Week 1: Introduce

→ vectors, vector addition, scalar multiplication, dot product, orthogonal vectors, parallel vectors

→ line, plane, vector form, parametric form, direction vector, normal vector, sets

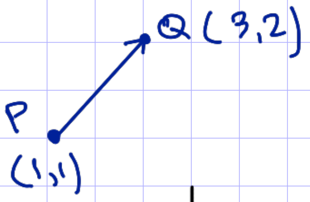
Reading: textbook Appendix A until dot products

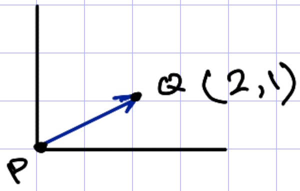
Introduction

Vectors

$$\vec{PQ} = \begin{bmatrix} q_1 - p_1 \\ q_2 - p_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

"move 2 units horizontally and 1 unit vertically"





- standard representation of a vector ... $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an arrow connecting the origin to (x_1, x_2)
- $\vec{x}$ is the position vector of point (x_1, x_2)

3-d

$P = (p_1, p_2, p_3)$
$Q = (q_1, q_2, q_3)$

$$\vec{PQ} = \begin{bmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - \boxed{p_3} \end{bmatrix}$$

these "entries" are called components

→ The collection of all column vectors with n components is denoted by $\mathbb{R}^n$ ← n-dimentional Euclidean vector space

$$\mathbb{R}^n = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| \, a_1 \cdots a_n \text{ are in } \mathbb{R} \right\}$$

Cartesian Plane

$R^2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right\}$

set of all vectors with 2 components $\rightarrow$ set of all points on Cartesian plane $\rightarrow$ R^2 is the Cartesian plane

$R^3 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right\} \rightarrow$ set of all vectors with 3 components $\rightarrow$ R^3 is the xyz-space

Example: Vector Addition

$\vec{v} = \begin{bmatrix} 1 \\ 2.5 \\ \pi \\ -2 \end{bmatrix} \quad \vec{\omega} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix}$ in R^4 and scalar $k = -4$ in R

a) $\vec{v} + \vec{\omega} = \begin{bmatrix} 1+0 \\ 2.5+3 \\ \pi+1 \\ -2-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5.5 \\ \pi+1 \\ -3 \end{bmatrix}$

b) $k\vec{v} = \begin{bmatrix} -4 \times 1 \\ -4 \times 2.5 \\ -4 \times \pi \\ -4 \times -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -10 \\ -4\pi \\ 8 \end{bmatrix}$

Parallel Vectors

$\vec{v}$ and $\vec{\omega}$ are parallel if there is a scalar k : $\vec{v} = k\vec{\omega}$

eg: $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix}$ and $\vec{\omega} = \begin{bmatrix} -3 \\ 6 \\ -9 \\ -12 \end{bmatrix}$ are parallel since $\vec{\omega} = -3\vec{v}$

PCE 1 Check-in Part 1

1. $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \vec{\omega} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad 2\vec{v} + \vec{\omega} = \begin{bmatrix} (2\times1)+1 \\ (2\times2)+2 \\ (2\times-1)+-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$

$0 = (\underset{x_1}{1}, \underset{y_1}{2}, \underset{z_1}{-2})$

$(2\vec{v} + \vec{\omega}) + 0 = (3+1, 6+2, -2-3)$

$= (4, 8, -5)$

$2\vec{v} + \vec{\omega} = (3, 6, -3)$

Week 1 - Solidify

Sep 13, 2023

Dot Product

$\vec{v}$ and $\vec{w}$ vectors denoted: $v_1, v_2 \cdots v_n$ and $w_1, w_2 \cdots w_n$

- The dot product is a scalar denoted by:

$$\boxed{\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n}$$

Norm

- length of vector (its magnitude)

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_3 \end{bmatrix}$$ is in $\mathbb{R}^n$ →

the norm is: $$\boxed{\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots v_n^2}}$$

Excersize

$$\begin{aligned} \vec{v} \cdot \vec{v} &= (v_n)(v_n) \\ &= v_n^2 \\ \|\vec{v}\| &= \sqrt{v_n^2} \\ &= \|\vec{v}\| \end{aligned}$$

Angle

Give $\vec{v}$ and $\vec{w}$, angle is: $$\boxed{\arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}\right)}$$

Perpendicular

$\vec{v}$ and $\vec{w}$ are perpendicular is $\vec{v} \cdot \vec{w} = 0$

Excersize

a) Yes

b) 90°, $0 \leq \theta \leq 180°$

c) Yes, because any vector in $\mathbb{R}^n$ can be "dot-producted" to get 0.

Sets Intro

Set: collection of objects ← elements or members of the set

Let S be a set...

Eg: "D" denotes set of even #s.

$a \in S$ ← a is element in the set

$a \notin S$ ← a is not element in set

$4 \in D$ and $3 \notin D$

Examples of Sets

$\mathbb{R}$: set of real numbers

$\mathbb{Q}$: set of rational #'s

$\mathbb{C}$: set of complex #'s

$\mathbb{Z}$: set of integers

$\mathbb{N}$: set of natural #'s

Describing Sets

- list all elements : $M = \{ \text{Owais}, \text{Zaeem}, \text{Saulih} \}$
- Eg: all even integers :

$$E = \{ 2k \mid k \in \mathbb{Z} \}$$

$$E = \{ z \in \mathbb{Z} \mid z = 2k, \text{ for some integer } k \}$$

Subsets

- A is a subset of B : $A \subseteq B$

↳ if all elements of A are also in B : $A \subseteq B$ for every $a \in A$, $a \in B$

Equality of Sets

- sets A and B are equal : $A = B$

↳ if A is a subset of B and B is a subset of A : $A = B$ if $A \subseteq B$ and $B \subseteq A$

Union and Intersection of Sets

- Let X be a set and A and B be subsets of X : $\boxed{A \subseteq X \text{ and } B \subseteq X}$

- The union of A and B is a set that contains all elements of A and B : $\boxed{A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}}$

- The intersection of A and B is the set of all common elements between A and B : $\boxed{A \cap B = \{x \in X, x \in A \text{ and } x \in B\}}$

Eg: Let $A = \{2, 5, 7, \pi\}$ and $B = \{4, \pi, 5\}$ be subsets of $\mathbb{R}$

→ $A \cup B = \{2, 5, 7, \pi\}$ … put all of them

→ $A \cap B = \{5, \pi\}$ … put only ones in common.

PCE Check-in Part 2

1. $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$$\|\vec{v}\| = \sqrt{1^2 + 2^2 + (-1)^2}$$
$$= \sqrt{1 + 4 + 1}$$
$$= \sqrt{6}$$

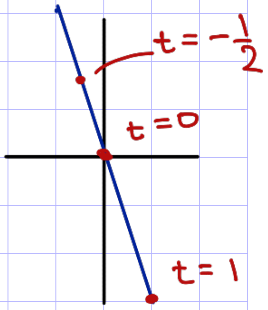
Week 1: Expand

Sep 13, 2023

Line in $\mathbb{R}^n$

- line $y = -2x$ ← line ℓ on Cart. plane, passing origin, slope -2

↳ in set notation: the line ℓ represents the set of all points (x,y) on the Cart. plane that satisfy the equation $y=-2x$



$$\ell = \{(x,y) \mid y=-2x\} = \{(t,-2t) \mid t \in \mathbb{R}\}$$

all points (x,y)

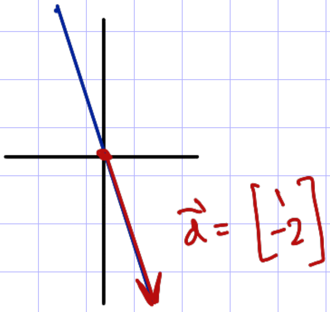
where

y equals $-2x$

★ Value of x is scalar t
so y is $-2t$
$(x,y) \rightarrow (t,-2t)$

$$\ell = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid y=-2x \right\} = \left\{ \begin{bmatrix} t \\ -2t \end{bmatrix} \mid t \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} 1 \\ -2 \end{bmatrix} \mid x \in \mathbb{R} \right\} \subseteq \mathbb{R}^2$$

contains all set of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ vectors : $\vec{d} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is a direction vector



Line Equations

- vector form or vector-parametric form of line : $\ell = \{ \vec{p} + t\vec{d} \mid t \in \mathbb{R} \}$

- parametric form : $\ell = \left\{ \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \right\} \longrightarrow \begin{matrix} \ell_1 = td_1 + p_1 \\ \ell_2 = td_2 + p_2 \\ \ell_3 = td_3 + p_3 \end{matrix}$

Line Definition

A line in $\mathbb{R}^n$ is any set of the form : $\boxed{L = \{ t\vec{m} + \vec{b} \mid t \in \mathbb{R} \}}$

... where $\vec{m}$ and $\vec{b}$ are non-zero vectors in $\mathbb{R}^n$

Plane Equations

↳ $\vec{n}$ ⊥ to plane P

↳ pick a point on P ← call this $\vec{p}$ (position vector)

↳ pick an arbitrary point on P ← call this $\vec{x}$

↳ $(\vec{x} - \vec{p})$ ← vector on plane, so ⊥ to $\vec{n}$

$\therefore \; \vec{n} \cdot (\vec{x} - \vec{p}) = 0$

⟹ plane P : $\boxed{P = \{ \vec{x} \in \mathbb{R}^3 \mid \vec{n} \cdot (\vec{x} - \vec{p}) = 0 \}}$

position vector for specific point on plane P

Eg: Write equation for plane P

↳ passing through origin : so $\vec{p} = \vec{0} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

↳ $\vec{n} = \begin{bmatrix}1\\2\\3\end{bmatrix}$

→ Let $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ ... so $\vec{x} \cdot \vec{n} = 0 = 1x_1 + 2x_2 + 3x_3$

$\vec{n} = \begin{bmatrix}n_1\\n_2\\n_3\end{bmatrix}$ passing through point $p = (p_1, p_2, p_3)$ is ...

$n_1x_1 + n_2x_2 + n_3x_3 = D$, where $D = n_1p_1 + n_2p_2 + n_3p_3$

Plane Definition

A plane in $\mathbb{R}^n$ is a set of the form ... : $\boxed{P = \left\{ t\vec{m} + s\vec{n} + \vec{b} \mid s,t \in \mathbb{R} \right\}}$

PCE Check-in Part 3

$\vec{v} = \begin{bmatrix}-1\\2\end{bmatrix}$ $\vec{w} = \begin{bmatrix}2\\3\end{bmatrix}$ $\vec{u} = \begin{bmatrix}1\\1\end{bmatrix}$

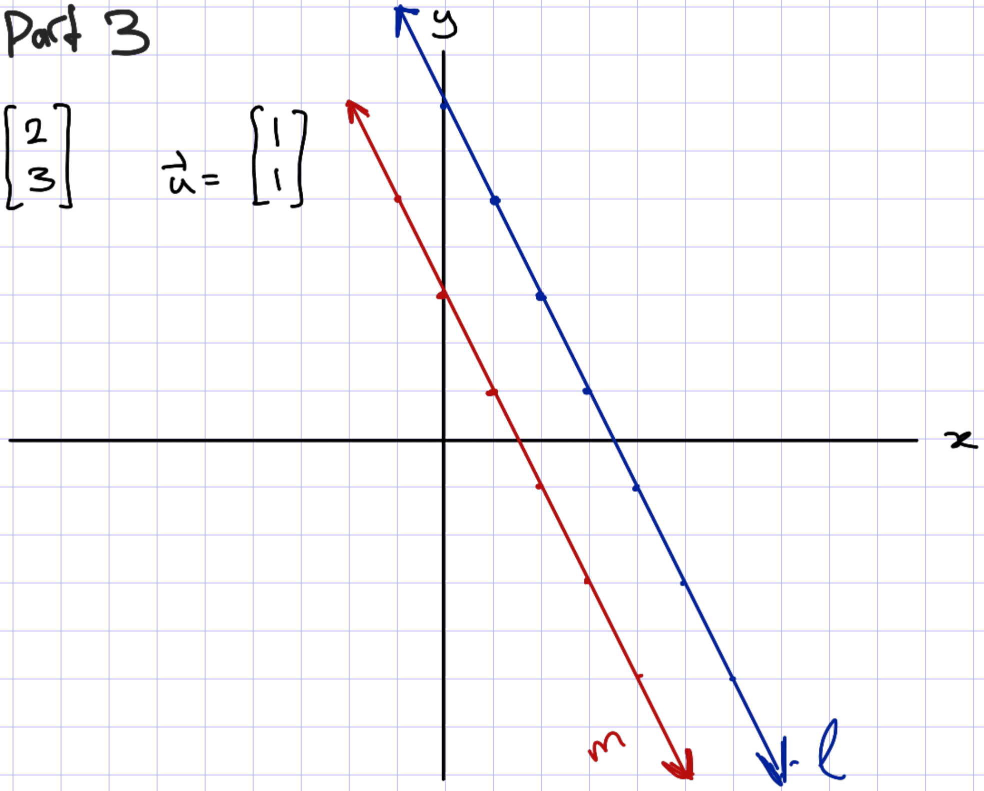
$l = t\vec{v} + \vec{w}$

$m = k\vec{v} + \vec{u}$

↓

$l = \vec{w} + t\vec{v}$

$m = \vec{u} + k\vec{v}$



Week 2: Introduce | Sep 17, 2023 | Systems of Linear Equations

duck = 4 = D

5 sparrows = 1 = S

rooster = 1 = R

100 birds for 100 coins

↓

$D + S + R = 100$ # of birds

$4D + \frac{1}{5}S + R = 100$ # cost

Linear Equation:

with variables $x_1, x_2 \cdots x_m$ → equation of the form...

$$a_1x_1 + a_2x_2 + \cdots a_mx_m = b$$

scalars

Linear Systems:

$$\begin{aligned} 2x_1 + 4x_2 &= 8 \\ 4x_1 + 3x_2 &= 10 \\ 2x_1 - x_2 &= 2 \end{aligned}$$

$$\begin{aligned} 4x_1 - 8x_2 &= 8x_3 \\ 22x_1 + 3x_3 &= 33 \\ 11x_2 + 2x_3 &= -2x_1 + 1 \end{aligned}$$

$2x_1 + 3x_2 = 5$ a line in x_1x_2-plane, or $\mathbb{R}^2$

$2x + 3y = 5$

$2x_1 + 3x_2 + 4x_3 = 0$ a plane in $\mathbb{R}^3$ passing through origin and $\perp$ to $\vec{n} = [2, 3, 4]$

System:

↳ collection of these objects

Solution:

↳ a point

↳ a position vector of a point

} that simultaneously satisfies all equations

Gauss - Jordan Elimination

- multiplying equation by scalar
- adding multiple of equation to another one
- switching order of equations

$\left.\right\}$ Do not change general sol. to system

Matix : n × m array of numbers

augmented column

$$\begin{aligned} 2x_1 + 4x_2 &= 8 \\ 4x_1 + 3x_2 &= 10 \\ x_1 - x_2 &= 2 \end{aligned} \qquad \begin{bmatrix} 2 & 4 \\ 4 & 3 \\ 1 & -1 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 2 & 4 & 8 \\ 4 & 3 & 10 \\ 1 & -1 & 2 \end{array}\right]$$

linear system — coefficient matrix — augmented matrix

Excersize:

$$\begin{aligned} 4x_1 - 8x_2 &= 8x_3 \\ 22x_1 + 3x_3 &= 33 \\ 11x_2 + 2x_3 &= -2x_1 \end{aligned} \longrightarrow \begin{aligned} 4x_1 - 8x_2 - 8x_3 &= 0 \\ 22x_1 + 0x_2 + 3x_3 &= 33 \\ 2x_1 + 11x_2 + 2x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 4 & -8 & -8 \\ 22 & 0 & 3 \\ 2 & 11 & 2 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 4 & -8 & -8 & 0 \\ 22 & 0 & 3 & 33 \\ 2 & 11 & 2 & 0 \end{array}\right]$$

Elementary Row Operations

1. Interchanging 2 rows
2. Multiplying one row by a non zero number
3. Replacing a row by itself plus a multiple of a different row.

Row Echelon Form (REF)

1. All zero rows are in the bottom
2. The leading entry in each row is to the right of the leading entry of the row above
3. All entries below a leading entry are zero

Reduced Row Echelon Form (RREF)

... all the entries in REF plus:

4. All the leading entries are 1. ... "leading 1's"
5. Each leading 1 is the only nonzero entry in its column

unique

* Given matrix B ⟶ many REF, but only one RREF

PCE2: Check-in 1

5 unknowns

$$B = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 2 & 9 \end{bmatrix} \longrightarrow \begin{aligned} x_1 + 4x_4 &= 0 \\ x_2 + 2x_3 + 9x_4 &= 0 \end{aligned}$$

Row Reduction

$$\begin{bmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & -12 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 4 & -12 & -4 & 24 \\ 0 & -1 & 3 & 4 & -12 \end{bmatrix}$$

Week 2: Solidify

leading entry (leading 1)

pivot position

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot column

free variables since corresponding variables in linears systems are not in pivot column

corresponding variables called: leading variable

basic variable

dependent variable

$$\begin{array}{ccccc} b & b & f & b & \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

x_1, x_2, x_4 ← basic variables

x_3 ← free variables

General Solution

x_3 : free variable bc. no pivot

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\begin{cases} x_1 + 2x_3 = 0 \\ x_2 - 3x_3 = 4 \\ x_4 = -2 \end{cases}$$

x_1, x_2, x_3 → basic variables

only show up in 1 equation

$$\begin{cases} x_1 + 2x_3 = 0 \\ x_2 - 3x_3 = 4 \\ x_4 = -2 \end{cases} \qquad \begin{cases} x_1 = -2x_3 \\ x_2 = 3x_3 + 4 \\ x_4 = -2 \end{cases}$$

x_3 is a free variables (I can choose any value for x_3)

let $x_3 = 0$

$x_1 = 0, x_2 = 4, x_3 = 0, x_4 = -2$

$\hookrightarrow \begin{bmatrix} 0 \\ 4 \\ 0 \\ 2 \end{bmatrix}$ vector in $\mathbb{R}^4$

$x_1 = -2t$
$x_2 = 3t + 4$
$x_4 = -2$

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ 3t+4 \\ t \\ -2 \end{bmatrix}, \ t \in \mathbb{R} \right\}$$

* let x_3 (the free variable) equal to t

$$\longrightarrow t \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 0 \\ -2 \end{bmatrix}$$

$$\downarrow$$

$$\left\{ t \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 0 \\ -2 \end{bmatrix} \right\}$$

Solution Types

consistent ... at least 1 solution

inconsistent ... no solutions

Excersize 1

RREF

$$\begin{cases} ax_1 + bx_2 = e \\ cx_1 + dx_2 = f \end{cases} \longrightarrow \left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & e \\ 0 & 1 & f \end{array}\right]$$

- parallel ~ no solution
- on each othes ~ infinite solutions
- intersect ~ 1 solution

Theorem 2.1 (Solution Type)

linear system constistent if (and only if):

↳ RREF has 0 = 1 nowhere

↳ or else : * infinety many sol. (free variab)

* 1 solution.

Definition : Rank

rank of matrix A is the number of leading ones in ref(A)

- A → coefficient matrix
  B → augmented matrix

- if rank(B) > rank(A) → inconsistent
- if rank(A) = rank(B) → consistent

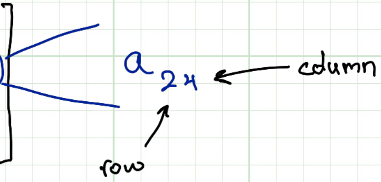
since in order for rank B to be higher, one of the leading ones had to be in the "answer" column of augmented matrix

Week 2: Expand

Matrix Algebra

- matrix : array of numbers
- size of matrix : (# of rows) x (# of columns)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & \boxed{a_{24}} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$



- entry a_{ij} : i'th row and j'th column
- 2 matrice A and B are equal if: $a_{ij} = b_{ij}$
- if A is $(n \times n)$, it's a square matrix
  - ↳ entries a_{11}, a_{22}, a_{33} ··· a_{nn} form main diagonal
  - ↳ if all entries above and below main diagonal are zero, it's called a diagonal : $a_{ij} = 0$ where $i \neq j$
  - ↳ upper triangle : all entries below main diagonal are 0 ($a_{ij} = 0$ where i exceed j)
  - ↳ lower triangle : all entries above main diagonal are 0 ($a_{ij} = 0$ where i is less than j)
- zero matrix : all entries are zero
- identity matrix : all diagonal entries are 1 → I_n, n indicates matrix is $n \times n$
- adding 2 matrices of same size: adding up corresponding entries
- multiplying scalar : multiplying it to all entries

Matrix Vector Product:

- Let A be $n \times m$ w/n row vectors $\vec{\omega}_1, \vec{\omega}_2 \cdots \vec{\omega}_n$
- Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$ be a vector in $\mathbb{R}^m$

$$A\vec{x} = \begin{bmatrix} \vec{\omega}_1 \\ \vec{\omega}_2 \\ \vdots \\ \vec{\omega}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \vec{\omega}_1 \cdot x_1 \\ \vec{\omega}_2 \cdot x_2 \\ \vdots \\ \vec{\omega}_n \cdot x_n \end{bmatrix}$$

- $A\vec{x}$ is column vector with n components
- i'th component of $A\vec{x}$ = i'th row of $A \cdot \vec{x}$

- linear system with augmented matrix $[A \mid \vec{b}\,]$ → rewrite as $A\vec{x} = \vec{b}$
- product of $A\vec{x}$ in terms of columns of A:

$$A\vec{x} = \begin{bmatrix} | & | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_m \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 \cdots x_m\vec{v}_m$$

Linear Combination

- linear comb. of vectors $\vec{v}_1, \vec{v}_2 \cdots \vec{v}_n$ in $\mathbb{R}^n$ is:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

each c_i is a scalar

Week 3: Introduce Linear Transformation

- **transformation**: another word for **function** or **mapping**
  ↳ takes inputs, applies rules, gives output
- **domain**: space from which inputs for transformation comes from **(source)**
- **codomain**: space in which output vectors lie **(target)**
- **range**: set of actual output vectors

→ **Eg:** $f(x) = x^2$

- input and output are real numbers → function has $\mathbb{R}$ as domain and codomain

$\boxed{f: \mathbb{R} \rightarrow \mathbb{R}}$ or $\boxed{f = \mathbb{R} \xrightarrow{f} \mathbb{R}}$

→ **Eg:** $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$ ← range is just x-axis of $\mathbb{R}^2$ since second coordinate is zero

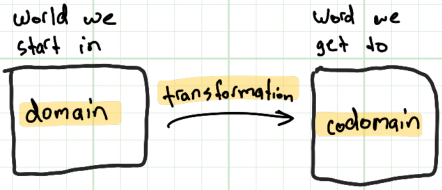
input is any vector in $\mathbb{R}^3$

output is vector in $\mathbb{R}^2$

$\boxed{T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2}$

or

$\boxed{T: \mathbb{R}^3 \xrightarrow{T} \mathbb{R}^2}$



→ **Eg:** $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ $\quad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ xy \end{bmatrix}$

plug in $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

→ **Eg:** $T: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

plug in $\begin{bmatrix}1\\1\end{bmatrix}$ $\quad T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x+2y\\3x+4y\\5x+6y\end{bmatrix}$ ... multiply matrix with vector

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1+1\\3+4\\5+6\end{bmatrix} = \begin{bmatrix}3\\7\\11\end{bmatrix}$$

Linear Transformations (a special class of transformation)

mapping $\mathbb{R}^m \xrightarrow{T} \mathbb{R}^n$ such that:

1. adding 2 vectors, transforming them = transform vector 1 + transform vector 2

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$$

2. multiply vector by k, transform = transform vector, multiply result by k

$$T(k\vec{u}) = kT(\vec{u}) ,$$

similar to matrix-vector multiplication

$$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} \qquad A(k\vec{x}) = kA(\vec{x})$$

→ **Eg:** Prove that $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x\\xy\end{bmatrix}$ is not linear transformation

$\vec{u} = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad T\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right)$ must equal $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$

$\vec{v} = \begin{bmatrix}0\\1\end{bmatrix}$ $\quad$ but $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = 0$

since they're not equal, T is not linear trans.

Week 3: Solidify

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}\frac{3}{2}x+y\\2y\end{bmatrix} \qquad \vec{u}=\begin{bmatrix}u_1\\u_2\end{bmatrix} \qquad \vec{v}=\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

Prop. 1: $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{u})$

$$T\left(\begin{bmatrix}u_1+v_1\\u_2+v_2\end{bmatrix}\right)=\begin{bmatrix}\frac{3}{2}(u_1+v_1)+(u_2+v_2)\\2(u_2+v_2)\end{bmatrix}$$

$$=\begin{bmatrix}\frac{3}{2}u_1+\frac{3}{2}v_1+u_2+v_2\\2u_2+2v_2\end{bmatrix}$$

$$=\begin{bmatrix}\frac{3}{2}u_1+u_2\\2u_2\end{bmatrix}+\begin{bmatrix}\frac{3}{2}v_1+v_2\\2v_2\end{bmatrix}$$

$$=T\left(\begin{bmatrix}u_1\\u_2\end{bmatrix}\right)+T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)$$

∴ Prop 1 holds

Lin. Trans. and Vector-Matrix Mult.

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}\frac{3}{2}x+y\\2y\end{bmatrix} \Longrightarrow T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}\frac{3}{2}&1\\0&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

vector-matrix multiplic.

$$T(\vec{x})=A\vec{x}$$

... so to test this out, let's try Prop. 1 and Prop. 2

$$\vec{u}=\begin{bmatrix}u_1\\u_2\end{bmatrix}=u_1\begin{bmatrix}1\\0\end{bmatrix}+u_2\begin{bmatrix}0\\1\end{bmatrix}$$

$$T(\vec{u})=T\left(u_1\begin{bmatrix}1\\0\end{bmatrix}\right)+T\left(u_2\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$=u_1T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)+u_2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

write as vector-matrix multip.

$$\begin{bmatrix}\underbrace{T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)} & \underbrace{T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)}\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}$$

columns of matrix

$$A\vec{x}=T(\vec{x})$$

Standard Vectors

- standard vector $\vec{e}_i \in \mathbb{R}^m$ is $\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ where there is a "1" in the "i" th entry

→ Eg: $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{3}{2}x + y \\ 2y \end{bmatrix}$

standard matrix representation : $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{3}{2} & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

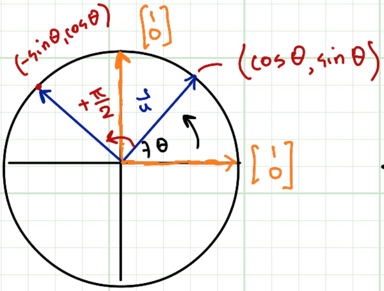
Week 3: Expand

Rotations (in the plane) in $\mathbb{R}^2$

Rotate $\vec{u} \in \mathbb{R}^2$ by θ radians in (+) direction:

$$T(\vec{u}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{u}$$

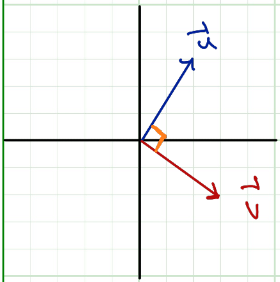
↳ $T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) =$ "rotating $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by θ degrees will land us at some point given by $(\cos\theta, \sin\theta)$"



$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \cos(\theta + \frac{\pi}{2}) \\ \sin(\theta + \frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Orthogonal Projections in $\mathbb{R}^2$

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} u_2 \\ -u_1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$



take any vector $\vec{x} \in \mathbb{R}^2$

$$\vec{x} = k\vec{u} + t\vec{v}$$

$\vec{x}$ can be written as any combin. of vectors parallel to $\vec{u}$ and $\vec{v}$.
"think vector x-y components"

solve $\vec{x} = k\vec{u} + t\vec{v}$:

$$k = \frac{\vec{u} \cdot \vec{x}}{|\vec{u}|^2} \implies \left.\begin{aligned} \vec{x}'' &= k\vec{u} \\ &= \left(\frac{\vec{u} \cdot \vec{x}}{|\vec{u}|^2}\right)\vec{u} \end{aligned}\right\} \text{projection of } \vec{x} \text{ on } \vec{u}$$

$$\text{proj}_{\vec{u}}(\vec{x}) = \left(\frac{\vec{u}\cdot\vec{x}}{|\vec{u}|^2}\right)\vec{u}$$

$$\text{proj}_{\vec{u}}(\vec{x}) = \left(\frac{\vec{u}\cdot\vec{x}}{|\vec{u}|^2}\right)\vec{u} = \begin{bmatrix} \frac{u_1^2}{|\vec{u}|^2} & \frac{u_1u_2}{|\vec{u}|^2} \\ \frac{u_1u_2}{|\vec{u}|^2} & \frac{u_2^2}{|\vec{u}|^2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

→ Eg: proj of $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$ on $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Let $\vec{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$\text{proj}_{\vec{u}}(\vec{x}) = \left(\frac{-5(3)+3(1)}{\left(\sqrt{3^2+1^2}\right)^2}\right)\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

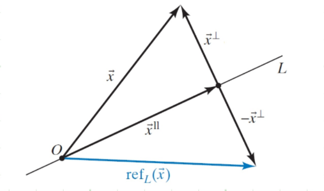
$$= -\frac{6}{5}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{18}{5} \\ -\frac{6}{5} \end{bmatrix}$$

Reflections

- line L passing through origin, direction of $\vec{u}$
- reflection of $\vec{x}$ about L

↳ $\text{ref}_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}$

$\text{ref}_L(\vec{x}) = 2\,\text{proj}_{\vec{u}}(\vec{x}) - \vec{x}$



Week 4: Introduce Composition and Inverse of Linear Transform.

Recall

linear trans. $T: \mathbb{R}^m \to \mathbb{R}^n$ has matrix representation where $T(\vec{x}) = A\vec{x}$, columns of A are found by evaluating T at corresponding standard vectors $\vec{e}_1, \vec{e}_m \cdots$

A ↙ $n \times m$ matrix

- composition of linear transformation:

① $\vec{v} \in \mathbb{R}^2 \to$ reflect across $y = 2x \to$ output

transformation L

② output $\to$ rotate ccw $\frac{\pi}{2}$ rad $\to$ answer

transformation R

} $R(L(\vec{v}))$ or $(R \circ L)(\vec{v})$

Composition and Order

$T: \mathbb{R}^3 \to \mathbb{R}^2$
$S: \mathbb{R}^2 \to \mathbb{R}^2$

} $T(S(\vec{x}))$ … undefined $T \circ S$ ✗
$S(T(\vec{x}))$ … defined $S \circ T$ ✓

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, bend left, "S"] & \mathbb{R}^2 \arrow[r, bend left, "\cancel{T}"] & \mathbb{R}^2 \end{tikzcd}$$

↑ codomain of S not in dom T

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, bend left, "T \checkmark"] & \mathbb{R}_2 \arrow[r, bend left, "S \checkmark"] & \mathbb{R}^2 \end{tikzcd}$$

↑ codomain of T in domain of S

→ Eg: $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 2x+y \\ x-y \end{bmatrix}$ $\quad S(x) = \begin{bmatrix} 4x \\ x \end{bmatrix}$

$T: \mathbb{R}^2 \to \mathbb{R}^2$ … $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 2(1)+0 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$S: \mathbb{R} \to \mathbb{R}$ … $S\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ undefined

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, bend left, "T"] & \mathbb{R}^2 \arrow[r, bend left, "\cancel{S}"] & \mathbb{R} \end{tikzcd}$$

↑ codom T not in dom of S

$\therefore S \circ T$ undefined

$$\begin{tikzcd} \mathbb{R} \arrow[r, bend left, "S"', "\checkmark"] & \mathbb{R} \arrow[r, bend left, "T"', "\checkmark"] & \mathbb{R}^2 \end{tikzcd}$$

$\therefore T \circ S$ defined

even though S outputs only some vectors in $\mathbb{R}^2$. It's okay as long as for every original input, we can compute composition

- Can any 2 transformations be composed?
  ↳ only if codomain of 1rst transformation is equal to the domain of second composition.

$$\begin{tikzcd} \mathbb{R}^{m} \arrow[r, "T"] \arrow[rr, bend right, "S\circ T"'] & \mathbb{R}^{n} \arrow[r, "S"] & \mathbb{R}^{p} \end{tikzcd}$$

For 2 transformations

$T: \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$

$S: \mathbb{R}^{n} \rightarrow \mathbb{R}^{p}$

$S\circ T: \mathbb{R}^{m} \rightarrow \mathbb{R}^{p}$

each $\vec{v} \in \mathbb{R}^{m}$, $(S\circ T)(\vec{v}) \in \mathbb{R}^{p}$

- we can not define $S\circ T$ unless codomain of T is same as domain of S
- order of composition matters

- Are compositions of linear transformations linear?
  ↳ prove that $(S\circ T)(\vec{u}+\vec{v}) = (S\circ T)(\vec{u}) + (S\circ T)(\vec{v})$
  and $(S\circ T)(k\vec{u}) = k(S\circ T)(\vec{u})$

- if S and T are linear, then $S\circ T$ (when it exists) is linear.

Matrix of Composition

- $(S\circ T)(\vec{x}) = C\vec{x}$, where $C = \begin{bmatrix} | & | & & | \\ (S\circ T)\vec{e}_1 & (S\circ T)\vec{e}_2 & \cdots & (S\circ T)\vec{e}_m \\ | & | & & | \end{bmatrix}$

  "left multiplication by a matrix"

has the output of the standard vectors in $\mathbb{R}^{m}$ as its column.

- size of matrix C?
  ↳ C has p rows and m colums (it's a $p \times m$ matrix)

Describing Matrix C

$T: \mathbb{R}^m \to \mathbb{R}^n \quad \cdots \quad T(\vec{x}) = A\vec{x}$ ← written left multiplication by matrices

$S: \mathbb{R}^n \to \mathbb{R}^p \quad \cdots \quad S(\vec{y}) = B\vec{y}$

① $S \circ T(\vec{x}) = S(T(\vec{x})) \quad \cdots \quad T(\vec{x}) = A\vec{x}$

$= S(A\vec{x}) \quad \cdots \quad S(\star) = B\star$

$= B(A\vec{x})$

② $(S \circ T)(\vec{x}) = C\vec{x} \quad \rightarrow \quad C\vec{x} = B(A\vec{x})$

$C = BA$

$\therefore \; S \circ T(\vec{x}) = (BA)\vec{x}$

Product of Matrices

$T: \mathbb{R}^m \to \mathbb{R}^n \quad \cdots \quad T(\vec{x}) = A\vec{x}$

$S: \mathbb{R}^n \to \mathbb{R}^p \quad \cdots \quad S(\vec{y}) = B\vec{y}$

→ BA is unique matrix associated with to composition:

$S \circ T: \mathbb{R}^m \longrightarrow \mathbb{R}^p$

Week 4: Solidify

Recall

$T: \mathbb{R}^m \to \mathbb{R}^n$ ... $T(\vec{x}) = A\vec{x}$

$S: \mathbb{R}^n \to \mathbb{R}^p$ ... $S(\vec{y}) = B\vec{y}$

We know

$S \circ T(\vec{x}) = B(A\vec{x})$

- First vector : $\vec{e}_1$

first column of matrix A

$$S\circ T(\vec{e}_1) = S(T(\vec{e}_1)) = S\left(\begin{bmatrix} a_{11} \\ a_{21} \\ a_{n1} \end{bmatrix}\right)$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{n1} & a_{n2} \end{bmatrix} \qquad = B\left(\begin{bmatrix} a_{11} \\ a_{21} \\ a_{n1} \end{bmatrix}\right)$$

$$B = \begin{bmatrix} | & | & | \\ \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \\ | & | & | \end{bmatrix} \qquad = \begin{bmatrix} | & | & | & \\ \vec{c}_1 & \vec{c}_2 & \vec{c}_3 & \cdots \\ | & | & | & \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{n1} \end{bmatrix}$$

$$= a_{11}\vec{c}_1 + a_{21}\vec{c}_2 \ldots$$

→ Eg: $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$B(A\vec{e}_1) = B\left(\begin{bmatrix} 4 \\ 2 \end{bmatrix}\right)$$

first column of A

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1(4) + 2(2) \\ 3(4) + 4(2) \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \end{bmatrix}$$

first column. of BA

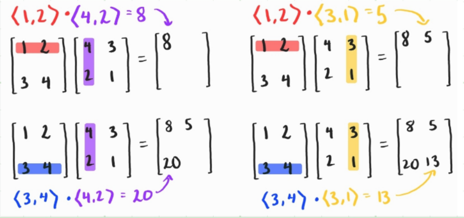
$$B(A\vec{e}_2) = B\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(3) + 2(1) \\ 3(3) + 4(1) \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$$

second column of BA

$$\therefore BA = \begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix}$$



→ Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $B = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 30 & 24 & 18 \\ 84 & 69 & 54 \\ 138 & 114 & 90 \end{bmatrix}$$

$$[9,6,3] \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1(9) + 2(6) + 3(3) = 30 \\ 4(9) + 5(6) + 6(3) = 84 \\ 7(9) + 8(6) + 9(3) = 138 \end{bmatrix}$$

$$[8,5,2] \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 8(1) + 5(2) + 2(3) \\ 8(4) + 5(5) + 2(6) \\ 8(7) + 5(8) + 2(9) \end{bmatrix} = \begin{bmatrix} 24 \\ 69 \\ 114 \end{bmatrix}$$

$$[7,4,1] \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 7(1) + 2(4) + 1(3) \\ 7(4) + 4(5) + 1(6) \\ 7(7) + 4(8) + 1(9) \end{bmatrix} = \begin{bmatrix} 18 \\ 54 \\ 90 \end{bmatrix}$$

Columns of Matrix Product

$A = \begin{bmatrix} \vert & \vert & & \vert \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \\ \vert & \vert & & \vert \end{bmatrix}$ is $n \times m$ matrix and

B is $p \times n$ matrix

$BA = \begin{bmatrix} \vert & \vert & & \vert \\ B\vec{a}_1 & B\vec{a}_2 & \cdots & B\vec{a}_m \\ \vert & \vert & & \vert \end{bmatrix}$ is $p \times m$ matrix

$n \times m : A$

$B : p \times n$

$BA : p \times m$

Entries of Matrix Product

$$B = \begin{bmatrix} \leftarrow & \vec{b}_1 & \rightarrow \\ \leftarrow & \vec{b}_2 & \rightarrow \\ & \vdots & \\ \leftarrow & \vec{b}_p & \rightarrow \end{bmatrix} \qquad A = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$BA = \begin{bmatrix} \vec{b}_1 \cdot \vec{a}_1 & \vec{b}_1 \cdot \vec{a}_2 & \cdots & \vec{b}_1 \cdot \vec{a}_m \\ \vec{b}_2 \cdot \vec{a}_1 & \vec{b}_2 \cdot \vec{a}_2 & \cdots & \vec{b}_2 \cdot \vec{a}_m \\ \vdots & \vdots & \cdots & \vdots \\ \vec{b}_p \cdot \vec{a}_1 & \vec{b}_p \cdot \vec{a}_2 & \cdots & \vec{b}_p \cdot \vec{a}_m \end{bmatrix}$$

→ Eg: $T: \mathbb{R}^2 \to \mathbb{R}^2$ rotation by θ Find matrix of $S \circ T$

$S: \mathbb{R}^2 \to \mathbb{R}^2$ rotation by ϕ ↳ rotation $\theta + \phi$

$$T(\vec{x}) = A\vec{x} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{A} \vec{x}$$

$$S(\vec{x}) = B\vec{x} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \vec{x}$$

$$S \circ T(\vec{x}) = S(T(\vec{x})) = B(A\vec{x})$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right)$$

$$= \begin{bmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & -\cos\phi\sin\theta - \sin\phi\cos\theta \\ \sin\phi\cos\theta + \cos\phi\sin\theta & -\sin\phi\sin\theta + \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

recall:

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$$

$$= \underbrace{\begin{bmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & -\cos(\phi+\theta) \end{bmatrix}}_{\text{matrix for rotation by } \phi + \theta} \begin{bmatrix} x \\ y \end{bmatrix}$$

- order matters : $AB \neq BA$
- when $AB = BA$, we say A and B commute

→ Eg: $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3(1)+1(2) & 3(0)+1(1) \\ 2(1)+1(2) & 2(0)+2(1) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$$

not equal

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1(3)+2(0) & 1(1)+0(1) \\ 2(3)+1(2) & 2(1)+1(1) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 3 \end{bmatrix}$$

Matrix Algebra

- Associative: A is $n \times m$ $\quad (AB)C = A(BC)$
  B is $m \times p$
  C is $p \times k$
- Distributive: A and B are $n \times m$ $\quad C(A+B) = CA + CB$
  C is $p \times n$ $\quad (A+B)D = AD + BD$
  D is $m \times q$
- Scalar Multiplication : A is $n \times m$
  B is $m \times p$ $\quad k(AB) = A(kB) = (AB)k$

Identity Transformation

- having zero for addition
- having 1 for multiplication
- identity transformation: maps every vector $\in \mathbb{R}^n$ to itself
  - ↳ $Id(\vec{x}) = \vec{x}$

since $Id(\vec{e}_1) = \vec{e}_1$

$$\left.\begin{aligned} Id(\vec{e}_1) &= \vec{e}_1 \\ Id(\vec{e}_2) &= \vec{e}_2 \\ \vdots \quad & \quad \vdots \\ Id(\vec{e}_n) &= \vec{e}_n \end{aligned}\right\}$$

matrix for Id is therefor

$$I_n = \begin{bmatrix} | & | & & | \\ \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \\ | & | & & | \end{bmatrix}$$

Week 4: Expand

We have $T: \mathbb{R}^n \to \mathbb{R}^n$ T^{-1} is inverse transformation

Can we find another lin. trans. that reverses?

$$T^{-1} \circ T(\vec{x}) = T \cdot T^{-1}(\vec{x}) = \vec{x} = id(\vec{x}) \longrightarrow \text{identity matrix}$$

Injective (one-to-one)

T is called injective if no two vectors in the domain get sent to same vector in codomain.

$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$

$T\left(\begin{bmatrix}3\\4\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$

since 2 diff. vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}3\\4\end{bmatrix}$ (in dom T) were sent to same vector $\begin{bmatrix}2\\3\end{bmatrix}$ in codom T, T is not injective

Surjective (onto)

T is called surjective if every vector in codomain of T has a vector in domain T that T maps it to.

$T(\vec{x}) \neq \begin{bmatrix}2\\2\end{bmatrix}$ and $\begin{bmatrix}2\\2\end{bmatrix}$ is in codom T, T is not surjective

- to be surjective, every vector in dom T must reach all vectors in codom T.

Bijective

- if transformation is both injective and surjective.
- if it is bijective, it is invertible

$T: X \to Y$ is invertible if and only if T is bijective

→ Eg: $\overset{\swarrow T}{T}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2}x + y \\ 0 + 2y \end{bmatrix}$ $\quad$ $\overset{\swarrow T^{-1}}{S}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x - \frac{1}{3}y \\ \frac{1}{2}y \end{bmatrix}$

do they satisfy $T(S(\vec{x})) = S(T(\vec{x})) = \vec{x}$

$$T(S(\vec{x})) = T\left(\begin{bmatrix} 2/3\,x - 1/3\,y \\ 0x + 1/2\,y \end{bmatrix}\right) = \begin{bmatrix} \frac{3}{2}\left(\frac{2}{3}x - \frac{1}{3}y\right) + \left(\frac{1}{2}y\right) \\ 2\left(\frac{1}{2}y\right) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$S(T(\vec{x})) = S\left(\begin{bmatrix} \frac{3}{2}x & + & y \\ 0 & & 2y \end{bmatrix}\right) = \begin{bmatrix} \frac{2}{3}\left(\frac{3}{2}x + y\right) - \frac{1}{3}(2y) \\ \frac{1}{2}(2y) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$\therefore$ S is inverse of T
T is inverse of S $\Big\}$ $S : T^{-1}$ or $T : S^{-1}$

$$\therefore T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad T^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$T(T^{-1}(x)) = \begin{bmatrix} \frac{3}{2} & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}\cdot\frac{2}{3} \cdot 1(0) & \frac{3}{2}\cdot -\frac{1}{3} + 1\cdot\frac{1}{2} \\ 0\cdot\frac{2}{3} + 2\cdot 0 & 0\cdot -\frac{1}{3} + 2\cdot\frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= I_2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Invertibility of Matrices

square matrix A is invertible is $T(\vec{x}) = A\vec{x}$ is invertible

$$T(\vec{x}) = A\vec{x} \xrightarrow[\text{inverse}]{} T^{-1}(y) = A^{-1}(y)$$

$$\left.\begin{array}{l} A \text{ is } n \times m \\ B \text{ is } n \times m \end{array}\right\} AB = BA = \underset{\substack{\uparrow \\ \text{identity} \\ \text{matrix}}}{I_n} \leftarrow \text{then } A \text{ and } B \text{ are inverses of each other}^{\bullet}$$

• as proved in above example

$$\left.\begin{array}{l} B = A^{-1} \\ A = B^{-1} \end{array}\right\} AB = BA = I_n$$

Matrix is Invertible if...

1. A is $n \times n$ matrix square matrix
2. RREF of A is I_n ← identity matrix

Inverting Matrix

To invert $n \times n$ matrix A

1. create augmented matrix $[A \mid I_n]$
2. get it to RREF (make it become I_n on left side)
3. stuff on right side is inverse matrix of A!

$$[I_n \mid A^{-1}]$$

→ Eg: find inverse matrix of $A = \begin{bmatrix} \frac{3}{2} & 1 \\ 0 & 2 \end{bmatrix}$

$= [A \mid I_n]$

$$= \left[\begin{array}{cc|cc} \frac{3}{2} & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \div 2 \end{array} \quad \left[\begin{array}{cc|cc} \frac{3}{2} & 1 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right] \begin{array}{l} -R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{cc|cc} \frac{3}{2} & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right] \begin{array}{l} \div \frac{3}{2} \\ \\ \end{array} \quad \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{3} & -\frac{2}{6} \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right] \xleftarrow{} \text{simplify}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right]$$

inverse of A is $A^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} \end{bmatrix}$

Week 5: Determinants Ch. 6.1

- investible linear maps don't collapse the space
- how do lin. transfor. change volume or area and orientation of space?

→ Let A be $n \times n$ matrix

- determinant: function $\det : A \mapsto \det A$
  - ↳ takes $n \times n$ matrix A as input
  - ↳ associates scalar called det A to matrix as output
- Dom: all $n \times n$ matrices
- Codom: $\mathbb{R}$

Determinant of 2×2 Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The determinant of A is scalar $ad - bc$

→ Eg: $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} {\color{red}+R_1} \quad \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} {\color{red}\div 5} \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} {\color{red}-R_2} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \therefore \operatorname{rref} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right] {\color{red}+R_1} \quad \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 5 & 1 & 1 \end{array}\right] {\color{red}\div 5} \quad \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array}\right] {\color{red}-2R_2}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3/5 & -2/5 \\ 0 & 1 & 1/5 & 1/5 \end{array}\right] \qquad \begin{array}{l} \frac{5}{5} - \frac{2}{5} = \frac{3}{5} \\ 0 - \frac{2}{5} = -\frac{2}{5} \end{array}$$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3+2 & -2+2 \\ -3+3 & 2+3 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

This proves that $A^{-1} = \frac{1}{\det A}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$

Theorem: Determinant and Invertibility of 2x2 Matrix

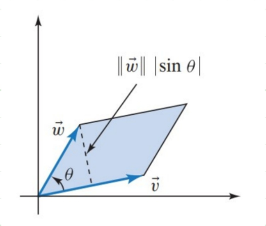
1. A 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if

   $\det A = ad - bc \neq 0$

2. If A is invertible, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

→ Eg: $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{x}) = A\vec{x}$, $A = \begin{bmatrix} 2 & 1 \\ 1 & 1.5 \end{bmatrix}$

- parallelogram created by $T(\vec{e}_1)$ and $T(\vec{e}_2)$
- let's find area of parallelogram



$T(\vec{e}_1) = \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $T(\vec{e}_2) = \vec{w} = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$

- height $= \|\vec{\omega}\| \sin\theta$
- base $= \|\vec{v}\|$
- $\theta = \arccos\left(\frac{\vec{\omega}\cdot\vec{v}}{\|\vec{\omega}\|\|\vec{v}\|}\right)$

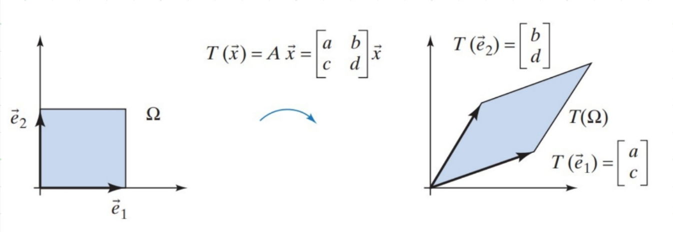
$$\text{Area} = \|\vec{v}\|\|\vec{\omega}\|\sin\theta$$
$$= (\sqrt{5})\left(\frac{\sqrt{13}}{2}\right)\left(\frac{4}{\sqrt{65}}\right) = 2$$

- equal to $|2(1.5) - 1(1)| = |\det A|$, absolute value of determinant A

- $\Omega = \{c_1\vec{e}_1 + c_2\vec{e}_2 \mid 0 \le c_1, c_2 \le 1\}$ unit square w/th $\vec{e}_1$, $\vec{e}_2$
- Let's feed all of Ω into T:

$$\{T(c_1\vec{e}_1+c_2\vec{e}_2) = c_1T(\vec{e}_1) + c_2T(\vec{e}_2) = c_1\vec{v} + c_2\vec{\omega} \mid 0 \le c_1, c_2 \le 1\}$$

- we transform unit square (sides $\vec{e}_1$ and $\vec{e}_2$) into parallelogram (sides $\vec{v}$ and $\vec{\omega}$)



- * area of $T(\Omega)$ (image of unit square) is exacly $|\det A|$
- \+ $|\det A|$ is the factor by which T changes the area.
- \+ The absolute value of the determinant of A is the expansion factor, or ratio, by which T changes the area of any subset Ω in $\mathbb{R}^2$

$$\frac{\text{area of } T(\Omega)}{\text{area of } \Omega} = |\det A|$$

- \+ relates to how 2×2 is invertible only if $\det \neq 0$
- \+ makes sense bc. non-zero area shape in dom $\mathbb{R}^2$ collapses when transformed into zero area (point or line) in codom $\mathbb{R}^2$.

Sign of Determinants

$$A = \begin{bmatrix} 2 & -1 \\ 1 & \frac{1}{2} \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 \\ \frac{1}{2} & 1 \end{bmatrix}$$

$T: \mathbb{R}^2 \to \mathbb{R}^2$ $\qquad$ $S: \mathbb{R}^2 \to \mathbb{R}^2$

$T(\vec{x}) = A\vec{x}$ $\qquad$ $S(\vec{x}) = B\vec{x}$

$\det A = 2$ $\qquad$ $\det B = -2$

$$|\det A| = |\det B| = 2$$

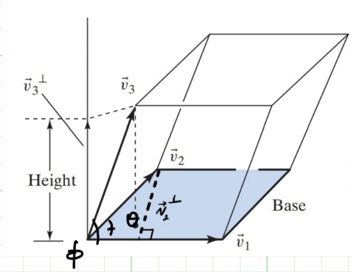
- since $|\det A| = |\det B| = 2$, both expand unit square by factor of 2.
- map T preserves order of input vectors
  - ↳ T preserves orientation of the space
- map S reverses order of input vectors
  - ↳ S reverses orientation of the space

(+) sign for determinant of map : preserves orientation of space

(−) sign for determinant of map : reverses orientation of space

Determinant of 3×3 Matrix

- let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $A = [v_1, v_2, v_3]$
- this means $T(\vec{e_1}) = \vec{v}_1$, $T(\vec{e_2}) = \vec{v}_2$, $T(\vec{e_3}) = \vec{v}_3$



volume of parallelepiped

base: $\|\vec{v}_1\| \times \|\vec{v}_2\| \sin\theta$

height: $\|\vec{v}_3\| \sin\phi$

$$\text{Volume} = \|\vec{v}_1\| \|\vec{v}_2\| \sin\theta \, \|\vec{v}_3\| \sin\phi$$
$$= \|\vec{v}_3^{\perp}\| \|\vec{v}_2^{\perp}\| \|\vec{v}_1\|$$

Cross Product in $\mathbb{R}^3$

→ The cross product $\vec{v} \times \vec{w}$ of two $\mathbb{R}^3$ vectors $\vec{v}$ and $\vec{w}$ is the vector in $\mathbb{R}^3$ with the following properties:

- $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$.
- $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\| \sin\theta$ — θ is angle between $\vec{v}$ and $\vec{w}$, $0 \leq \theta \leq \pi$
- magnitude of cross product is the area of parallelogram $\vec{v}$ and $\vec{w}$
- direction of $\vec{v} \times \vec{w}$ follows right hand rule
- volume of parallelepiped defined by columns of A is expressed as:

$$\left| \vec{v}_3 \cdot (\vec{v}_1 \times \vec{v}_2) \right|$$

- determinant of upper or lower triangular matrix:
  - ↳ product of diagonal elements

3 x 3 Determinant

If $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$ is 3x3 matrix, then

$$\det A = \vec{v}_3 \cdot (\vec{v}_1 \times \vec{v}_2)$$

* $|\det A|$ is the volume of parallelepiped formed by $\vec{v}_1$ $\vec{v}_2$ $\vec{v}_3$.

- $\det A = 0$ when T_A maps standard cube to a volumless object in $\mathbb{R}^3$ codom (line, point, plane)

  ↳ T_A is ∴ not bijective and ∴ not invertibe

Determinant of n x n Matrix

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be given by $T(\vec{x}) = A\vec{x}$

* determinant of A is a scalar called det A.
* $|\det A|$ is the factor by which T changes the n-dimensional volume of any subset of $\mathbb{R}^n$
* sign of det A determines if T preserves orientation of space.

Co-factor Expansion

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

* a_{ij} → entry in i row and j column

  eg: $a_{12} = -1$, $a_{33} = 3$

* A_{ij} → 2x2 matrix formed when you remove i'th row and j'th column.

  eg: $A_{23} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

* $|A_{ij}| = \det(A_{ij})$

- det A : cofactor expansion along any row or column

cofactor expansion of A along the i'th row $= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \left| A_{ij} \right|$

Along the first row $(i = 1)$

$$A = \begin{array}{c} \begin{matrix} 11 & 12 & 13 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \end{array}$$

- $\det A = (-1)^{1+1}(1)\begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix} + (-1)^{1+2}(2)\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} + (-1)^{1+3}(0)\begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix}$

$$= \left[-1(3) - 0(1)\right] + -2\left[0(3) - 1(0)\right]$$

$$= -3 - 0 + 2(0)$$

$$= -3$$

$\therefore \det A = -3$

Along the first column $(j=1)$

$$A = \begin{array}{cc} 11 & \multirow{3}{*}{$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$} \\ 21 & \\ 22 & \end{array}$$

- $\det A = (-1)^{1+1}(1)\begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix} + (-1)^{2+1}(0)\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + (-1)^{2+2}(0)\begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix}$

$$= 1\left[-1(3) + 0(1)\right] + 0 + 0$$

$$= -3$$

$\therefore \det A = -3$

Theorem: Invertibility and Determinant

- square matrix A is invertible if and only if det $A \neq 0$

Transpose

- transpose of matrix A, called A^T → switch rows and columns of A

eg: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Theorem: Determinant of transpose

- taking transpose of matrix doesn't change det.
- if A is square matrix, $\det A = \det A^T$

Determinants of Products and Powers

A and B are $n \times n$ matrices

- $\det(AB) = \det A \det B$
- $\det(A^m) = (\det A)^m$

Determinant of Inverse

If A is invertible, then:

- $\det(A^{-1}) = \dfrac{1}{\det A}$

Week 5: Expand

Theorem: Elementary Row Operations and Determinants

If A is $n \times n$ matrix and B is what u get afer row red.

- dividing a row of A by scalar k : $\det B = \frac{1}{k} \det A$
- row swap A : $\det B = -\det A$
- adding multiple of a row A to another row: $\det B = \det A$

→ **Eg:**

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix} \rightleftarrows \quad A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \div 3 \quad A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - R_2 \quad A_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

↷ row swap ↷ divide by 3 ↷ subtract R_2 from R_1

$\det A_2 = -\det A_1$ $\quad$ $\det A_3 = -\frac{1}{3} \det A_1$ $\quad$ $\det A_4 = -\frac{1}{3} \det A_1$

$$\text{Since } \det A_4 = (-1)^{1+1}(1)(1 \cdot 1 - 0 \cdot 0) = 1$$

$$\det A_4 = -\frac{1}{3} \det A_1$$

$$1 = -\frac{1}{3} \det A_1$$

$$1 = -\frac{1}{3} \square$$

$$\square = \frac{1 \cdot 3}{-1} = -3$$

$$\therefore \det A_1 = -3$$

Notes: $\det I_n = 1$

↖ $1 = \det I = \det A_3 = \frac{1}{3} \det A_2 = -\frac{1}{3} \det A$

Elementary Matrix

- an elementary matrix is a matrix you get by applying an elementary row reduction step to the identity matrix.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

← yes, it's an elementary matrix because we just did one operation (swapped R_1 and R_2) to an identity matrix I_3.

Linearity of Determinants

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 2 & 0 \\ 1 & 4 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 3 & 3 \\ -1 & 8 & 0 \\ 1 & 10 & 3 \end{bmatrix}$$

- 3 matrices only differ in second column.
- second column of C is linear combination of 2nd col. A, B.

↳ 2nd col. C = 2 [second col. A] + 1 [second col. B]

∴ **det C = 2 det A + 1 det B**

Given column vectors:

→ $\det \begin{bmatrix} \vdots & & \vdots \\ \vdots & \vec{x} + k\vec{y} & \vdots \\ \vdots & & \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots & & \vdots \\ \vdots & \vec{x} & \vdots \\ \vdots & & \vdots \end{bmatrix} + k \det \begin{bmatrix} \vdots & & \vdots \\ \vdots & \vec{y} & \vdots \\ \vdots & & \vdots \end{bmatrix}$ $\vdots$ $\begin{matrix} | \\ \vec{v}_1 \\ | \end{matrix}$

Given row vectors

→ $\det \begin{bmatrix} \cdots \\ \vec{x} + k\vec{y} \\ \cdots \end{bmatrix} = \det \begin{bmatrix} \cdots \\ \vec{x} \\ \cdots \end{bmatrix} + k \det \begin{bmatrix} \cdots \\ \vec{y} \\ \cdots \end{bmatrix}$ $\cdots$ $— \vec{v}_1 —$

Cramer's Rule

- consider equation $A\vec{x} = \vec{b}$, A is $n \times n$ matrix

↳ it has a unique solution

↳ we can solve equation by mutiplying both sides by A^{-1}

$$\begin{aligned} &= \vec{x} \\ &= I_n \vec{x} \\ &= A^{-1}A\vec{x} \\ &= A^{-1}\vec{b} \end{aligned}$$

$$A\vec{x} = \vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

- Cramer's rule allows us to find a particular component of the solution without entirely solving the system.

- if the coefficients of the matrix are not known, we can still find the rate of change of a certain component of the solution w.r.t. one of the parameters.

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$A\vec{x} = \vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = \frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} b_1 a_{22} - b_2 a_{12} \\ -b_1 a_{21} + b_2 a_{11} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} b_1 a_{22} - b_2 a_{12} \\ b_2 a_{11} - b_1 a_{21} \end{bmatrix}$$

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{\det A} \qquad x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{\det A}$$

$$x_1 = \frac{\det \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}}{\det A} \qquad x_2 = \frac{\det \begin{bmatrix} b_1 & a_{11} \\ b_2 & a_{21} \end{bmatrix}}{\det A}$$

- let $\vec{b}_{,i}$ denote the matrix we get by replacing the i'th column of A by $\vec{b}$.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$$

$$x_1 = \frac{\det \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}}{\det A} \qquad x_2 = \frac{\det \begin{bmatrix} b_1 & a_{11} \\ b_2 & a_{21} \end{bmatrix}}{\det A}$$

$$x_1 = \frac{\det \vec{b}_{,1}}{\det A} \qquad x_2 = \frac{\det \vec{b}_{,2}}{\det A}$$

Theorem: Cramer's Rule

- for A invertible n×n matrix and $A\vec{x} = \vec{b}$
- the components x_i of solution vector $\vec{x}$ are:

$$\vec{x}_i = \frac{\det \vec{b}_{,i}}{\det A} \longleftarrow$$

$\det \vec{b}_{,i}$ is the matrix we get by replacing the i'th column of A by $\vec{b}$

→ Eg: $$\left| \begin{array}{l} (b-1)x_1 + ax_2 = 0 \\ -ax_1 + (b-1)x_2 = c \end{array} \right|$$

$$A\vec{x} = \vec{b}, \quad A = \begin{bmatrix} (b-1) & a \\ -a & (b-1) \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

$$x_1 = \frac{\det \begin{bmatrix} 0 & a \\ c & (b-1) \end{bmatrix}}{\det \begin{bmatrix} (b-1) & a \\ -a & (b-1) \end{bmatrix}} \qquad x_2 = \frac{\det \begin{bmatrix} (b-1) & 0 \\ -a & c \end{bmatrix}}{\det \begin{bmatrix} (b-1) & a \\ -a & (b-1) \end{bmatrix}}$$

Week 6: Introduce

Subspace

- subspace of $\mathbb{R}^n$ is a copy of $\mathbb{R}^k$ for $k \leq n$ that sits inside $\mathbb{R}^n$

eg: a line ℓ in $\mathbb{R}^3$ is a copy of $\mathbb{R}$ sitting in $\mathbb{R}^3$

↳ ℓ is subspace of $\mathbb{R}^3$

→ Eg: plane V in $\mathbb{R}^3$. $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| : x + 2y + 3z = 0 \right\}$

- set V is subset of $\mathbb{R}^3$, describes plane through origin
- V is a copy of $\mathbb{R}^2$ sitting inside $\mathbb{R}^2$
- 2 vectors $\vec{u}_1$ and $\vec{u}_2$ on V : $\vec{u}_1 + \vec{u}_2$ and $c_1\vec{u}_1$ and $c_2\vec{u}_2$ are also on V.
- We say V is closed under vector addition and scalar multiplication

↳ any linear comb. $c_1\vec{u}_1 + c_2\vec{u}_2$ is on V.

Def: Subspace

- a subspace of $\mathbb{R}^n$ is a subset W of $\mathbb{R}^n$ which contains $\vec{0}$ and is closed under addition and scalar multiplication
- a subspace of $\mathbb{R}^n$ is a subset $W \subseteq \mathbb{R}^n$ s.t.

1. $\vec{0} = W$
2. if $\vec{x}, \vec{y} \in W$, then also $\vec{x} + \vec{y} \in W$
3. if $\vec{x} \in W$ and k any scalar, then also $k\vec{x} \in W$

→ Eg: zero set $\{0\}$ and $\mathbb{R}^n$ are substances of $\mathbb{R}^n$

$\{0\}$ ← trivial subspace of $\mathbb{R}^n$

→ plane V in $\mathbb{R}^3$: $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| : x + 2y + 3z = 0 \right\}$

$$V = \left\{ t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} , \; s, t \in \mathbb{R} \right\}$$

↑ V is the set of all linear comb. of $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Span

- let $\vec{v}_1, \vec{v}_2 \cdots \vec{v}_m$ be vectors in $\mathbb{R}^n$
- span: set of all linear combinations of $\vec{v}_1, \vec{v}_2 \cdots \vec{v}_m$

$$\text{span}(\vec{v}_1, \vec{v}_2 \cdots \vec{v}_m) = \left\{ c_1\vec{v}_1 + c_2\vec{v}_2 \cdots c_m\vec{v}_m \,\middle|\, c_1, c_2 \cdots c_m \in \mathbb{R} \right\}$$

→ Eg: V plane : $x_1 + 2y_1 + 3z_1 = 0$, $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

- V is spanned by $\{\vec{v}_1, \vec{v}_2\}$
- span$(\vec{v}_1, \vec{v}_2)$ is V
- $\{\vec{v}_1, \vec{v}_2\}$ is the spanning set for V
- every vector on plane V is in span$(\vec{v}_1, \vec{v}_2)$

↳ bc. every vector on plane V can be written as linear comb. of $\vec{v}_1$ and $\vec{v}_2$.

Span is a Subspace

- given vectors $\vec{v}_1, \vec{v}_2 \cdots \vec{v}_m$ in $\mathbb{R}^n$ ...

span$(\vec{v}_1, \vec{v}_2 \cdots \vec{v}_m)$ is a subspace of $\mathbb{R}^n$

Eg: span$(\vec{v}_1, \vec{v}_2)$ is plane V, which is subspace of $\mathbb{R}^3$

Image

- linear trans. $T: \mathbb{R}^m \to \mathbb{R}^n$, set of all outputs of T creates a subset of the codomain of T

  this subset is a subspace of codom

- image of $f: X \to Y$ is the set

$$\text{im}(f) = \{f(x) : x \in X\} = \{y \in Y \mid f(x) = y \text{ for some } x \in X\}$$

→ Eg: $T: \mathbb{R}^3 \to \mathbb{R}^3$ proj. onto x-axis. $T(\vec{x}) = A\vec{x}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- since T projects vectors onto x-axis, every output T is on x-axis

- every vector on x-axis is projection of some $\mathbb{R}^3$ vector

$$\therefore \quad \text{im}(T) = \{T(\vec{x}) \mid \vec{x} \in \mathbb{R}^3\} = x\text{-axis}$$

∴ im(T) is a subspace of $\mathbb{R}^3$

$$\text{im}(T) = \{T(\vec{x}) \mid \vec{x} \in \mathbb{R}^3\}$$

$$= \{A(\vec{x}) \mid x_i \in \mathbb{R}^3\}$$

$$= \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad '' \right\}$$

$$= \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad '' \right\}$$

$$= \left\{ x_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad '' \right\} = \text{span}(\vec{e}) = x\text{-axis}$$

Image is a Subspace

$A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_m]$ is $n \times m$ matrix, $T(\vec{x}) = A\vec{x}$, $T: \mathbb{R}^m \to \mathbb{R}^n$

$$\begin{aligned} im(T) &= \{T(\vec{x}) \mid \vec{x} \in \mathbb{R}^m\} \\ &= \{A\vec{x} \mid \vec{x} \in \mathbb{R}^m\} \\ &= \{x_1\vec{a}_1 + x_2\vec{a}_2 \cdots x_m\vec{a}_m \mid x_1, x_2 \cdots x_m \in \mathbb{R}^m\} \\ &= span(\vec{a}_1, \vec{a}_2 \cdots \vec{a}_m) \end{aligned}$$

* image of T is span of columns of A or column space A

$Col(A) \leftarrow$ column space of A

- $im(T) = Col(A)$ is $\underbrace{\text{span of columns of } A}_{\text{span of vectors is always a subspace}}$

$\therefore$ $im(T)$ is subspace $\mathbb{R}^n$

Theorem: image is a subspace

Let $T: \mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$

$im(T) = Col(A)$ is subspace of $\mathbb{R}^n$

Week 6: Solidify

Kernel

- the **kernel** of a linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is the set of all vectors $\vec{v}$ in dom T s.t. $T(\vec{v}) = \vec{0}$

$$\ker(T) = \{\vec{v} \in \mathbb{R}^m \mid T(\vec{v}) = \vec{0}\} \qquad \qquad A\vec{v} = 0$$

- the set of vectors that are mapped to zero is exactly the solution to system $A\vec{x} = \vec{0}$:

$$\ker(T) = \{\vec{x} \in \mathbb{R}^m \mid T(\vec{x}) = \vec{0}\}$$

general solution to

$T(\vec{x}) = 0 \implies A\vec{x} = 0$

- (T) is null space for A: Null(A)

→ **Eg:** $T: \mathbb{R}^3 \to \mathbb{R}^3$ proj. into x-axis, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(T) is general solution to $A\vec{x} = 0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= 0 \\ x_2 &= t \\ x_3 &= s \end{aligned} \qquad \vec{x} = \begin{bmatrix} 0 \\ t \\ s \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

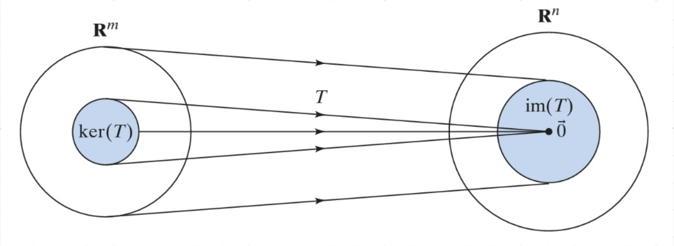
$$\therefore \ker(T) = \left\{ t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \,\middle|\, s,t \in \mathbb{R} \right\} = \text{span}(\vec{e}_2, \vec{e}_3) = yz\text{-plane}$$

plugging in anything from here gets us to 0 in the codom

Kernel is a Subspace

1. kernel of linear trans. can always be described as span of vectors
2. kernel of a linear trans. is a subspace of its domain

$T: \mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$ (T) = Null (A) is a subspace of $\mathbb{R}^m$



- T is surjective exactly when im (T) is the entie codomain.
- T is injective exactly when ker (T) has only one element: $\vec{0}$.

Theorem: For $T: \mathbb{R}^m \to \mathbb{R}^n$

- T is injective if and only if $\ker(T) = \{\vec{0}\}$
- T is surjective if and only if $\mathrm{im}(T) = \mathbb{R}^n$

Linear Independance

$$A = \begin{array}{cccc} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \\ \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_4 \end{array}$$

we know that image of T is given by the span of the columns of A.

$\therefore \mathrm{im}(T) = \mathrm{span}(\vec{v}_1 \cdots \vec{v}_4)$, $\{\vec{v}_1 \cdots \vec{v}_4\}$ is the spanning set for im (T)

- we know that:

$\vec{v}_2 = 2\vec{v}_1$ and $\vec{v}_4 = \vec{v}_1 + \vec{v}_3$

this means $\vec{v}_2, \vec{v}_4 \in \mathrm{span}(\vec{v}_1, \vec{v}_3)$

$\therefore \mathrm{span}(\vec{v}_1, \underset{\nwarrow}{\vec{v}_2}, \vec{v}_3, \underset{\nearrow}{\vec{v}_4}) = \mathrm{span}(\vec{v}_1, \vec{v}_3)$

redundant

- How can we find redundant vectors? (find vectors that can be removed from spanning set without changing subspace?)
  ↳ find ker (T) and non-pivot columns are redundant

$$A = \begin{array}{cccc} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \\ \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_4 \end{array} \xrightarrow{RREF} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Bigg\} \begin{aligned} x_1 &= -2t - 1s \\ x_2 &= t \\ x_3 &= -1s \\ x_4 &= s \end{aligned}$$

$$\therefore \ker(T) = \left\{ t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \middle| \, t, s \in \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right)$$

$$A \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \vec{0} \qquad A \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \vec{0}$$

$$-2\vec{v}_1 + \vec{v}_2 = \vec{0} \qquad -\vec{v}_1 - \vec{v}_3 + \vec{v}_4 = \vec{0}$$

each is a non-trivial linear relation among the colums of A.

isolating for $\vec{v}_2$ and $\vec{v}_4$ (vectors whose column in A is NOT a pivot column) tells us that they can be written as lin. comb. of $\vec{v}_1$ and $\vec{v}_3$ and are $\therefore$ redundant

Linear Relation

- a linear relation among $\vec{v}_1, \vec{v}_2 \cdots \vec{v}_n \in V$ (subspace V of $\mathbb{R}^m$) is any eq. of form

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_n\vec{v}_n = \vec{0}$$ where c_i are scalars

Linearly Dependent

not all are 0

- linearly dependent if for $\underline{c_1}\vec{v}_1 + \underline{c_2}\vec{v}_2 + \cdots \underline{c_n}\vec{v}_n = \vec{0}$
- if and only if there are non-trivial (not zero) relations among them.

Linearly Independent

all are 0

- linearly independant if for $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_n\vec{v}_n = \vec{0}$

basically that equ. has 1 solution (they're all 0) ... trivial solution

- if and only if the only relation is the trivial (zero) relation.

→ Eg: $\vec{u}_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix}2\\1\\-1\end{bmatrix}$ $\vec{u}_3 = \begin{bmatrix}-2\\1\\4\end{bmatrix}$ linearly independent

- to show that, we prove that there is unique solution that $=\vec{0}$

$$\begin{bmatrix}1 & 2 & -2\\0 & 1 & 1\\-1 & -1 & 4\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

∴ $c_1 = c_2 = c_3 = 0$ is sol. so vectors are linearly independent.

$$\downarrow$$

$$\left[\begin{array}{ccc|c}1 & 2 & -2 & 0\\0 & 1 & 1 & 0\\-1 & -1 & 4 & 0\end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{array}\right]$$

Week 6: Expand | PCE #6 | Ch. 3.1-3.2

- For a list of vectors $\vec{v}_1, \vec{v}_2 \cdots \vec{v}_m$ in $\mathbb{R}^n$, the following are the same.

1. $\vec{v}_1, \vec{v}_2 \cdots \vec{v}_m$ are linearly independent
2. the set $\{\vec{v}_1, \vec{v}_2 \cdots \vec{v}_m\}$ contains no redundant vector
3. none of the vectors can be written as a lin. comb. of the rest.
4. none of the vectors are in the span of the rest
5. the only relation among $\vec{v}_1, \vec{v}_2 \cdots \vec{v}_m$ is the trivial solution.
6. the only solution to $c_1\vec{v}_1 + c_2\vec{v}_2 \cdots c_m\vec{v}_m$ is $c_1 = c_2 \cdots = 0$
7. $\text{Null}(A) = \{\vec{0}\}$ for $A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}$
8. $\text{rank}(A) = m$
9. map of T is injective
10. $\ker(T) = \{\vec{0}\}$

- the ideal way to describe subspace $V \in \mathbb{R}^n$ is with a linearly independent spanning set

Basis

- a basis of a subspace V of $\mathbb{R}^n$ is a linearly independent set of vectors in V that spans V.
- a basis is the spanning set for V that is linearly independent

→ Suppose $\vec{v}_1, \vec{v}_2 \cdots \vec{v}_m$ form basis for V in $\mathbb{R}^n$

(1) $\vec{v}_1, \vec{v}_2 \cdots \vec{v}_m$ are linearly independent

(2) given any vector $\vec{x} \in \text{span}(\vec{v}_1, \vec{v}_2 \cdots \vec{v}_m)$ we can write $\vec{x}$ as lin. comb. of $\vec{v}_1, \vec{v}_2 \cdots \vec{v}_m$

→ Eg: $\vec{e_1}, \vec{e_2} \cdots \vec{e_n}$ form standard basis of $\mathbb{R}^n$

Proof :

(1) $\vec{e_1}, \vec{e_2} \cdots \vec{e_n}$ are linearly independent because only solution to $c_1\vec{e_1} + c_2\vec{e_2} \cdots$ is that $c_1 = c_m \cdots = 0$

(2) any vector $\vec{x} \in \mathbb{R}^n$ can be described as lin. comb. of $\vec{e_1}$ and $\vec{e_2} \cdots$ vectors.

→ Eg: $\vec{u_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\vec{u_2} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ $\vec{u_3} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ form $\overset{a}{\wedge}$ basis of $\mathbb{R}^3$

(1) RREF $= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ so they are linearly independent

(2) let $\vec{x} \in \mathbb{R}^3$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ ∴ prove that $c_1\vec{u_1} + c_2\vec{u_2} + c_3\vec{u_3} = \vec{x}$

since RREF of $[A|\vec{0}]$ is identity, we know there will be one solution for $[A|\vec{x}]$, so $\vec{x} \in \text{span}(\vec{u_1}, \vec{u_2}, \vec{u_3})$

Theorem :

+ vectors $\vec{v_1}, \vec{v_2} \cdots \vec{v_m}$ are in subspace $\mathbb{R}^n$.
+ they form a basis for V if and only if there is only one solution to $c_1\vec{v_1} + c_2\vec{v_2} + \cdots c_m\vec{v_m} = \vec{x}$ where $\vec{x} \in V$.
- if every vector in V can be uniquely described as linear combination of $\vec{v_1}, \vec{v_2} \cdots \vec{v_m}$ (the basis vectors)
- in that case, coefficients $c_1, c_2 \cdots c_m$ called coordinate $\vec{x}$ w.r.t. the basis $\vec{v_1}, \vec{v_2} \cdots \vec{v_m}$

Week 7: Introduce | PCE 7 | Sec 3.3 - 3.4

→ Eg: $P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \middle| \; x+y+z=0 \right\}$ ← plane P is subspace of $\mathbb{R}^3$

let $\vec{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -2 \\ -2 \\ 4 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3 \; \vec{v}_4 \in P$

$\vec{\omega} \in P$, $\vec{\omega} = \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix}$ for $y, z \in \mathbb{R}$ ← $\vec{\omega}$ is general form of vectors on P

- To check if $\vec{\omega} \in \text{span}(\vec{v}_1 \cdots \vec{v}_4)$, check if

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_4 \\ | & & | \end{bmatrix} \vec{c} = \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix}$$

is consistent for every y and z

basically augment it and solve

$$\left[\begin{array}{cccc|c} -1 & -2 & 1 & 0 & -y-z \\ -1 & -2 & -1 & 1 & y \\ 2 & 4 & 0 & -1 & z \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -\frac{1}{2} & \frac{z+y}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{z+y}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- consistent, so we know $\vec{\omega}$ can be expressed as some lin. comb. of $\vec{v}_1 \cdots \vec{v}_3$ and $\therefore \vec{\omega} \in P$
- and since $\vec{\omega}$ represents all vectors on P, $\{\vec{v}_1 \cdots \vec{v}_4\}$ is the spanning set of P

- We can span P with only $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\left[\begin{array}{ccc|c} -1 & -2 & 1 & -y-z \\ -1 & -2 & -1 & y \\ 2 & 4 & 0 & z \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & \frac{z+y}{2} \\ 0 & 0 & 1 & \frac{z+y}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- We can span P with only $\{\vec{v}_1, \vec{v}_3\}$

$$\left[\begin{array}{cc|c} -1 & 1 & -y-z \\ -1 & -1 & y \\ 2 & 0 & z \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & \frac{z+y}{2} \\ 0 & 1 & \frac{z+y}{2} \\ 0 & 0 & 0 \end{array}\right]$$

- But we cannot go lower than that, or else it'll be inconsistent
  - ↳ eg, removing $\vec{v}_2$ yields

$$\left[\begin{array}{c|c} -1 & -y-z \\ -1 & y \\ 2 & z \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{c|c} 1 & \frac{z+y}{2} \\ 0 & \frac{z+y}{2} \\ 0 & 0 \end{array}\right] \leftarrow \underline{\text{inconsistent}}$$

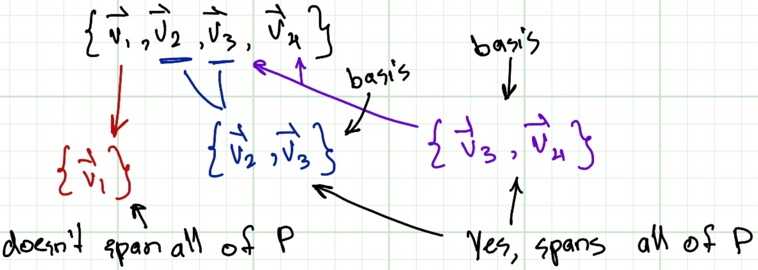
$$\therefore P = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{span}\{\vec{v}_1, \vec{v}_2\}$$

$$\neq \text{span}\{\vec{v}_1\}$$

Takeways

- started with 4 vectors, but only need 2 to span P.
- fewer than 2 linearly independent vectors fails to span P.
- given a spanning set for subspace, we can alway remove redundant vectors until we get $\boxed{\text{linearly independent}}$ set of vectors

→ Eg: $P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mid x+y+z=0 \right\}$



Theorem:

- every spanning set is larger or equal to every linearly independent set in V
- consider $\vec{v}_1 \cdots \vec{v}_p$ and $\vec{\omega}_1, \vec{\omega}_q$ in subspace V
- if $\vec{v}_1 \cdots \vec{v}_p$ are linearly independent and
- $\vec{\omega}_1, \vec{\omega}_q$ are vectors that span V, then ... $q \geq p$

Theorem: Number of Vectors in a Basis

- All bases of a subspace V of $\mathbb{R}^n$ consist of the same number of vectors.

Dimension

- consider subspace V in $\mathbb{R}^n$
- the # of vectors in any basis of V is called the dimension of V
- $\dim(V)$: # of vectors in the basis

→ Eg: $T_A : \mathbb{R}^4 \rightarrow \mathbb{R}^3$, $T(\vec{x}) = A\vec{x}$, $A = \begin{bmatrix} -1 & -2 & 1 & 0 \\ -1 & -2 & -1 & 1 \\ 2 & 4 & 0 & -1 \end{bmatrix} = [\vec{v}_1 \cdots \vec{v}_4]$

- $\operatorname{im}(T) = \text{span}\{\vec{v}_1, \vec{v}_2\}$

↖ basis has 2 vectors, which is equal to the number of pivot columns in RREF (A)

- $\ker(T) = \left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix} \right)$ ← these vectors are linearly independent

$\therefore \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix} \right\}$ is basis for $\ker(T)$

↖ # of vectors in basis = # of the non-pivot columns

Summary

- dim im (T) ← # of pivot columns in RREF (A)
- dim ker (T) ← # of non-pivot columns in RREF (A)
- # of columns in A ← dimension of dom T

Rank and Nullity $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $T_A(\vec{x}) = A\vec{x}$

- rank of matrix A ← dimension of im (T)
- nullity of matrix A ← dimension of ker (T)
- dim im (T) + dim ker (T) = n
- Rank (A) + Nullity (A) = n

Week 7: PCE 7 : | Solidify | Sec. 3.3, 3.4

Basis and Unique Representation

- vectors $\vec{v}_1, \cdots \vec{v}_m$ are in subspace V of $\mathbb{R}^n$
- $\vec{v}_1, \cdots \vec{v}_m$ form basis of V I.A.O.I every vector $\vec{x}$ in V can be expressed uniquely as a lin. comb. $\vec{x} = c_1\vec{v}_1 + \cdots c_m\vec{v}_m$
- fixing an ordered basis $\mathcal{B}$ of subspace V…
  ↳ every vector on V can be expressed by $c_1, \cdots c_m$ that depend on V.

→ **Eg:** P is subspace in $\mathbb{R}^3$, P: $x+y+z=0$, $\vec{b}_1 = \begin{bmatrix}-1\\-1\\2\end{bmatrix}$ $\vec{b}_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$

- ordered basis → $\mathcal{B} = (\vec{b}_1, \vec{b}_2)$
- we have $\vec{w}$ on plane P where $\vec{w} = \begin{bmatrix}2\\-4\\2\end{bmatrix}$ … so $\vec{w} = 1\vec{b}_1 + 3\vec{b}_2$
- coordinates of $\vec{w}$ w.r.t. $\mathcal{B}$ are $\begin{bmatrix}1\\3\end{bmatrix}$

$$[\vec{w}]_\mathcal{B} = \begin{bmatrix}1\\3\end{bmatrix}$$

Coordinates

- $\mathcal{B} = (\vec{v}_1, \cdots \vec{v}_n)$ ← ordered basis of subspace V
- the $\mathcal{B}$-coordinates of $\vec{v} \in V$ are the unique scalars a_i s.t.:

$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$$

- $\mathcal{B}$-coordinates arranged into column vector: $[\vec{v}]_\mathcal{B} = \begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}$

→ **Eg:** $V = \mathbb{R}^n$ and $\mathcal{E} = (\vec{e}_1, \vec{e}_2, \cdots \vec{e}_n)$ standard basis of $\mathbb{R}^n$

- every vector $\vec{v} \in \mathbb{R}^n$ is lin. comb. of standard basis of $\mathbb{R}^n$

$\therefore [\vec{v}]_\mathcal{E} = \begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}$ standard coordinates of $\vec{v}$ w.r.t. $\mathcal{E}$

since $\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n$

→ **Eg:** P is subspace in $\mathbb{R}^3$, P: $x+y+z=0$, $\vec{u}_1 = \begin{bmatrix}-2\\-2\\4\end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix}2\\-4\\2\end{bmatrix}$

- ordered basis ⟶ $\mathcal{C} = (\vec{u}_1, \vec{u}_2)$
- we have $\vec{w}$ on plane P where $\vec{w} = \begin{bmatrix}2\\-4\\2\end{bmatrix}$ ... so $\vec{w} = -1\vec{u}_1 - 6\vec{u}_2$

$$\therefore [\vec{w}]_{\mathcal{C}} = \begin{bmatrix}-1\\-6\end{bmatrix}$$

- so we have 3 representations of $\vec{w}$ ...

$$[\vec{w}]_{\mathcal{E}} = \begin{bmatrix}2\\-4\\2\end{bmatrix} \qquad [\vec{w}]_{\mathcal{B}} = \begin{bmatrix}1\\3\end{bmatrix} \qquad [\vec{w}]_{\mathcal{C}} = \begin{bmatrix}-1\\-6\end{bmatrix}$$

change-of-coordinate matrix

change-of-coordinate matrix

Change-of-coordinates Matrix

- $\mathcal{B} = (\vec{b}_1 \cdots \vec{b}_n)$
- $\mathcal{C} = (\vec{c}_1 \cdots \vec{c}_n)$

2 ordered bases of subspace V

- chang-of-coordinate matrix S: $S[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{C}}$ for all $\vec{v} \in V$
- $S_{\mathcal{B}\to\mathcal{C}}$ ← means matrix S changes $\mathcal{B}$-coord. to $\mathcal{C}$-coord.
- the i'th column of S is $[\vec{b_i}]_{\mathcal{C}}$

→ **Eg:** P has $\mathcal{B} = \left(\vec{b}_1 = \begin{bmatrix}-1\\-1\\2\end{bmatrix}, \vec{b}_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}\right)$ and $\mathcal{C} = \left(\vec{u}_1 = \begin{bmatrix}-2\\-2\\4\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}0\\1\\-1\end{bmatrix}\right)$

$[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix}a_1\\a_2\end{bmatrix}$ st. $\vec{b}_1 = a_1\vec{u}_1 + a_2\vec{u}_2$

$$\begin{bmatrix}-1\\-1\\2\end{bmatrix} = a_1\begin{bmatrix}-2\\-2\\4\end{bmatrix} + a_2\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}-2\\-2\\4\end{bmatrix} + 0\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$$[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}$$

$$[\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix}a_1\\a_2\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\-2\end{bmatrix}$$

$$\begin{bmatrix}1\\-1\\0\end{bmatrix} = a_1\begin{bmatrix}-2\\-2\\4\end{bmatrix} + a_2\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

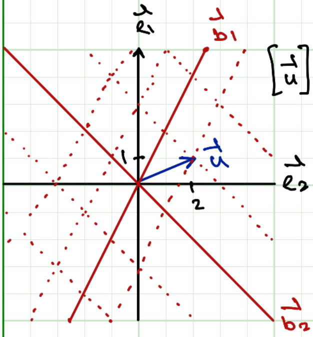
$$= -\frac{1}{2}\begin{bmatrix}-2\\-2\\4\end{bmatrix} - 2\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

∴ $S_{\mathcal{B}\to C} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & -2 \end{bmatrix}$ so let's see ... $\nearrow$ $S_{\mathcal{B}\to C}[\vec{w}]_{\mathcal{B}} = [\vec{w}]_C$

$$S_{\mathcal{B}\to C}[\vec{w}]_{\mathcal{B}} = [\vec{w}]_C$$

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}(3) \\ 0 - 2(3 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \end{bmatrix} = [\vec{w}]_C$$

→ **Eg:** consider $\mathbb{R}^2$: $\mathcal{C} = (\vec{e}_1, \vec{e}_2)$ and $\mathcal{B} = \left(\vec{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$



$[\vec{u}]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad$ $[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- $\vec{u} = \vec{b}_1 + \vec{b}_2$
- consider change-of-basis matrix

$S_{B\to\mathcal{C}}$ ← collum. are $[\vec{b}_i]_{\mathcal{C}}$

$[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\qquad$ $[\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\vec{b}_1 = a_1\vec{e}_1 + a_2\vec{e}_2$ $\qquad$ $\vec{b}_2 = a_1\vec{e}_1 + a_2\vec{e}_2$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\vec{e}_1 + 2\vec{e}_2$ $\qquad$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\vec{e}_1 - 1\vec{e}_2$

$[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad$ $[\vec{u}]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

∴ $S_{B\to\mathcal{C}} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ ... and we know $S_{B\to\mathcal{C}}[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{C}}$

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Week 7: PCE 3 Expand

- For $T:\mathbb{R}^n \to \mathbb{R}^n$, $\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots v_n\vec{e}_n$ is vector $\in \mathbb{R}^n$
- then $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $T(\vec{v}) = T(v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots v_n\vec{e}_n)$

$$= v_1(T\vec{e}_1) + v_2(T\vec{e}_2) + \cdots v_nT(\vec{e}_n)$$

$$= \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix}\vec{v}$$

def. of lin. transformation

vector-matrix multiplication is linear comb. of columns

- since all coordinates in this case are w.r.t. the standard basis $(\vec{e}_1, \vec{e}_2 \cdots)$, we can say:

$$[T(\vec{v})]_{\mathcal{E}} = \underbrace{\begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix}}_{\text{standard matrix of } T}[\vec{v}]_{\mathcal{E}}$$

which means based on standard basis

$\mathcal{B}$-Matrix

- $T:\mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{B} = (\vec{v}_1, \vec{v}_2 \cdots \vec{v}_n)$ is basis of $\mathbb{R}^n$
- $\mathcal{B}$-matrix is unique matrix A s.t. $[T(\vec{v})]_{\mathcal{B}} = A[\vec{v}]_{\mathcal{B}}$
- denoted $[T]_{\mathcal{B}}$
- its i'th column is $[T(\vec{v}_i)]_{\mathcal{B}}$

→ **Eg:** $T:\mathbb{R}^2 \to \mathbb{R}^2$, projection on span$(\vec{u})$ where $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

- take basis $\mathcal{U} = \left(\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right)$
- $T(\vec{u}_1) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{U}}$ ∴ matrix based on $\mathcal{U}$ is: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

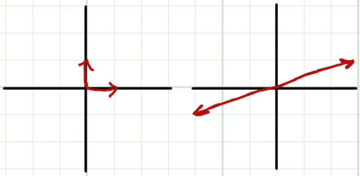
  $T(\vec{u}_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\mathcal{U}}$

Similar Matrices

- 2 n × n matrices A and B are similar if ∃ an invertible n × n matrix S s.t. $B = S^{-1}AS$
- matrix representation of the same linear transformation but w.r.t (or based on) different bases are similar

→ Eg: $T: \mathbb{R}^2 \to \mathbb{R}^2$, projection on $\text{span}(\vec{u})$ where $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

- standard matrix : $\mathcal{E} = (\vec{e_1}, \vec{e_2})$



$$[T(\vec{e_1})]_{\mathcal{E}} = \left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\right]_{\mathcal{E}}$$

$$\text{proj}_{\vec{u}}\, \vec{e_1} = \left(\frac{\vec{e_1}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$$

$$= \left(\frac{3}{3^2+1^2}\right)\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9/10 \\ 3/10 \end{bmatrix}$$

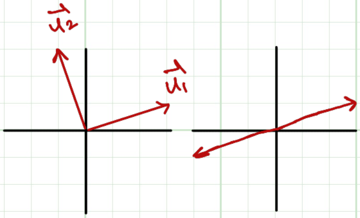
$$[T(\vec{e_2})]_{\mathcal{E}} = \text{proj}_{\vec{u}}\, \vec{e_2}$$

$$= \left(\frac{1}{3^2+1^2}\right)\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/10 \\ 1/10 \end{bmatrix}$$

∴ standard matrix that represents lin. trans. $A = \begin{bmatrix} \frac{9}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix}$

- $\mathcal{U}$-matrix : $\mathcal{U} = \Big(\underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\vec{u_1}}, \underbrace{\begin{bmatrix} -1 \\ 3 \end{bmatrix}}_{\vec{u_2}}\Big)$



$$[T(\vec{u_1})]_{\mathcal{U}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$[T(\vec{u_2})]_{\mathcal{U}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

∴ $\mathcal{U}$-matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

similar matrices

Week 8: Introduce | Orthonormal bases | Sec 5.1 - 5.2

Motivation

- 2 $\mathbb{R}^2$ bases: $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ $\qquad \mathcal{W} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$
- which basis more useful? see how easy it is to represent a vector w.r.t. that basis coordinate.

let $\vec{v} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$

- in $\mathcal{B}$-coordinate: $\left[\vec{v}\right]_{\mathcal{B}} = c_1 \begin{bmatrix} 3 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ too hard to solve entire linear syst.
- in $\mathcal{W}$-coordinate:

$$\left[\vec{v}\right]_{\mathcal{W}} = c_3 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + c_4 \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} c_3 \\ c_4 \end{bmatrix}$$

... take dot prod. of both sides with $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = c_3 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + c_4 \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$6(1/\sqrt{2}) + 5(1/\sqrt{2}) = c_3\left[(1/\sqrt{2})^2 + (1/\sqrt{2})^2\right] + c_4\left[(-1/\sqrt{2})(1/\sqrt{2}) + (1/\sqrt{2})^2\right]$$

$$11/\sqrt{2} = \underline{c_3(1)} + \underline{c_4(0)}$$

$$11/\sqrt{2} = c_3$$

$\therefore c_3 = \frac{11}{\sqrt{2}}$

... instead dot producting both sides with $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ ... $c_4 = \begin{bmatrix} 6 \\ 5 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = -\frac{1}{\sqrt{2}}$

$\therefore c_4 = -\frac{1}{\sqrt{2}}$

* How did this happen?

↳ bc. $\mathcal{W}$ is orthonormal basis

Orthogonal Set

- set of vectors $\vec{u}_1 \cdots \vec{u}_n$ is orthogonal if $\vec{u}_i \cdot \vec{u}_j = 0$ for $i \neq j$

  perpendicular ↗

Orthonormal Set

- set of vectors $\vec{u}_1 \cdots \vec{u}_n$ is orthonormal if …
  - $\vec{u}_i \cdot \vec{u}_j = 0$ for $i \neq j$ ← means they're perpendicular
  - $\vec{u}_i \cdot \vec{u}_i = 1$ for all i ← means they're unit vectors

Orthogonal & Orthonormal Basis

- orthogonal basis of subspace V is orthogonal set in V that is also a basis of V.
- orthonormal basis of subspace V is orthonormal set in V that is also a basis of V.

→ Eg: $W = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$

1. all vectors perpendicular since $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = 0$
2. all vectors are unit vectors since …

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = 1 \qquad \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = 1$$

$\therefore W$ is orthonormal basis of $\mathbb{R}^2$

- property 1 ← explains why c_4 vanished
- property 2 ← explains why c_1 had coefficient of 1

Coordinate W.R.T. Orthonormal Basis

$\mathcal{U} = \{ \vec{u}_1 \cdots \vec{u}_n \}$ ← orthonormal basis of $\mathbb{R}^n$ (alway linearly independent)

- for all $\vec{v} \in \mathbb{R}^n$, $[\vec{v}]_{\mathcal{U}} = \begin{bmatrix} \vec{v} \cdot \vec{u}_1 \\ \vec{v} \cdot \vec{u}_2 \\ \vec{v} \cdot \vec{u}_n \end{bmatrix}$

Projecting Onto Subspace

subspace of $\mathbb{R}^2$

- $\mathbb{R}^2$ line $L = \{ \vec{x} \in \mathbb{R}^2 \mid \vec{x} = t\begin{bmatrix}1\\2\end{bmatrix}, t \in \mathbb{R} \} = \text{span}\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$
- consider $L^\perp$ (contains all vectors ⊥ to L) ... find $\vec{x}$ ⊥ to $\begin{bmatrix}1\\2\end{bmatrix}$

also subspace of $\mathbb{R}^2$

$$L^\perp = \{ \vec{x} \in \mathbb{R}^2 \mid \vec{x} \cdot \begin{bmatrix}1\\2\end{bmatrix} = 0 \} = \{ \vec{x} \in \mathbb{R}^2 \mid \vec{x} = t\begin{bmatrix}-2\\1\end{bmatrix} \mid t \in \mathbb{R} \} = \text{span}\left(\begin{bmatrix}-2\\1\end{bmatrix}\right)$$

"orthogonal complement" to L

- $\mathcal{B} = \{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix} \}$ ← forms $\mathbb{R}^2$ basis bc. lin. ind.
- $\mathcal{B}$ is orthogonal basis ... since $\begin{bmatrix}1\\2\end{bmatrix} \cdot \begin{bmatrix}-2\\1\end{bmatrix} = 0$
- $\mathcal{B}$ can be made orthonormal by scaling each vector to length 1

$$\left| k\begin{bmatrix}1\\2\end{bmatrix} \right| = 1$$

$$\left| \begin{bmatrix}k\\2k\end{bmatrix} \right| = 1$$

$$1 = \sqrt{k^2 + 4k^2}$$

$$1 = \sqrt{5k^2}$$

$$k = \pm\sqrt{\frac{1}{5}} = \pm \frac{1}{\sqrt{5}}$$

$\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}$ and $\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\end{bmatrix}$

$$\mathcal{B} = \left\{ \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}, \begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix} \right\}$$

orthonormal basis

What this Means ?

$$[\vec{x}]_{\mathcal{B}} = \overbrace{\underbrace{c_1\begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}}_{\in L}}^{\text{proj. of } \vec{x} \text{ on } L} + \overbrace{\underbrace{c_2\begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}}_{\in L^{\perp}}}^{\text{proj. of } \vec{x} \text{ on } L}$$

original line ↗ ↖ line ⊥ to original line

finding c_1

dot product $\begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$ with both sides ...

$$\vec{x}\cdot\begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix} = c_1 + 0$$

$\text{proj}_L \vec{x}$ ↓

$\therefore$ proj. of $\vec{x}$ onto L given by $\left(\vec{x}\cdot\begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}\right)\begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$

... $c_1\begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$

finding c_2

dot product $\begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$ with both sides ...

$$\vec{x}\cdot\begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix} = 0 + c_2$$

$\text{proj}_{L^\perp} \vec{x}$ ↓

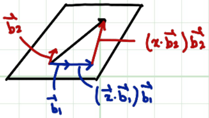
$\therefore$ proj. of $\vec{x}$ onto $L^{\perp}$ given by $\left(\vec{x}\cdot\begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}\right)\begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$

... $c_2\begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$

In general,

$$\vec{x} = (\vec{x}\cdot\vec{b}_1)\vec{b}_1 + (\vec{x}\cdot\vec{b}_2)\vec{b}_2 + \cdots \qquad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix}\vec{x}\cdot\vec{b}_1\\\vec{x}\cdot\vec{b}_2\\\vdots\end{bmatrix}$$

if $\mathcal{B} = \{\vec{b}_1, \vec{b}_2 \cdots\}$ is an orthonormal basis



Orthogonal Complement

- if W is subspace in $\mathbb{R}^n$, orthogonal complement is the set

$$W^\perp = \left\{ \vec{v} \in \mathbb{R}^n : \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W \right\}$$

- $\mathcal{B}$ is basis for W, so $W^\perp$ is the set of all vectors that are $\perp$ to vectors in $\mathcal{B}$

→ Eg: find orthogonal complement of subspace W: $x+y+z=0$

$\vec{n}$ for plane : $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ← span of this gives us all the vectors $\perp$ to all vectors on W

$$\therefore W^\perp = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \left\{ \vec{x} \in \mathbb{R}^3 \mid \vec{x} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, t \in \mathbb{R} \right\}$$

→ Eg: find orthogonal complement of subspace $V = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$

$$V^\perp = \left\{ \vec{v} \in \mathbb{R}^3 \mid \vec{v} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0 \right\}$$ ... mutiply out the dot product

$$V^\perp = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \mid v_1 + 2v_2 + 3v_3 = 0 \right\}$$

$\therefore V^\perp$ is plane with $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Orthogonal Decomposition

- vector $\vec{x}$ in $\mathbb{R}^n$ and subspace V of $\mathbb{R}^n$

↳ $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$

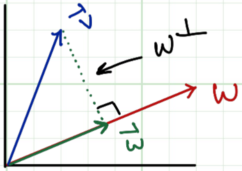
$\vec{x}$ is in V

$\vec{x}^\perp$ is in $V^\perp$

Orthogonal Projections

- W is subspace of $\mathbb{R}^n$ and $\vec{v} \in \mathbb{R}^n$, orthogonal projection of $\vec{v}$ onto W is unique $\vec{w} \in W$ s.t. $\vec{v} - \vec{w} \in W^{\perp}$

$\text{proj}_W(\vec{v})$



- orthonormal basis for V: $\{\vec{u_1}, \dots, \vec{u_k}\}$ for V and $\vec{v} \in \mathbb{R}^n$

$$\text{Proj}_V(\vec{v}) = (\vec{v} \cdot \vec{u_1})\vec{u_1} + (\vec{v} \cdot \vec{u_2})\vec{u_2} + \cdots + (\vec{v} \cdot \vec{u_k})\vec{u_k}$$

→ **Eg:** find orthogonal proj. of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto subspace P: $x + y + z = 0$

1. Find orthonormal basis for P ... Gram-Schmidt process

$1 - 1 + 0 = 0$ $\qquad$ $1 + 1 - 2 = 0$

$\therefore \vec{b_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is in P $\qquad$ $\therefore \vec{b_2} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ is in P

$\vec{b_1} \cdot \vec{b_2} = 0$, so they're orthogonal

Now make their magnitude 1 to have orthonormal basis

$$\left\| k\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\| = 1$$
$$\left\| \begin{bmatrix} k \\ -k \\ 0 \end{bmatrix} \right\| = 1$$
$$\sqrt{k^2 + k^2} = 1$$
$$1 = \sqrt{2k^2}$$
$$1 = 2k^2$$
$$k = \pm \frac{1}{\sqrt{2}}$$

$$\left\| k\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\| = 1$$
$$\left\| \begin{bmatrix} k \\ k \\ -2k \end{bmatrix} \right\| = 1$$
$$1 = \sqrt{k^2 + k^2 + (-2k)^2}$$
$$1 = \sqrt{6k^2}$$
$$k = \frac{1}{\sqrt{6}}$$

$$\therefore \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \vec{b_1}$$

$$\therefore \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} = \vec{b_2}$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} \right\}$$

2. Apply theorem:

$$\text{proj}_P\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = \left(\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}\right)\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix} + \left(\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{bmatrix}\right)\begin{bmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{bmatrix}$$

$$= \left(\frac{1-2}{\sqrt{2}}\right)\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix} + \left(\frac{1+2-6}{\sqrt{6}}\right)\begin{bmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{bmatrix}$$

$$= \begin{bmatrix}-1\\0\\1\end{bmatrix} \longleftarrow \text{notice } \underset{\overset{\uparrow}{\vec{v}}}{\begin{bmatrix}1\\2\\3\end{bmatrix}} - \underset{\overset{\uparrow}{\vec{w}\in P}}{\begin{bmatrix}-1\\0\\1\end{bmatrix}} = \underset{\overset{\uparrow}{\vec{n}\in P^{\perp}}}{\begin{bmatrix}2\\2\\2\end{bmatrix}}$$

- Let V be subspace of $\mathbb{R}^n$

$\dim V + \dim V^{\perp} = n$

PCE 8: Expand | Orthogonal Projections | Sec 5.1, 5.2

- subspace V of $\mathbb{R}^{n}$ ← basis $\mathcal{U} = \{\vec{b}_1, \vec{b}_2, \cdots \vec{b}_k\}$
- orthogonal complement $V^\perp$ of V ← basis will have $n - k$ vectors

  so $\mathcal{U}^\perp = \{\vec{b}_{k+1}, \cdots, \vec{b}_n\}$

- basis for entire $\mathbb{R}^n$: $\mathcal{B} = \{\underbrace{\vec{b}_1, \cdots, \vec{b}_k}_{\in V}, \underbrace{\vec{b}_{k+1}, \cdots, \vec{b}_n}_{\in V^\perp}\}$

  orthonormal

- basically, number of basic vectors in V and $V^\perp$ must add up to n, or the dimension of the domain.

→ **Eg**: find orthogonal decomposition of $\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ w.r.t $V: x+y+z=0$

- orthonormal basis of V: $\mathcal{B} = \left\{\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{bmatrix}\right\}$
- orthonormal basis of $V^\perp$: $\mathcal{C} = \left\{\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}\right\}$
- $\text{proj}_V \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$ and $\text{proj}_{V\perp} \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}$

$\therefore$ orthogonal decomposition of $\vec{v}$ w.r.t V : $\begin{bmatrix}1\\2\\3\end{bmatrix} = \vec{x}^{\parallel} + \vec{x}^{\perp} = \begin{bmatrix}-1\\0\\1\end{bmatrix} + \begin{bmatrix}2\\2\\2\end{bmatrix}$

$\mathcal{B} = \{\vec{u}_1, \cdots, \vec{u}_k\}$ orthogonal complement $\vec{y}$ s.t. $\vec{y} \cdot \vec{v} = 0$ for $\vec{v} \in V$

basis for subspace V

we know $A^T\vec{y} = \vec{0}$

$A = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_k \\ | & & | \end{bmatrix}$ so $A^T = \begin{bmatrix} — & \vec{u}_1 & — \\ & \vdots & \\ — & \vec{u}_k & — \end{bmatrix}$

so if $\vec{y}$ is orthogonal to $\mathcal{B}$ vectors, then $\vec{y} \in \text{Null}(A^T)$

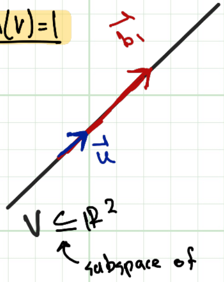
- Let A be a matrix whose columns are $\vec{b}_i$'s → $A = \begin{bmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_r \\ | & & | \end{bmatrix}$

  $(\text{Coll}(A))^\perp = \text{Null}(A^T)$

Gram-Schmidt Process (finding orthonormal bases)

+ How do we find orthonormal bases for given subspace

dim(V)=1

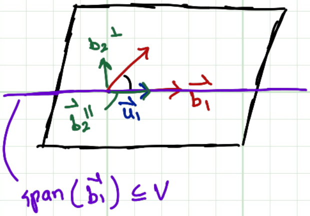


$$\mathcal{B} = \{ \vec{b}_1 \}$$

$$\vec{u} = \frac{\vec{b}_1}{\|\vec{b}_1\|} \qquad \mathcal{U} = \{ \vec{u} \}$$

orthonormal basis for V

dim(V) = 2



$$\mathcal{B} = \{ \vec{b}_1, \vec{b}_2 \}, \quad \mathcal{U} = \{ \vec{u}_1, \vec{u}_2 \}$$

$$\vec{u}_1 = \frac{\vec{b}_1}{\|\vec{b}_1\|} \qquad \vec{u}_2 = \frac{\vec{b}_2^{\perp}}{\|\vec{b}_2^{\perp}\|}$$

$$\vec{b}_2 = b_2^{\parallel} + b_2^{\perp}$$

$$\vec{b}_2^{\perp} = \vec{b}_2 - \vec{b}_2^{\parallel}$$

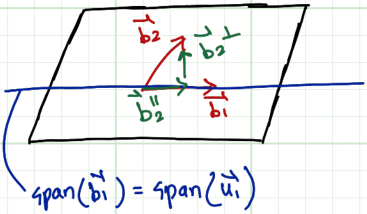
$$\vec{b}_2^{\perp} = \vec{b}_2 - \text{proj}_{\text{span}(\vec{b}_1)} \vec{b}_2$$

$$= \vec{b}_2 - (\vec{b}_2 \cdot \vec{u}_1)\, \vec{u}_1$$

Steps

1. make the first vector into unit by dividing by its magnitude
2. orthogonally decompose second given basis and get the perpendicular complement to subspace, divide this by its magnitude

→ Eg: $V = \mathbb{R}^2$, $\mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \vec{b}_2 = \begin{bmatrix}-1\\2\end{bmatrix} \right\}$



① $\vec{u}_1 = \dfrac{\vec{b}_1}{\|\vec{b}_1\|} = \boxed{\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}}$

$\|\vec{b}\| = \sqrt{1^2+1^2} = \sqrt{2}$

$\vec{b}_2 = \vec{b}_2^{\parallel} + \vec{b}_2^{\perp}$

② $b_2^{\perp} = \vec{b}_2 - \vec{b}_2^{\parallel}$

$= \vec{b}_2 - \text{proj}_{sp(\vec{u}_1)} \vec{b}_2$

$= \vec{b}_2 - (\vec{b}_2 \cdot \vec{u}_1)\vec{u}_1$

$= \begin{bmatrix}-1\\2\end{bmatrix} - \left(\begin{bmatrix}-1\\2\end{bmatrix} \cdot \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}\right) \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

$= \begin{bmatrix}-1\\2\end{bmatrix} - \left(\frac{1}{\sqrt{2}}\right) \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

$= \begin{bmatrix}-1\\2\end{bmatrix} - \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix} = \begin{bmatrix}-3/2\\3/2\end{bmatrix}$

③ $\vec{u}_2 = \dfrac{b_2^{\perp}}{\|\vec{b}_2^{\perp}\|}$

$\|\vec{b}_2\| = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2}$

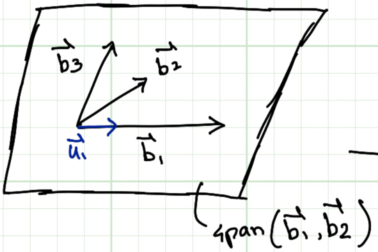
$= \dfrac{3\sqrt{2}}{2}$

$\therefore \vec{u}_2 = \boxed{\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}}$

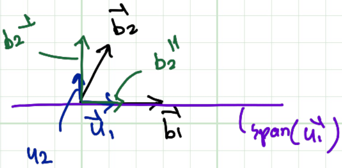
$\therefore \mathcal{U} = \{\vec{u}_1, \vec{u}_2\}$ ← orthonormal basis for $\mathbb{R}^2$

dim(V) = 3 $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$

① $\vec{u}_1 = \dfrac{\vec{b}_1}{\|\vec{b}_1\|}$



→ look at from top



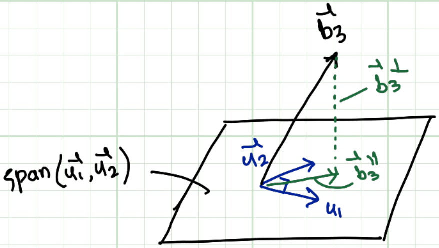
$$\vec{b}_2 = \vec{b}_2^{\parallel} + b_2^{\perp}$$

$$b_2^{\perp} = \vec{b}_2 - b_2^{\parallel}$$

$$= \vec{b}_2 - \text{proj}_{sp(\vec{u}_1)} \vec{b}_2$$

$$= \vec{b}_2 - (\vec{b}_2 \cdot \vec{u}_1)\vec{u}_1$$

② $\vec{u}_2 = \dfrac{b_2^{\perp}}{\|b_2^{\perp}\|}$



$\vec{b}_3^{\parallel} \in \text{span}(\vec{u}_1, \vec{u}_2)$

$\vec{b}_3^{\perp} \in \underbrace{\text{span}(\vec{u}_1, \vec{u}_2)}_{\text{"orthogonal complement to span}(\vec{u}_1,\vec{u}_2)\text{"}}{}^{\perp}$

$$\vec{b}_3 = \vec{b}_3^{\parallel} + \vec{b}_3^{\perp}$$

$$\vec{b}_3^{\perp} = \vec{b}_3 - \vec{b}_3^{\parallel}$$

$$= \vec{b}_3 - \text{proj}_{sp(\vec{u}_1,\vec{u}_2)} \vec{b}_3$$

$$= \vec{b}_3 - \left((\vec{b}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{b}_3 \cdot \vec{u}_2)\vec{u}_2 \right)$$

③ $\vec{u}_3 = \dfrac{\vec{b}_3^{\perp}}{\|\vec{b}_3^{\perp}\|}$

$\mathcal{U}$

Gram-Schmidt

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_m\}$ basis for V, $\dim(V) = m$

$\mathcal{U} = \{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_m\}$ orthonormal basis for V

$$\vec{u}_1 = \frac{\vec{b}_1}{\|\vec{b}_1\|}$$

$$\vec{u}_i = \frac{\vec{b}_i^{\perp}}{\|\vec{b}_i^{\perp}\|}$$

$$\vec{b}_i^{\perp} = \vec{b}_i - \vec{b}_i^{\parallel}$$

$$= \vec{b}_i - \text{proj}_{\text{span}(\vec{b}_1, \vec{b}_2, \cdots \vec{b}_{i-1})} \vec{b}_i$$

$$= (\vec{b}_1 \cdot \vec{u}_1)\vec{u}_1 + \cdots + (\vec{b}_{i-1} \cdot \vec{u}_{i-1})\vec{u}_{i-1}$$

QR- Factorization

- since $\mathcal{U}$ is (orthonorm.) basis for V, vectors in $\mathcal{B}$ can be written as lin. comb. of $\mathcal{U}$ vectors

- QR-factorization
$$\underset{}{\begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_k \end{bmatrix}} = \underset{Q}{\begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_k \end{bmatrix}} \underset{R}{\begin{bmatrix} [\vec{b}_1]_{\mathcal{U}} & \cdots & [\vec{b}_k]_{\mathcal{U}} \end{bmatrix}}$$

invertible, upper triangular

→ Eg: $\mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ for $V: x+y+z=0$ in $\mathbb{R}^3$

① $\vec{u}_1 = \dfrac{\vec{b}_1}{\|\vec{b}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$

$\|\vec{b}_1\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\vec{u}_2 = \dfrac{\vec{b}_2^{\perp}}{\|\vec{b}_2^{\perp}\|}$

$\therefore \mathcal{U} = \{\vec{u}_1, \vec{u}_2\}$ ← orthonormal basis

$$\begin{aligned}
\vec{b}_2^{\perp} &= \vec{b}_2 - \vec{b}_2 \\
&= \vec{b}_2 - \operatorname{proj}_{sp(\vec{u}_1)} \vec{b}_2 \\
&= \vec{b}_2 - (\vec{b}_2 \cdot \vec{u}_1)\vec{u}_1 \\
&= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{1}{\sqrt{2}} \right) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\|\vec{b}_2^{\perp}\| &= \sqrt{\left(\tfrac{1}{2}\right)^2 + 1^2 + \left(\tfrac{1}{2}\right)^2} \\
&= \sqrt{3/2}
\end{aligned}$$

$$\begin{aligned}
\vec{u}_2 &= \frac{1}{\sqrt{3/2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}
\end{aligned}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} = \underset{Q}{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}} \underset{R}{\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{6}/2 \end{bmatrix}}$$

$$\vec{b}_1 = \sqrt{2}\,\vec{u}_1 + 0\,\vec{u}_2$$

$$\vec{b}_2 = \frac{1}{\sqrt{2}}\vec{u}_1 + \frac{\sqrt{6}}{2}\vec{u}_2$$

Week 9: Introduce | Orthogonal Lin. Trans.

Orthogonal Linear Transformations

- orthogonal trasformations preserve magnitudes of vectors
- $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if $\|T(\vec{x})\| = \|\vec{x}\|$ for all $\vec{x} \in \mathbb{R}^n$
- the matrix of orthogonal trasformation is called orthogonal matrix
- Eg: rotations, reflections
- columns of orthogonal matrix form orthonormal basis for $\mathbb{R}^n$
- orthogonal trasformations preserve angle between between 2 vectors and $\therefore$ preserve the dot product
- inverse of orthogonal matrix is its transpose : $A^{-1} = A^T$
  ↳ for "A" $n \times n$, A is orthogonal exactly when $A^TA = I_n$ or $A^{-1} = T$
- orthogonal matrix is always square

Orthogonal Projections

- orthonormal basis $\mathcal{B} = \{\vec{u}_1, \cdots, \vec{u}_m\}$ for subspace $V \subseteq \mathbb{R}^n$

$$\text{Proj}_V(\vec{x}) = (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + \cdots + (\vec{x}\cdot\vec{u}_m)\vec{u}_m$$

lin. comb. of $\vec{u}_1$ to $\vec{u}_m$ with $(\vec{x}\cdot\vec{u}_m)$ as coeff.

$$= \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix} \begin{bmatrix} \vec{x}\cdot\vec{u}_1 \\ \vdots \\ \vec{x}\cdot\vec{u}_m \end{bmatrix}$$

rows of matrix are the transpose of $\vec{u}_m$

$$= \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix} \begin{bmatrix} — & \vec{u}_1 & — \\ & \vdots & \\ — & \vec{u}_m & — \end{bmatrix} \vec{x}$$

$$\boxed{\text{Proj}_V(\vec{x}) = QQ^T\vec{x}}$$

← standard matrix for orthogonal proj. onto V

Q: matrix with basis vectors as columns

Q^T: transpose of Q

- Q is <u>not</u> orthogonal matrix unless it's square

→ **Eg:** $\mathcal{U} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\} = V$ proj. of $\vec{x} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$ onto V is...

Using Old Method

$$\text{Proj}_V \vec{x} = (\vec{x} \cdot \vec{u_1})\vec{u_1} + (\vec{x} \cdot \vec{u_2})\vec{u_2}$$

$$= \left(\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} + \left(\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right) \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} + 0 \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix}$$

Using Matrix Multiplication

$$\text{Proj}_V(\vec{x}) = QQ^T \vec{x} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix}$$

Week 9: Solidify

- We have $\mathcal{B} = \{\vec{u}_1, \cdots, \vec{u}_m\}$ for subspace V of $\mathbb{R}^n$
- define matrix $A = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}$ ← $n \times m$ matrix

  $Col(A) = V$

  $= QR$

  ← $m \times m$ upper triangular, invertable matrix

  $n \times m$ matrix with orthonormal columns

$A^TA = (QR)^TQR$ ... Let $A = QR$

$= R^TQ^TQR$ ...order of matrix product switches when we do transpose

$Q^TQ = I_m$ ... since Q is $n \times m$, Q^TQ gives us $m \times m$ identity

$$\boxed{A^TA = R^TR}$$

- if A is $n \times m$ with lin. indep. columns, then A^TA is $m \times m$ invertible matrix

Standard Matrix of Orthogonal Projection

subspace $V \in \mathbb{R}^n$ and $\mathcal{B} = \{\vec{v}_1, \cdots, \vec{v}_m\}$

if A is orthonormal, then $A^TA = I_m$ and $proj_V(\vec{b}) = AA^T\vec{b}$

if $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$, then $\boxed{proj_V(\vec{b}) = A(A^TA)^{-1}A^T\vec{b}}$

Closest Vector

- $V \leq \mathbb{R}^n$ and $\vec{x} \in \mathbb{R}^n$, the closest vector in V to $\vec{x}$ is given by $Proj_V(\vec{x})$
- $\|Proj_V(\vec{x}) - \vec{x}\| \leq \|\vec{v} - \vec{x}\|$ for any $\vec{v} \in V$

Least Squares

- (1,2), (2,4), (3,6) ; how can we find equ. passing through?
- let $y = c_0 + c_1 x$

$$\hookrightarrow \left.\begin{aligned} c_0 + 1x &= 2 \\ c_0 + 2x &= 4 \\ c_0 + 3x &= 6 \end{aligned}\right\} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{\text{matrix } A} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

solutions: $c_0 = 0$
$c_1 = 2$

- What if we have this?

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$$

slightly changed

- least square solution : find $\begin{bmatrix} c_0^* \\ c_1^* \end{bmatrix}$ which minimizes the squared error

$$\left\| \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} - \underbrace{\begin{bmatrix} c_0 + c_1 \\ c_0 + 2c_1 \\ c_0 + 3c_2 \end{bmatrix}} \right\|^2$$

every vector of that form is in Coll(A), so we seek $\vec{v}$ in Coll(A) which minimizes $\left\| \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} - \vec{v} \right\|^2$

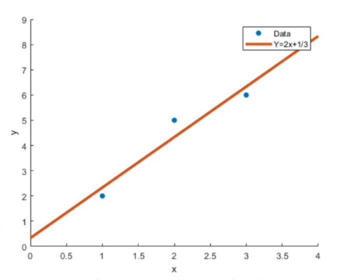
vector in subspace V which is closest to $\vec{x}$ is $\mathrm{Proj}_V(\vec{x})$

∴ c_0^* and c_1^* must satisfy

$$A\begin{bmatrix} c_0^* \\ c_1^* \end{bmatrix} = A(A^TA)^{-1}A^T\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} c_0^* \\ c_1^* \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 2 \end{bmatrix}$$



Example: $T: \mathbb{R}^3 \to \mathbb{R}^3$, $[T]_{\mathcal{E}} = A = \begin{bmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix}$ $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

let $\mathcal{B} = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Note: $A\vec{v}_1 = -2\vec{v}_1$
$A\vec{v}_2 = 2\vec{v}_2$
$A\vec{v}_3 = 4\vec{v}_3$

Hence: $[T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$

$[T(\vec{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$[T(\vec{v}_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$

$\therefore [T]_{\mathcal{B}} = B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

B is much easier to work with because it's diagonal

only if it has an eigenbasis

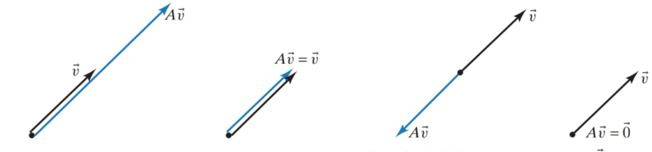
- matrix A is diagonalizable exactly when A is similar to some diagonal matrix B. Basically, A in a good basis is diagonal.
- this good basis is made of vectors that go to a multiple of themselves under linear map T.
- diagonal matrix B would have these multiples as diagonal entries

Eigenvector, Eigenvalue, Eigenbasis

- an eigenvector of a lin. trans. is any non-zero vector $\vec{v} \in \mathbb{R}^n$ s.t. $T(\vec{v}) = \lambda \vec{v}$ for some scalar λ.

eigenvalue of T corresponding to eigenvector $\vec{v}$

A nonzero vector $\vec{v}$ in $\mathbb{R}^n$ is an eigenvector of $T(\vec{x}) = A\vec{x}$ if $A\vec{v}$ is parallel to $\vec{v}$.



$A\vec{v} = 2\vec{v}$ eigenvalue: 2

$A\vec{v} = \vec{v} = 1\vec{v}$ eigenvalue: 1

$A\vec{v} = -\vec{v} = (-1)\vec{v}$ eigenvalue: –1

$A\vec{v} = \vec{0} = 0\vec{v}$ eigenvalue: 0

Week 10: Solidify | Eigenbasis

Finding Eigenvectors

$A\vec{v} = \lambda \vec{v}$ … λ = eigenvalue

$A\vec{v} - \lambda \vec{v} = \vec{0}$

$A\vec{v} - \lambda I_n \vec{v} = \vec{0}$

$\underbrace{(A - \lambda I_n)}_{\text{square matrix}} \vec{v} = \vec{0}$

λ is eigenvalue for eigenvector $\vec{v}$ exactly when $\vec{v} \in \text{Null}(A - \lambda I_n)$

→ since kernel contains the non-zero vector $\vec{v}$, the matrix is not invertible and det = 0

- λ is eigenvalue of nxn A exactly when: $\boxed{\det(A - \lambda I_n) = 0}$

n-degree polynomial called the characteristic polynomial

$$f_A(x) = \det(A - x I_n)$$

Eg 1:

- given $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

$$\det(A - \lambda I_n) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)(2-\lambda) - 6$$

$$= 2 - \lambda - 2\lambda + \lambda^2 - 6$$

characteristic polynomial → $= \lambda^2 - 3\lambda - 4$ … set it $= 0$ to find eigenvalues

$0 = \lambda^2 - 3\lambda - 4$

$0 = (\lambda - 4)(\lambda + 1)$ $\quad \boxed{\therefore \lambda_1 = 4, \ \lambda_2 = -1}$

Eg 2: $A = \begin{bmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix}$

$$\det(A - \lambda I_n) = \det \begin{bmatrix} -\lambda & -1 & -3 \\ 2 & 3-\lambda & 3 \\ -2 & 1 & 1-\lambda \end{bmatrix}$$

$$= -\lambda^3 + 4\lambda^2 + 4\lambda - 16$$

$$= (\lambda - 4)(\lambda - 2)(\lambda + 2)$$

eigenvalues $\lambda_1 = 4$
$\lambda_2 = 2$
$\lambda_3 = -2$

Eg 3: $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ projection onto plane $x + y + z = 0 : P$

$x = -t - s$
$y = t$
$z = s$

$$\vec{x} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad P : \mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\vec{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \frac{\vec{b}_2^{\perp}}{\|\vec{b}_2^{\perp}\|} \qquad \vec{b}_2^{\perp} = \vec{b}_2 - (\vec{b}_2 \cdot \vec{u}_1)\vec{u}_1$$

$$= b_2 - \left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right) \vec{u}_1$$

$$= b_2 - \frac{1}{\sqrt{2}} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \frac{2}{\sqrt{6}} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/2\sqrt{6} \\ -2/2\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

After finding the orthonormal basis for P, find proj. onto P of $\vec{e}_1, \vec{e}_2, \vec{e}_3$ to find standard matrix of T

$$\text{proj}_P(\vec{e}_1) = (\vec{e}_1 \cdot \vec{u}_1)\vec{u}_1 + (\vec{e}_1 \cdot \vec{u}_2)\vec{u}_2 = \begin{bmatrix} 2/3 \\ -1/3 \\ 1/3 \end{bmatrix}$$

$$\vdots$$

$$A = \begin{bmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{bmatrix}$$

$$\det(A - \lambda I_n) = \det \begin{bmatrix} 2/3 - \lambda & -1/3 & 1/3 \\ -1/3 & \frac{2}{3} - \lambda & 1/3 \\ 1/3 & 1/3 & 2/3 - \lambda \end{bmatrix} = (\lambda - 1)(\lambda - 1)\lambda = (\lambda - 1)^2 \lambda$$

$\therefore \lambda_1 = 1 \quad \lambda = 0$

repeated twice, so we say eigenvalue $\lambda = 1$ has algebraic multiplicity two

Notice

$\lambda = 1$ … vectors on P

$\lambda = 0$ … vectors on $P^{\perp}$

Week 10: Expand

Finding Eigenvectors

eigenvalue

- nonzero $\vec{v} \in \mathbb{R}^n$ is eigenvector if $T(\vec{v}) = \lambda \vec{v}$
- $\vec{v}$ is an eigenvector when : $T(\vec{v}) = \lambda \vec{v}$

$$T(\vec{v}) - \lambda \vec{v} = \vec{0}$$

$$A\vec{v} - \lambda I_n \vec{v} = \vec{0}$$

$$\vec{v}(A - \lambda I_n) = \vec{0}$$

$\vec{v}$ is eigenvector exactly when $\vec{v} \in Null(A - \lambda I_n)$

Eigenspace

- if λ is eigenvalue for $T: \mathbb{R}^n \to \mathbb{R}^n$, then λ - eigenspace is:

$$V_\lambda = \left\{ \vec{v} \in \mathbb{R}^n \,\middle|\, T(\vec{v}) = \lambda \vec{v} \right\} \longleftarrow \text{contains } \vec{0}$$

Eg: $A = \begin{bmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix}$ and $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = -2$

- for $\lambda_1 = 4$

  find kernel of $(A - 4I_n) = \left[\begin{array}{ccc|c} -4 & -1 & -3 & 0 \\ 2 & -1 & 3 & 0 \\ -2 & 1 & -3 & 0 \end{array}\right] \ldots E_4 = span\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$

- for $\lambda_2 = 2$

  find kernel of $(A - 2I_n) = \left[\begin{array}{ccc|c} -2 & -1 & -3 & 0 \\ 2 & 1 & 3 & 0 \\ -2 & 1 & -1 & 0 \end{array}\right] \ldots E_2 = span\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$

- for $\lambda_3 = -2$

  find kernel of $(A + 2I_n) = \left[\begin{array}{ccc|c} 2 & -1 & -3 & 0 \\ 2 & 5 & 3 & 0 \\ -2 & 1 & 3 & 0 \end{array}\right] \ldots E_{-2} = span\left\{\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}\right\}$

Notes

* eigenvectors lin. indep, and for basis $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

the $\mathcal{B}$-matrix of T will be $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Week 11: Introduce | Diagonalization | Sec. 7.3, 7.4, 8.1

→ **Theorem:** $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$

- eigenvectors of T that correspond to distinct eigenvalues are linearly independent.
- If we have "n" distinct eigenvalues, then we can always create an eigenbasis by picking a vector from each eigenspace.
- If "n" distinct eigenvalues, T is diagonalizable.
- dimension of eigenspaces must add up to "n".

→ **Eg:** Is $T(\vec{x}) = A\vec{x}$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ diagonalizable?

$$\det(A - \lambda I) = (\lambda)(\lambda) - 0$$

$$0 = \lambda^2$$

$$\boxed{\lambda = 0 \ , \ \text{almu} = 2}$$

$$E_0: \left.\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]\right\} \begin{array}{l} x_1 = x_1 \\ x_2 = 0 \end{array} \quad \vec{x}_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$E_0 = \text{span}\{\vec{e}_1\} \leftarrow \boxed{\text{gemu} = 1}$$

since $\dim(E_0) \neq 2$,

T is <u>NOT diagonalizable</u>

→ <u>Geometric Multiplicity</u>

- gemu of λ is the dimension of λ-eigenspace
- $\text{gemu}(\lambda) \leq \text{gemu}(\lambda)$ (gemu $<\neq$ almu caused a problem above)

Let's summarize all of our observations in one single theorem

Theorem. *Let $T: \mathbb{R}^n \to \mathbb{R}^n$ $T(\vec{x}) = A\vec{x}$. The following are equivalent*

1. *The linear transformation T is diagonalizable.*
2. *There exists an eigenbasis for T.*
3. *There exists a diagonal matrix D and an invertible matrix S such that $A = SDS^{-1}$.*
4. *The dimensions of the eigenspaces of T add up to n.*
5. *The geometric multiplicities of the eignevalues of T add up to n.*
6. *The algebraic multiplicities of the eignevalues of T add up to n AND for every eigenvalue λ of T $\text{gemu}(\lambda) = \text{almu}(\lambda)$.*

Week 11: 2. Expand

→ **Eg 1:** $P: \mathbb{R}^3 \to \mathbb{R}^3$ proj. onto plane $V: x+y+z=0$

- **bad basis**: standard basis (hard to compute)
- **good basis**: basis containing orthonormal basis for V
  a) can compute $\text{proj}_V \vec{x}$ easily ... $= (\vec{x}\cdot\vec{u_1})\vec{u_1} + (\vec{x}\cdot\vec{u_2})\vec{u_2}$
  b) basis vectors on V stay put under lin. trans., so $[P]_{\mathcal{B}}$ clean.
- **better basis**: orthonormal basis of $\mathbb{R}^3$ containing orthonormal basis for V and orthonormal basis for $V^\perp$
  a) it's orthonormal basis for $\mathbb{R}^n$
  b) it's an eigenbasis
  c) $[P]_{\mathcal{B}}$ is diagonal with $\lambda_1, \lambda_2, \lambda_3$ on main diagonal.

→ **Eg 2:** $L: \mathbb{R}^3 \to \mathbb{R}^3$, $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

- standard basis bad, use eigenbasis $\mathcal{B} = \Big\{ \overbrace{\vec{v_1} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}^{\in E_1}, \overbrace{\vec{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}^{\in E_2} \Big\}$
- $\vec{v_1} \in E_1$, $\vec{v_2}, \vec{v_3} \in E_2$ for $\lambda_1 = 1$, $\lambda_2 = 2$
- $\therefore [L]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

→ **Eg 3:** $T: \mathbb{R}^3 \to \mathbb{R}^3$ $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$\lambda_1 = 0$, $\vec{v_1}, \vec{v_2} \in E_0$

$\lambda_2 = 3$, $\vec{v_3} \in E_3$

- stadard basis not bad, but not best
- eigenbasis $\mathcal{B} = \Big\{ \vec{v_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Big\}$
- $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ ⟵ since E_0 basis $\perp$ to E_3 basis, use Gram Schmidt to find orthonormal basis

- **orthonormal eigenbasis** $\mathcal{U} = \Big\{ \vec{u_1}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \vec{u_2} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \vec{u_3} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \Big\}$

  for T, with $[T]_{\mathcal{U}} = [T]_{\mathcal{B}}$

Orthogonally Diagonalizable : $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$

- if $T: \mathbb{R}^n \to \mathbb{R}^n$ has orthogonal eigenbasis
- $A = QDQ^{-1}$ Q has orthonormal columns (is orthogonal)

  $A = QDQ^T$ and $\therefore$ $Q^{-1} = Q^T$

- if $Q^T AQ$ is diagonal, A is orthogonally diagonalizable

  $\uparrow$ orthogonal $n \times n$ matrix

Symmetric Matrices

- when the matrix and its transpose are equal
- eigenspaces are $\perp$ to one another
- matrix is orthogonally diagonalizable

Spectral Theorem $T: \mathbb{R}^n \to \mathbb{R}^n$, A is orthogonally diagonalizable IAOI A is symmetical

- every symmetric matrix is diagonalizable
- eigenspaces of symmetric matrix are orthogonal to each other.
- To generate orthonormal eigenbasis ...

  (1) find orthonormal basis for each eigenspace

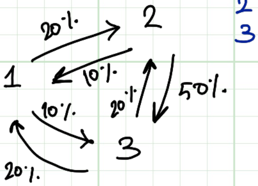
  (2) build their union (combine them).

Quiz

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 3 \\ 3 & 3 & 5 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 3 \\ 3 & 3 & 5 \end{bmatrix}$

Week 11: Expand

→ Eg 1. Pt. I



$$\begin{array}{c} pg.\# \\ 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} 0.7x_1 + 0.1x_2 + 0.2x_3 \\ 0.2x_1 + 0.4x_2 + 0.2x_3 \\ 0.1x_1 + 0.5x_2 + 0.6x_3 \end{bmatrix}$$

↑ distribution vector

$$A = \begin{array}{c|ccc} pg. & 1 & 2 & 3 \\ \hline & 0.7 & 0.1 & 0.2 \\ & 0.2 & 0.4 & 0.2 \\ & 0.1 & 0.5 & 0.6 \end{array}$$

↑

70% stay pg. 1

20% go pg. 2

10% go pg. 3

→ distribution vector:

- all components add up to 1.
- all components positive or 0.

→ Transition Matrix (stochastic)

- square matrix
- all columns distribution vectors

Problem

- to find state after 3000 transitions, we must do $A^{3000}x_0$ ... very hard

→ Eg 1. Pt. II

- diagonalize A to ease computations
- $\lambda_1 = 1 \longrightarrow E_1 = \text{span}\left(\begin{bmatrix} 7 \\ 5 \\ 8 \end{bmatrix}\right)$

$\lambda_2 = 0.5 \longrightarrow E_{0.5} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$

$\lambda_3 = 0.2 \longrightarrow E_{0.2} = \text{span}\left(\begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}\right)$

} eigenbasis ... construct S

$$S = \begin{bmatrix} 7 & 1 & -1 \\ 5 & 0 & -3 \\ 8 & -1 & 4 \end{bmatrix}$$
$$\quad \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$$

$$\therefore A = SDS^{-1} \rightarrow A = \begin{bmatrix} 7 & 1 & -1 \\ 5 & 0 & -3 \\ 8 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \begin{bmatrix} 7 & 1 & -1 \\ 5 & 0 & -3 \\ 8 & -1 & 4 \end{bmatrix}^{-1}$$

Easy to do $A^{3000} = (SDS^{-1})^{3000}$

$$= SD^{3000}S^{-1}$$

$$= S \begin{bmatrix} (1)^{3000} & 0 & 0 \\ 0 & (0.5)^{3000} & 0 \\ 0 & 0 & (0.2)^{3000} \end{bmatrix} S^{-1}$$

Eg 1 : Pt III Let $\vec{x}_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ … find $\lim\limits_{t\to\infty} A^t x_0$ ← # of transitions

- find close-formed expression for $A^t x_0$ … write $\vec{x}_0$ as lin. comb. of eigenbasis
- $$\left[\begin{array}{ccc|c} 7 & 1 & -1 & 1/3 \\ 5 & 0 & -3 & 1/3 \\ 8 & -1 & 4 & 1/3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/20 \\ 0 & 1 & 0 & -2/45 \\ 0 & 0 & 1 & -1/36 \end{array}\right]$$

$$\therefore \vec{x}_0 = \frac{1}{20}\vec{v}_1 - \frac{2}{45}\vec{v}_2 - \frac{1}{36}\vec{v}_3$$

Note

$A\vec{v}_i = \lambda_i \vec{v}_i$

$A^t\vec{v}_i = \lambda_i^t \vec{v}_i$

$$A^t\vec{x}_0 = \frac{1}{20}A^t\vec{v}_1 - \frac{2}{45}A^t\vec{v}_2 - \frac{1}{36}A^t\vec{v}_3$$

$$= \frac{1}{20}(1)^t\vec{v}_1 - \frac{2}{45}(0.5)^t\vec{v}_2 - \frac{1}{36}(0.2)^t\vec{v}_3$$

$$\lim_{t\to\infty} A^t\vec{x}_0 = \lim_{t\to\infty}\left[\frac{1}{20}(1)^t\vec{v}_1\right] - \lim_{t\to\infty}\left[\frac{2}{45}\underline{(0.5)}^t\vec{v}_2\right] - \lim_{t\to\infty}\left[\frac{1}{36}\underline{(0.2)}^t\vec{v}_3\right]$$

base is less than 0, so exponential decay as $t\to\infty$

$$= \lim_{t\to\infty} \frac{1}{20}\vec{v}_1$$

$$= \frac{1}{20}\vec{v}_1 = \begin{bmatrix} 35\% \\ 25\% \\ 49\% \end{bmatrix}$$ ← equilibrium distribution

- unique distribution vector that is eigenvector of A with $\lambda = 1$

↳ s.t. $A\vec{x} = \vec{x}$

$$\boxed{\lim_{t\to\infty} A^t x_0 = x_{equ}}$$ ← equilibrium distribution vector

Definition (Regular Transition Matrices)

A transition matrix is said to be **positive** if all its entries are positive (meaning that all the entries are greater than 0). A transition matrix A is said to be **regular (or eventually positive)** if the matrix A^m is positive for some positive integer m

Theorem (Equilibria for regular transition matrice). *Let A be an $n \times n$ regular transition matrix. Then*

(1) *There exists exactly one distribution vector $\vec{x}$ in $\mathbb{R}^n$ such that $A\vec{x} = \vec{x}$, meaning that $\vec{x}$ is the unique eigenvector with eigenvalue 1 which is also a distribution vector. This vector is called the equilibrium distribution for A, and is denoted $\vec{x}_{equ}$.*

(2) *If $\vec{x}_0$ is any initial distribution of $\lim_{t\to\infty} A^t\vec{x}_0 = \vec{x}_{equ}$.*