MAT187 Main Concepts

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A1. Integration By Parts

- derived from product rule

$$\int u\,dv = uv - \int v\,du \implies \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

- what to pick for "u" ? — Logariths, Inverse trig, Algebraic, Trig, Exponential
  - → "u" easy to differentiate
  - → "dv" easy to integrate

A2. Trig sub

- identites : $\cos^2 x + \sin^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$
- build triangle when needed
- remember to change bounds and variable of integration
- watch out for domain restrictions when going from "x" to "θ"
- common trig subs

$$\int \sqrt{a^2 - x^2}\,dx \implies \text{sub } x = a\sin\theta \,,\; dx = a\cos\theta\,d\theta$$

$$\int \sqrt{a^2 + x^2}\,dx \implies \text{sub } x = a\tan\theta \,,\; dx = a\sec^2\theta\,d\theta$$

$$\int \sqrt{x^2 - a^2}\,dx \implies \text{sub } x = a\sec\theta \,,\; dx = a\sec\theta\tan\theta\,d\theta$$

A3. Partial Fractions

- rational if polynomial in numerator and denominator
- proper rational if deg(numerator) < deg (denominator)
  - ↳ must be proper to apply PFD
  - ↳ if not, use long division to make rational

→ Distinct Linear Factors

$$\frac{11x+64}{(x+5)(x+8)} = \frac{A}{x+5} + \frac{B}{x+8}$$

→ Distinct Quadratic Factors

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\frac{x^3-3x+8}{(x^2-1)^2(x^2+1)(x+\pi)^3(x^2+x+1)} = \frac{x^3-3x+8}{\underbrace{(x^2-1)}_{(x+1)(x-1)}\underbrace{(x^2-1)}_{(x+1)(x-1)}(x^2+1)(x+\pi)^3(x^2+x+1)}$$

$$= \frac{x^3-3x+8}{(x+1)^2(x-1)^2(x^2+1)(x+\pi)^3(x^2+x+1)}$$

9 terms

11 coefficients

$$= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{Ex+F}{x^2+1} + \frac{G}{x+\pi} + \frac{H}{(x+\pi)^2}$$

$$+ \frac{I}{(x+\pi)^3} + \frac{Jx+K}{x^2+x+1}$$

A4. Numeric Integration

$$\int_a^b f(x)dx \approx R_n = \sum_{k=1}^{n} f(x'_k)\Delta x \qquad \Delta x = \frac{b-a}{n}, \quad x'_k = x_0 + k\Delta x$$

$$\int_a^b f(x)dx \approx L_n = \sum_{k=1}^{n} f(x'_{k-1})\Delta x \qquad \Delta x = \frac{b-a}{n}, \quad x'_{k-1} = x_0 + (k-1)\Delta x$$

$$\int_a^b f(x)dx \approx M_n = \sum_{k=1}^{n} f(x'_k)\Delta x \qquad \Delta x = \frac{b-a}{n}, \quad x'_k = x_0 + \frac{(2k-1)\Delta x}{2}$$

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

$$\Delta x = \frac{b-a}{n}, \quad \text{also } T_n = \frac{1}{2}\left[L_n + R_n\right]$$

* Error = |exact value − approximation|

* Relative Error = $\dfrac{\text{Error}}{|\text{exact value}|}$

* For concave down functions, the midpoint rule is an overestimate and the trapezoidal rule is an under estimate.

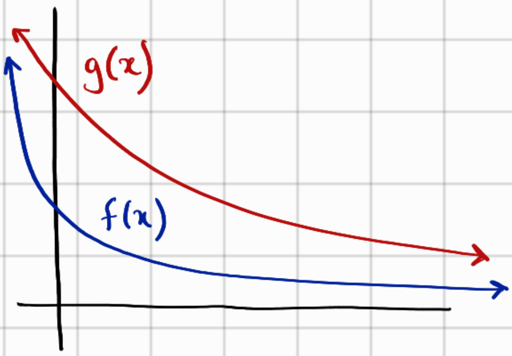
* Midpoint rule is more accurate than trapezoidal rule for concave down functions.

A5. Improper Integrals

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx$$

* improper integral converges if limit exists.
* improper integral diverges if limit doesn't exist.

* Comparison Theorem



$$0 \le \int_a^t f(x)\,dx \le \int_a^t g(x)\,dx$$

* if $\int_0^{\infty} g(x)\,dx$ converges, then $\int_0^{\infty} f(x)\,dx$ also converges

* if $\int_0^{\infty} f(x)\,dx$ diverges, then $\int_0^{\infty} g(x)\,dx$ also diverges

* Family of Comparision Functions

$$\int_1^{\infty} \frac{1}{x^p}\,dx \text{ is } \begin{cases} \text{convergent if } p>1 \\ \text{divergent if } p\le 1 \end{cases}$$

$$\int_0^{\infty} e^{-ax}\,dx \text{ is } \begin{cases} \text{convergent if } a>0 \\ \text{divergent if } a\le 0 \end{cases}$$

B2. Separable ODEs

- ODE can be written as product of 2 functions

$$\frac{dy}{dt} = f(t)\,g(y)$$

$$\frac{1}{g(y)}\,dy = f(t)\,dt \implies \int \frac{1}{g(y)}\,dy = \int f(t)\,dt$$

B3. Linear First Order ODEs

- standard form: $y' + p(t)y = q(t)$
- integrating factor $\mu(t) = e^{\int p(t)dt}$ that lets you use product rule after multiplying both sides of ODE with $\mu(t)$

$$y' - 2y = 4 - t \qquad \mu(t) = e^{-\int 2dt} = e^{-2t}$$

$$e^{-2t}(y' - 2y) = (4-t)e^{-2t}$$

$$\int (e^{-2t}y)'\,dt = \int (4-t)e^{-2t}\,dt$$

$$e^{-2t}y = -\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t} + C$$

$$\to\ y = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t}$$

B4. Modelling with ODEs

$$\frac{dQ}{dt} = \text{rate at which substance enters} - \text{rate at which substance leaves}$$

$$\text{concentration} = \frac{Q(t)}{V(t)}$$

solve ODE with given initial values

C1.2 Complex Numbers

$z = a + bi$ $\quad a = r\cos\theta$, $b = r\sin\theta$ $\quad \theta = \arctan\left(\frac{b}{a}\right)$ for $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$z = re^{i\theta}$ $\quad r = \sqrt{a^2 + b^2}$

Euler Identities

$$e^{\pi i} = -1 \qquad e^{2\pi i} = 1$$

D1. Homogenous Equations

- Given ODE : $ay'' + by' + cy = 0$, you let $y = e^{rt}$

$y' = re^{rt}$

$y'' = r^2 e^{rt}$

$\Rightarrow ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$

char. poly. $\longrightarrow ar^2 + br + c = 0$ $\quad e^{rt} > 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- General Solutions

↳ $\Delta > 0$: r_1, r_2 unique : $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ $\quad C_1, C_2 \in \mathbb{R}$

↳ $\Delta = 0$: $r_1 = r_2 = r$: $y = C_1 e^{rt} + C_2 t e^{rt}$ $\quad C_1, C_2 \in \mathbb{R}$

↳ $\Delta < 0$:

roots $\alpha \pm \beta i \longrightarrow y = e^{\alpha t}\left[C_1 \cos\beta t + C_2 \sin\beta t\right]$ $\quad C_1, C_2 \in \mathbb{C}$

D2. Nonhomogenous Equations

- Given ODE : $ay'' + by' + cy = f(t)$

↳ complementary solution $y_c(t)$ solves homogeneous equation

↳ particular solution $y_p(t)$ solves nonhomogeneous equation

↳ general solution $y(t) = y_c(t) + y_p(t)$

$r(x)$	**Initial guess for $y_p(x)$**
k (a constant)	A (a constant)
$ax + b$	$Ax + B$ (*Note*: The guess must include both terms even if $b = 0$.)
$ax^2 + bx + c$	$Ax^2 + Bx + C$ (*Note*: The guess must include all three terms even if b or c are zero.)
Higher-order polynomials	Polynomial of the same order as $r(x)$
$ae^{\lambda x}$	$Ae^{\lambda x}$
$a\cos\beta x + b\sin\beta x$	$A\cos\beta x + B\sin\beta x$ (*Note*: The guess must include both terms even if either $a = 0$ or $b = 0$.)
$ae^{\alpha x}\cos\beta x + be^{\alpha x}\sin\beta x$	$Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$
$(ax^2 + bx + c)\,e^{\lambda x}$	$(Ax^2 + Bx + C)\,e^{\lambda x}$
$(a_2x^2 + a_1x + a_0)\cos\beta x + (b_2x^2 + b_1x + b_0)\sin\beta x$	$(A_2x^2 + A_1x + A_0)\cos\beta x + (B_2x^2 + B_1x + B_0)\sin\beta x$
$(a_2x^2 + a_1x + a_0)\,e^{\alpha x}\cos\beta x + (b_2x^2 + b_1x + b_0)\,e^{\alpha x}\sin\beta x$	$(A_2x^2 + A_1x + A_0)\,e^{\alpha x}\cos\beta x + (B_2x^2 + B_1x + B_0)\,e^{\alpha x}\sin\beta x$

Table 7.2 Key Forms for the Method of Undetermined Coefficients

E1.2.3 Series, Sequences, Convergence

- infinite sequence : ordered list of numbers
- infinite series : $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

Convergence/Divergence Tests

1) Divergence Test : given $\sum_{n=0}^{\infty} a_n$ and $\lim_{n\to\infty} a_n \neq 0$ then series diverges

2) Ratio Test : given $\sum_{n=0}^{\infty} a_n$

$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ and
- if $\rho > 1$ — diverges
- if $\rho < 1$ — converges
- if $\rho = 1$ — we don't know

3) P-series Test : given $\sum_{n=0}^{\infty} \frac{1}{n^p}$

if $p \leq 1$ — diverges and if $p > 1$ — converges

4) Alternating Series Test: given $\sum_{n=0}^{\infty} (-1)^n b_n$

if $b_n > 0$ and b_n's are non increasing and $\lim_{n\to\infty} b_n = 0$ — converges

5) Geometric Series Test: if $|x| < 1$, then

$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots = \frac{1}{1-r}$ but if $|x| \geq 1$ — diverges

E4. Taylor Series

- We approximate nasty functions with Taylor Polynomials
- remainder $R_n(x) = f(x) - P_n(x) \Longrightarrow f(x) = p_n(x) + R_n(x)$
- $f(x)$ equals the Taylor approx plus the remainder
- Error $= |R_n(x)| = |f(x) - P_n(x)|$

- To use the pre-built Maclaurin series, get your function to look like the Maclaurin series and plug in your "x" used to get there. Don't forget to do the same for the interval of convergence

F1. Parametric Equations

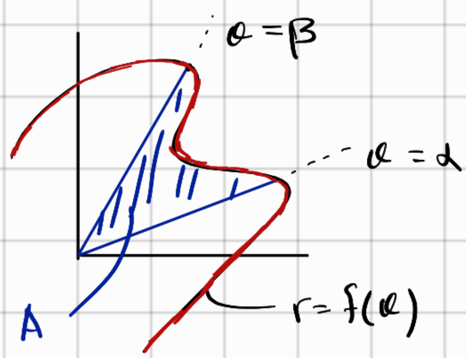
* Given continuous functions $f(t)$ and $g(t)$ of parameter t, then

$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned}\right\} \text{parametric equations}$$

$$\longrightarrow (f(t), g(t)) \longleftarrow \text{points on curve}$$

* derivative: $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{y'(t)}{x'(t)}$

F2.3.4 Calculus in Polar Coordinates / Vector-Valued Functions



$$\boxed{A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta}$$

Limits

$$\lim_{t\to\infty} \vec{r}(t) = \left< \lim_{t\to\infty} f(t), \lim_{t\to\infty} g(t), \lim_{t\to\infty} h(t) \right>$$

Derivatives

$$\vec{r}\,'(t) = \left< f'(t), g'(t), h'(t) \right>$$

Indefinite Integral

$$\int \vec{r}(t)\,dt = \left< \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right>$$

Definite Integral

$$\int_a^b \vec{r}(t)\,dt = \left< \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right>$$

F5. Arc Length

* Given function $< f(t), g(t), h(t) >$, the arc length L is:

$$L = \int_{t_1}^{t_2} \|\vec{r}\,'(t)\|\, dt = \int_{t_1}^{t_2} \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

* Arc length as function of time:

$$s(t) = \int_{t_0}^{t} \|\vec{r}\,'(t')\|\, dt'$$

* We say a curve $\vec{r}$ is parametrized w.r.t arc length if $\|\vec{r}\,'(t)\| = 1$ at all time t.

* Inverse parametrization: changing a function of time to a function of distance

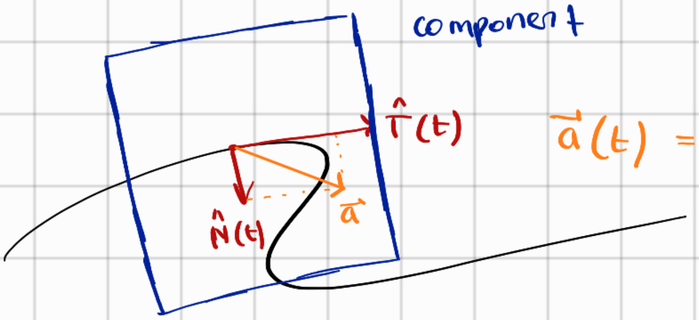
F6.7 Normal / Tangential Vectors and Motion

* the curvature "k" a circle radius r is defined as

$$\boxed{k = \frac{1}{r}} \qquad \text{curvature at time } t: \quad \boxed{k(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|}}$$

* when $\vec{r}(s)$ is in arclength parametization, $k(t) = \left\| \frac{d\vec{T}}{ds} \right\|$

* acceleration: $\vec{a}(t) = \underbrace{a_T \hat{T}(t)}_{\text{tangential component}} + \underbrace{a_N \hat{N}(t)}_{\text{normal component}}$



$\vec{a}(t) = \cup\, \hat{T}(t) + \cup\, \hat{N}(t)$