MAT188 Essential Concepts

Zaeem Ahmad

2. Systems of Linear Equations & their Solutions

- a collection of linear equations
- elementary row operations do not change general solution of system
- pivot column has a leading one ~ leading/basic/dependent variable
- non pivot column has free/independent variables

3. Unique/Infinete/No Solutions

- A is $n \times m$, given $A\vec{x} = \vec{b}$ and augmented matrix $[A|\vec{b}]$
  - ↳ system is consistent with unique solution if $\text{rank}(A) = \text{rank}(A|\vec{b}) = n$
  - ↳ system is consistent with infinite solution if $\text{rank}(A) = \text{rank}(A|\vec{b}) < n$
  - ↳ system is inconsistent if $\text{rank}(A) < \text{rank}(A|\vec{b})$

4. Matrix Vector Mutiplication

a) Linear combination of matrix columns

$$\begin{bmatrix} 4 & 8 & 12 \\ 5 & 9 & 13 \\ 6 & 10 & 14 \\ 7 & 11 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 9 \\ 10 \\ 11 \end{bmatrix} + x_3 \begin{bmatrix} 12 \\ 13 \\ 14 \\ 15 \end{bmatrix}$$

b) dot product vector with each row of matrix

$$\begin{bmatrix} - & \vec{v}_1 & - \\ - & \vec{v}_2 & - \\ - & \vec{v}_3 & - \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \cdot \vec{v}_1 \\ x_2 \cdot \vec{v}_2 \\ x_3 \cdot \vec{v}_3 \end{bmatrix}$$

6. Linear Transformation

$T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation represented by $T(\vec{x}) = A\vec{x}$ if

1) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
2) $T(k\vec{x}) = kT(\vec{x})$

7. Kernel & Image of Transformation

* Kernel of $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(\vec{x}) = A\vec{x}$
  ↳ $\ker(T) = \text{Null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\} = \{\vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = 0\}$
  "all the vectors in the domain that map to $\vec{0}$ in codomain"
  ↳ $\ker(T)$ is a subspace of the domain

* Image of $f: X \to Y$
  ↳ $\text{im}(f) = \{f(x) : x \in X\} = \{y \in Y \mid f(x) = y \text{ for some } x \in X\}$
  "all the outputs of the linear transformation"

  ↳ for a lin. trans. $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(\vec{x}) = A\vec{x}$

  $\text{im}(T) = \text{col}(A)$ ← linear comb of the column vectors of A
  ↳ $\text{im}(T)$ is a subspace of the codomain

8. Rank - Nullity

Given a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ and $T(\vec{x}) = A\vec{x}$
a) $\text{rank}(A)$ is $\dim(\text{im}(T))$ ~ # of pivot columns in $\text{rref}(A)$
b) $\text{nullity}(A)$ is $\dim(\ker(T))$ ~ # of non-pivot colums in $\text{rref}(A)$
→ $\dim(\text{dom}(T))$ ~ # of colums in matrix A (AKA "n")
→ $\dim(\text{codom}(T))$ ~ # of rows in matrix A (AKA "m")

<u>Rank Nullity Theorem:</u> $\text{Rank}(A) + \text{Nullity}(A) = n$
"dim of image and dim of kernel must add up to dim domain of T"

9. Injective / Surjective / Invertible ~ for $T: \mathbb{R}^n \to \mathbb{R}^m$ and $T(\vec{x}) = A\vec{x}$

a) T is injective exactly when $\ker(T) = \{\vec{0}\}$ ~ one-to-one
b) T is surjective exactly when $\text{im}(T) = \mathbb{R}^m$ ~ onto
* T is invertible exactly when T is bijective, which is exactly when T is both injective and surjective
  ↳ T invertible when <u>1) A is square</u> and <u>2) $\text{rref}(A) = I_n$</u>

10. Matrix - Matrix Multiplication

$T: \mathbb{R}^m \to \mathbb{R}^n \quad \ldots \quad T(\vec{x}) = A\vec{x}$

$S: \mathbb{R}^n \to \mathbb{R}^p \quad \ldots \quad S(\vec{x}) = B\vec{x}$

$S \circ T = S(T(\vec{x})) = BA$

$S \circ T : \mathbb{R}^m \to \mathbb{R}^p$

11. Determinants

* General notes on determinants
  - ↳ the absolute value of the determinant of A is the expansion factor, or ratio, by which T changes the area of any subset Ω in dom

$$\boxed{|\det A| = \frac{\text{area of } T(\Omega)}{\text{area of } \Omega}}$$

(+) sign ~ preserves orientation

(−) sign ~ reverses orientation

a) 2x2 matrices ... $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

↳ $\det(A) = ad - bc$

↳ A is invertible only if $\det(A) \neq 0$

b) 3x3 matrix ... $A = [\vec{v}_1 \ \ \vec{v}_2 \ \ \vec{v}_3]$

↳ $\det(A) = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)$

↳ A is invertible only if $\det(A) \neq 0$

Co-factor Expansion

cofactor expansion of A along the i'th row =

$$\boxed{\sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|}$$

* Some more random det notes
  - ↳ if A is square, then $\det A = \det A^T$ ~ transposing square matrix doesn't change determinant
  - ↳ if A and B both square, then...

$$\det(AB) = \det A \det B \quad \text{and} \quad \det(A^m) = (\det A)^m$$

  - ↳ inverses... $\det(A^{-1}) = \dfrac{1}{\det A}$
  - ↳ Elementary Row Operations ~ A is $n \times n$, B is result of row operation
    - dividing row by scalar k: $\det B = \frac{1}{k} \det A$
    - row swap A: $\det B = -\det A$
    - adding multiple of another row: $\det B = \det A$
  - ↳ determinants are linear in rows and columns

12. Subspaces

"a subspace of $\mathbb{R}^n$ is a copy of $\mathbb{R}^k$ for $k \leq n$ sitting inside $\mathbb{R}^n$"

Def: a subspace of $\mathbb{R}^n$ is a subset $W \subseteq \mathbb{R}^n$ s.t.

a) $\vec{0} \in W$

b) if $\vec{x}, \vec{y} \in W$, then also $\vec{x} + \vec{y} \in W$

c) if $\vec{x} \in W$, then also $k\vec{x} \in W$ for any scalar k

13. Linearly Dependent

- a set of vectors are linearly dependent if there is a linear relation among them
- "Put vectors into matrix and rref it. If there are non-pivot columns, then those corresponding vectors are redundant and can be expressed as a linear combination of the pivot column vectors"

14. Linearly Independent

- only if the trivial solution is the only linear relation.
- "rref the matrix gives you identity matrix"
- none of the vectors can be expressed as linear combination of the rest.

15. Span

- "span is the set of all linear combinations of a set of vectors"

$\mapsto$ $\text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m \mid c_1, c_2, \dots, c_m \in \mathbb{R}\}$

16. Spanning Set

- if $\text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m) = V$, then the set $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_m\}$ is spanning set for V (the vectors span V)

17. Basis

- a basis of a subspace V of $\mathbb{R}^n$ is a linearly independent set of vectors that span V. ~ Basis is the linearly independent spanning set
- $\dim(V)$ = # of vectors in Basis

17.5 Coordinates

let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ be basis for $\mathbb{R}^m$

$\therefore \vec{x} \in \mathbb{R}^m$, $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m$ and so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$

18. Change of Basis

$\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$

$C = \{\vec{c}_1, \dots, \vec{c}_n\}$

$$S_{\mathcal{B}\to C} = \begin{bmatrix} [\vec{b}_1]_C & \cdots & [\vec{b}_n]_C \end{bmatrix}$$

"old basis vectors expressed in new basis form columns of matrix"

change of coordinates matrix S s.t. $S[\vec{x}]_{\mathcal{B}} = [\vec{x}]_C$

19. Matrix of Transformation W.R.T. Basis

$$[T]_{\mathcal{E}} = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix}$$

$\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \dots \vec{e}_n\}$

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & | & & | \\ T(\vec{b}_1) & T(\vec{b}_2) & \cdots & T(\vec{b}_n) \\ | & | & & | \end{bmatrix}$$

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$

20. Similar Matrices

$$\begin{tikzcd} {[\vec{x}]_{\mathcal{E}}} \arrow[r, "A"] \arrow[d, "S^{-1}"'] & {[T(\vec{x})]_{\mathcal{E}}} \\ {[\vec{x}]_{\mathcal{B}}} \arrow[r, "B"] & {[T(\vec{x})]_{\mathcal{B}}} \arrow[u, "S_{\mathcal{B}\to\mathcal{E}}"'] \end{tikzcd}$$

since $S_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix} [b_1]_{\mathcal{E}} & \cdots & [b_n]_{\mathcal{E}} \end{bmatrix}$

$P_{\mathcal{E}\to\mathcal{B}} = S^{-1}$

$\therefore A = SBS^{-1}$

similar matrices

21. Orthogonal vectors

- set of vectors $\{\vec{u}_1 \dots \vec{u}_n\}$ are orthogonal if $\vec{u}_i \cdot \vec{u}_j = 0$ for $i \neq j$

perpendicular

22. Orthonormal Basis

- a basis for a subspace V that is orthonormal
- for $\mathcal{U} = \{\vec{u}_1, \dots, \vec{u}_n\}$, a) $\vec{u}_i \cdot \vec{u}_i = 1$ all vectors of magnitude 1

  b) $\vec{u}_i \cdot \vec{u}_j = 0$ for $i \neq j$ all vectors $\perp$

$$\boxed{[\vec{v}]_{\mathcal{U}} = \begin{bmatrix} \vec{v} \cdot \vec{u}_1 \\ \vdots \\ \vec{v} \cdot \vec{u}_n \end{bmatrix}}$$

23. Orthogonal Projection

- Let W be a subspace of $\mathbb{R}^n$ and $\vec{v}$ be any vector in $\mathbb{R}^n$
- orthogonal projection of $\vec{v}$ onto W is the unique vector $\vec{w}$ s.t. $\vec{v} - \vec{w} \in W^\perp$

$$\boxed{\text{proj}_W(\vec{v}) = (\vec{v}\cdot\vec{u}_1)\vec{u}_1 + (\vec{v}\cdot\vec{u}_2) + \cdots + (\vec{v}\cdot\vec{u}_n)}$$

→ <u>Orthogonal Complement:</u>

Let W be subspace of $\mathbb{R}^n$. Orthogonal complement of W is a subspace that contains all the vectors $\perp$ to all vectors in W.

$$W^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x}\cdot\vec{w} = 0 \text{ for all } \vec{w} \in W\}$$

- $\dim W + \dim W^\perp = n$ for W subspace of $\mathbb{R}^n$
- # of basis vectors in $W^\perp$ and W must add up to n.

23.1 Orthogonal Decomposition

Given any vector $\vec{x} \in \mathbb{R}^n$ and subspace V of $\mathbb{R}^n$

↳ $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$

$\vec{x}^{\parallel} \in V$ $\quad$ $\vec{x}^{\perp} \in V^\perp$

<u>More on Orthogonal Complements</u>

- Let $V \leq \mathbb{R}^n$ and $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_k\}$ be orthonormal basis for V
- orthogonal complement $V^\perp$ contains vector $\vec{y}$ s.t. $\vec{y}\cdot\vec{v} = 0$ for all $\vec{v}$ in V. "$\vec{y}$ is $\perp$ to all vectors in V"
- Let matrix $A = \begin{bmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_k \\ | & & | \end{bmatrix}$ so $A^T = \begin{bmatrix} - & \vec{b}_1 & - \\ & \vdots & \\ - & \vec{b}_k & - \end{bmatrix}$

we know that

$A^T\vec{y} = \vec{0}$

$$\begin{bmatrix} - \vec{b}_1 - \\ - \vec{b}_k - \end{bmatrix}\begin{bmatrix} | \\ \vec{y} \\ | \end{bmatrix} = \begin{bmatrix} \vec{b}_1\cdot\vec{y} \\ \vec{b}_k\cdot\vec{y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \vec{y} \in \text{Null}(A^T)$, so ...

$$\boxed{(\text{Col}(A))^\perp = \text{Null}(A^T)}$$

23.2 Gram Schmidt

- Used to find orthonormal basis for subspace V of $\mathbb{R}^n$

Given

$\mathcal{B} = \{\vec{b_1}, \cdots, \vec{b_m}\}$

basis for V and

$\dim(V) = m$

$\xrightarrow{\text{Gram Schmidt}}$

Output

$\mathcal{U} = \{\vec{u_1}, \cdots, \vec{u_m}\}$

orthonormal basis for V

step 1: a) normalize $\vec{b_1}$ ... $\vec{u}_1 = \dfrac{\vec{b_1}}{\|\vec{b}_1\|}$

step 2: a) find $\vec{b_2}^{\perp}$ ... $\vec{b_2}^{\perp} = \vec{b_2} - \vec{b_2}^{\parallel}$

$$\vec{b_2}^{\perp} = \vec{b_2} - \text{proj}_{\text{span}(\vec{u_1})}(\vec{b_2})$$

$$\vec{b_2}^{\perp} = \vec{b_2} - (\vec{b_2} \cdot \vec{u_1})\,\vec{u_1}$$

b) normalize $b_2^{\perp}$ ... $\vec{u_2} = \dfrac{\vec{b_2}^{\perp}}{\|\vec{b_2}\|}$

Repeat

24. QR Factorization

→ From above, since $\mathcal{U}$ is orthonormal basis for V, $\mathcal{B}$-basis vectors can be written as linear combination of $\mathcal{U}$-basis vectors.

→ QR factorization:

$$\begin{bmatrix} | & & | \\ \vec{b_1} & \cdots & \vec{b_m} \\ | & & | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ \vec{u_1} & \cdots & \vec{u_m} \\ | & & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} | & & | \\ [\vec{b_1}]_{\mathcal{U}} & \cdots & [\vec{b_m}]_{\mathcal{U}} \\ | & & | \end{bmatrix}}_{R}$$

R is also change of basis matrix $S_{\mathcal{U}\to\mathcal{B}}$

R ~ invertible, upper triangular matrix

25. Orthogonal Matrix

- matrix of an orthogonal linear transformation (one that preserves magnitudes of vectors) $\|T(\vec{x})\| = \|\vec{x}\|$
- columns form orthonormal basis for $\mathbb{R}^n$
- always square ... $n \times n$
- inverse is its transpose: $\boxed{A^{-1} = A^T}$ $\boxed{A^TA = I_n}$

25.1 Orthogonal Projections & Matrix Multiplication

Case #1: We have Orthonormal Basis

- let $\mathcal{U} = \{\vec{u}_1, \dots, \vec{u}_m\}$ be orthonormal basis for subspace V of $\mathbb{R}^n$, then

$$\boxed{\text{proj}_V(\vec{x}) = Q\,Q^T\vec{x}}$$

standard matrix for orthogonal projection onto V

Q: matrix with basis vectors as columns

Q^T: transpose of Q

Note: Q is not orthogonal matrix unless it is square

Case #2: We have a regular basis for V

- let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_m\}$ be a basis for subspace V of $\mathbb{R}^n$, then

$$\boxed{\text{proj}_V(\vec{x}) = A(A^TA)^{-1}A^T\vec{x}}$$

standard matrix for orthogonal projection onto V

A: matrix with basis vectors as columns

26. Least Squares Solution

Given $V \subseteq \mathbb{R}^n$ and $\vec{x} \in \mathbb{R}^n$, the closest vector to $\vec{x}$ on V is $\text{proj}_V(\vec{x})$

$$\boxed{\|\text{proj}_V(\vec{x}) - \vec{x}\| \leq \|\vec{v} - \vec{x}\| \text{ for any } \vec{v} \in V}$$

→ After finding matrix A and vector $\vec{b} \notin \text{Col}(A)$, the $\vec{x}^*$ that gets you closest to $\vec{b}$ is given by

$$\boxed{\vec{x}^* = (A^TA)^{-1}A^T\vec{b}}$$

27. Data Fitting Example ~ fit a line $y = ax + b$ through data pts

x	y
1	2
4	6
8	9

$(1,2)$: $2 = 1x + b$

$(4,6)$: $6 = 4x + b$

$(8,9)$: $9 = 8x + b$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 1 \\ 8 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ 6 \\ 9 \end{bmatrix}}_{\vec{b}}$$

Find $\vec{x}^* = \begin{bmatrix} a^* \\ b^* \end{bmatrix}$

$$\vec{x}^* = (A^TA)^{-1}A^Tb = \begin{bmatrix} 73/74 \\ 103/74 \end{bmatrix}$$

$$\therefore y = \frac{73}{74}x + \frac{103}{74}$$

30. Eigenvalues and Eigenvectors

- any nonzero vector $\vec{v} \in \mathbb{R}^n$ s.t. $T(\vec{v}) = \lambda \vec{v}$ for some scalar λ
- $\vec{v}$ is $\therefore$ an eigenvector with eigenvalue of λ.

Finding Eigenvalues

$A\vec{v} = \lambda \vec{v}$ ... λ is eigenvalue

$A\vec{v} - \lambda \vec{v} = 0$

$A\vec{v} - \lambda I_n \vec{v} = 0$

$\underbrace{(A - \lambda I_n)}_{\text{square matrix}} \vec{v} = 0$

λ is an eigenvalue for eigenvector $\vec{v}$ exactly when $\vec{v} \in \text{Null}(A - \lambda I_n)$

since kernel contains something other than $\vec{0}$ (since we're saying it also contains $\vec{v}$) the matrix is not invertible and det $= 0$

λ is eigenvalue of $n \times n$ A when

$$\boxed{\det(A - \lambda I_n) = 0}$$

- characteristic polynomial: $f_A(x) = \det(A - xI_n)$ ← n'th degree polynomial

λ-Eigenspace

$V_\lambda = \{\vec{v} \in \mathbb{R}^n \mid T(\vec{v}) = \lambda \vec{0}\}$ ← basically, find kernel of $(A - \lambda I_n)$

31. Eigenbasis

a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_3\}$ made of eigenvectors and spans all of $\mathbb{R}^n$

↳ $\mathcal{B}$-matrix of T will be $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

32. Diagonalization ... for $T: \mathbb{R}^n \to \mathbb{R}^n$ & $T(\vec{x}) = A\vec{x}$

- Matrix A is diagonalizable IAOI $A = SDS^{-1}$ for diagonal matrix D
- If we have "n" distinct eigenvalues
- Dimension of eigenspaces must add up to "n"

→ almu (algebraic multiplicity) ~ almu(λ) is # of times λ is factor of charpol.

→ gemu (geometric multiplicity) ~ gemu(λ) is dimension of λ-eigenspace

$$\text{gemu}(\lambda) \leq \text{almu}(\lambda)$$

Eigenvectors

$A\vec{v_1} = \lambda_1 \vec{v_1}$

$A\vec{v_2} = \lambda_2 \vec{v_2}$

$A\vec{v_3} = \lambda_3 \vec{v_3}$

$A = SDS^{-1}$

$$A = \begin{bmatrix} | & | & | \\ \vec{v_1} & \vec{v_2} & \vec{v_3} \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{v_1} & \vec{v_2} & \vec{v_3} \\ | & | & | \end{bmatrix}^{-1}$$

Is A diagonalizable?

Are there n distinct eigenvalues? $\xrightarrow{\text{Yes}}$ A is diagonalizable

No ↓

For every eigenvalue with almmu > 1, is gemu = almu? $\xrightarrow{\text{Yes}}$ A is diagonalizable

No ↓

A is not diagonalizable

32.1 Trace of 2x2 Matrix ~ sum of diagonal

Let $\chi_A(\lambda) = (-\lambda)^n + \underbrace{a_1}_{\text{trace}(A)}(\lambda)^{n-1} + \cdots + \underbrace{a_n}_{\det(A)}$ be char. poly. of $n \times n$ A

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det(A - \lambda I_2) = \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$

$0 = (a-\lambda)(d-\lambda) - bc$

$0 = ad - \lambda a - \lambda d + \lambda^2 - bc$

$0 = \lambda^2 + (-\lambda)(a+d) + ad - bc$

$0 = \lambda^2 + \underbrace{(a+d)}_{\text{trace}(A)}(-\lambda) + \underbrace{ad-bc}_{\det(A)}$

31. Orthogonally Diagonalizable ~ $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$

- if T has orthogonal eigenbasis
- $A = QDQ^{-1}$ ~ since Q has orthonormal columns, Q is an orthogonal matrix, so $Q^{-1} = Q^T$

  $A = QDQ^T$
- if Q^TAQ is diagonal, A is orthogonally diagonalizable

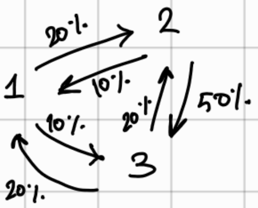
32. Spectral Theorem

$T: \mathbb{R}^n \to \mathbb{R}^n$, A is orthogonally diagonalizable IAOI A is symmetric

Symmetric Matrices

* when matrix and its transpose are equal
* eigenspaces are $\perp$ to one another
* matrix is orthogonally diagonalizable

33. Dynamic Systems



$$\begin{array}{c} \text{pg.\#} \\ 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} 0.7x_1 + 0.1x_2 + 0.2x_3 \\ 0.2x_1 + 0.4x_2 + 0.2x_3 \\ 0.1x_1 + 0.5x_2 + 0.6x_3 \end{bmatrix}$$

distribution vector

$$\text{pg.} \quad A = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.1 & 0.5 & 0.6 \end{bmatrix} \end{array}$$

70% stay pg. 1
20% go pg. 2
10% go pg. 3

distribution vector:

* all components add up to 1.
* all components positive or 0.

Transition Matrix (stochastic)

* square matrix
* all columns distribution vectors

Problem

* to find state after 3000 transitions, we must do $A^{3000} x_0$ ... very hard

* diagonalize A to ease computations
* $\lambda_1 = 1 \longrightarrow E_1 = \text{span}\left(\begin{bmatrix} 7 \\ 5 \\ 8 \end{bmatrix}\right)$

$\lambda_2 = 0.5 \longrightarrow E_{0.5} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$

$\lambda_3 = 0.2 \longrightarrow E_{0.2} = \text{span}\left(\begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}\right)$

eigenbasis ... construct S

$$S = \begin{bmatrix} 7 & 1 & -1 \\ 5 & 0 & -3 \\ 8 & -1 & 4 \end{bmatrix}$$
$$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$$

$\therefore A = SDS^{-1}$

$$A = \begin{bmatrix} 7 & 1 & -1 \\ 5 & 0 & -3 \\ 8 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \begin{bmatrix} 7 & 1 & -1 \\ 5 & 0 & -3 \\ 8 & -1 & 4 \end{bmatrix}^{-1}$$

try to do $A^{3000} = (SDS^{-1})^{3000}$

$$= SD^{3000}S^{-1}$$

$$= S \begin{bmatrix} (1)^{3000} & 0 & 0 \\ 0 & (0.5)^{3000} & 0 \\ 0 & 0 & (0.2)^{3000} \end{bmatrix} S^{-1}$$

Let $\vec{x}_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ ... find $\lim\limits_{t \to \infty} A^t x_0$ ← # of transitions

* find close-formed expression for $A^t x_0$ ... write $\vec{x}_0$ as lin. comb. of eigenbasis
* $\left[\begin{array}{ccc|c} 7 & 1 & -1 & 1/3 \\ 5 & 0 & -3 & 1/3 \\ 8 & -1 & 4 & 1/3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/20 \\ 0 & 1 & 0 & -2/45 \\ 0 & 0 & 1 & -1/36 \end{array}\right]$

$\therefore \vec{x}_0 = \frac{1}{20}\vec{v}_1 - \frac{2}{45}\vec{v}_2 - \frac{1}{36}\vec{v}_3$

Note

$A\vec{v}_i = \lambda_i \vec{v}_i$

$A^t \vec{v}_i = \lambda_i^t \vec{v}_i$

$$A^t \vec{x}_0 = \frac{1}{20} A^t \vec{v}_1 - \frac{2}{45} A^t \vec{v}_2 - \frac{1}{36} A^t \vec{v}_3$$

$$= \frac{1}{20}(1)^t \vec{v}_1 - \frac{2}{45}(0.5)^t \vec{v}_2 - \frac{1}{36}(0.2)^t \vec{v}_3$$

$$\lim_{t \to \infty} A^t \vec{x}_0 = \lim_{t \to \infty}\left[\frac{1}{20}(1)^t \vec{v}_1\right] - \lim_{t \to \infty}\left[\frac{2}{45}(0.5)^t \vec{v}_2\right] - \lim_{t \to \infty}\left[\frac{1}{36}(0.2)^t \vec{v}_3\right]$$

* unique distribution vector that is eigenvector of A with $\lambda = 1$
  ↳ s.t. $A\vec{x} = \vec{x}$

$$\boxed{\lim_{t \to \infty} A^t x_0 = x_{equ}} \leftarrow \text{equilibrium distribution vector}$$

$$= \lim_{t \to \infty} \frac{1}{20}\vec{v}_1$$

$$= \frac{1}{20}\vec{v}_1 = \begin{bmatrix} 35\% \\ 25\% \\ 49\% \end{bmatrix} \text{ equilibrium distribution}$$