_R
- For $f(t)$ when $t \geq 0$, close up the contour to the left

$$\Rightarrow \frac{1}{2\pi i}\oint_{C+C_R} \frac{se^{st}}{(s+1)^3(s-1)^2}\,ds = \frac{1}{2\pi i}\int_{C_1} \frac{se^{st}}{(s+1)^3(s-1)^2}\,ds + \frac{1}{2\pi i}\int_{CR} \frac{se^{st}}{(s+1)^3(s-1)^2}\,ds$$

$$= \left(\frac{1}{2\pi i}\right)\left[2\pi i \sum_{k=\phi}^{2} \text{Res}\left(F(s)e^{st}, s_k\right)\right]$$

$$\boxed{f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)e^{st}\,ds = \sum_k \text{Res}\left(F(s)e^{st}, s_k\right)}$$

↖ sum of ~~all the~~ residues of all poles of F(s)

$$\text{Res}\left(F(s)e^{st}, s_k\right) = \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\left[(s-s_k)^n F(s)e^{st}\right]\Bigg|_{s=s_k}$$

For $F(s) = \frac{s}{(s+1)^3(s-1)^2}$

t is a constant since der. w.r.t. s

$$\text{Res}\left(F(s)e^{st}, s=1\right) = \lim_{s\to 1}\frac{1}{(2-1)!}\frac{d}{ds}\left[\frac{se^{st}\cancel{(s-1)^2}}{(s+1)^3\cancel{(s-1)^2}}\right] = \frac{1}{16}e^{t}(2t-1)$$

$$\text{Res}\left(F(s)e^{st}, s=-1\right) = \lim_{s\to -1}\frac{1}{(3-1)!}\frac{d^2}{ds^2}\left[\frac{se^{st}\cancel{(s+1)^3}}{\cancel{(s+1)^3}(s-1)^2}\right] = \frac{1}{16}e^{-t}(1-2t^2)$$

$$\therefore f(t) = \mathcal{L}^{-1}\{F(s)\} = \left[\frac{1}{16}e^{t}(2t-1) + \frac{1}{16}e^{-t}(1-2t^2)\right]u(t) \quad //$$

valid for $t \geq 0$ only

$$\underline{\text{Ex:}}\quad X(x) = \underset{\uparrow G(s)}{\frac{4se^{-s}}{(s+2)(s+1)^2}} + \underset{\uparrow H(s)}{\frac{5s}{(s+1)^2}}$$

$$f(t) = \sum_k \operatorname{Res}(F(s)e^{st}, s_k)$$

$$\operatorname{Res}(G(s)e^{st}, s=-2] = \lim_{s\to-2}\left[\frac{4se^{-s}e^{st}\cancel{(s+2)}}{\cancel{(s+2)}(s+1)^2}\right]$$

$$= \lim_{s\to-2}\left[\frac{4se^{s(t-1)}}{(s+1)^2}\right] = -8e^{-2(t-1)}u(t-1)$$

$t-1 \geq 0$

valid for $t-1 \geq 0$

or $t \geq 1$

$$\operatorname{Res}(G(s)e^{st}, s=-1) = 4e^{-(t-1)}u(t-1)(3-t)$$

$$\operatorname{Res}(H(s)e^{st}, s=-1) = 5e^{-t}(1-t)u(t)$$

$$\therefore x(t) = 5e^{-t}(1-t)u(t) + \left[4e^{-(t-1)}(3-t) - 8e^{-2(t-1)}\right]u(t-1) \quad //$$