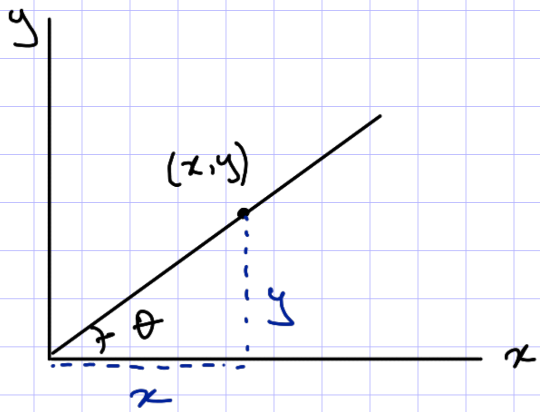
A1. Trigonometric Functions

sine and cosine ⟶ radians ⟶ graphs ⟶ identities ⟶ solving equations

Sine and Cosine

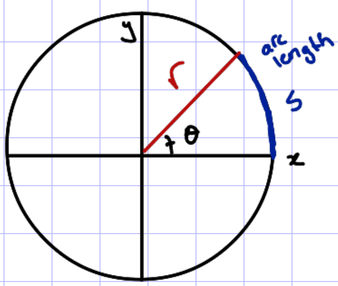


$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

Radian Measure

" θ has radian measure s "

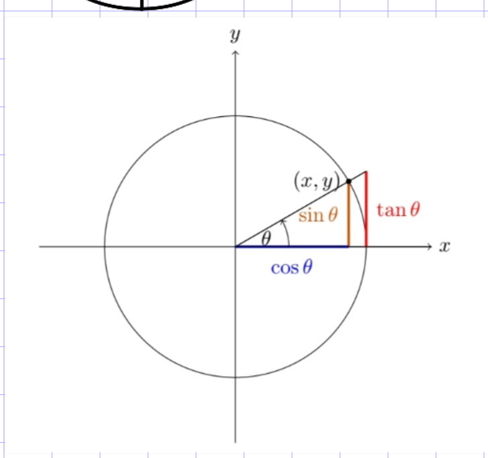


$$\text{rad} = \frac{s}{r}$$

Conversions

$$\text{rad} = \frac{\pi}{180}\ \text{degrees}$$

$$\text{degrees} = \frac{180}{\pi}\ \text{rad}$$



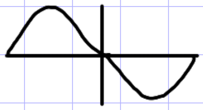
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Even and Odd

- even ~ $f(-x) = f(x)$ ~ symetric about y-axis
- odd ~ $f(-x) = -f(x)$ ~ symmetric about origin

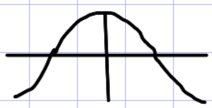
Activity

$\sin\theta \to$



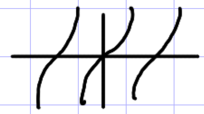
"odd"

$\cos\theta \to$



"even"

$\tan\theta \to$



"odd"

Trigonometric Identities

$$\boxed{\sin^2 x + \cos^2 x = 1}$$

$$\boxed{\tan x = \frac{\sin x}{\cos x}}$$

$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$$

$$\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta}$$

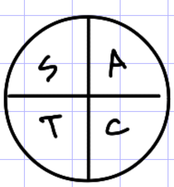
$$\boxed{\csc\theta = \frac{1}{\sin\theta}} \quad \boxed{\sec\theta = \frac{1}{\cos\theta}}$$

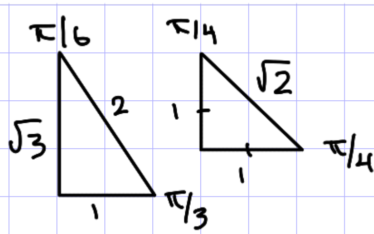
$$\boxed{\cot\theta = \frac{1}{\tan\theta}}$$

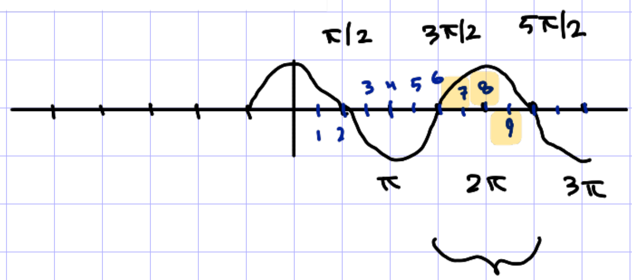
$$\boxed{\begin{aligned} \sin(x \pm y) &= \sin x\cos y \pm \cos x \sin y \\ \cos(x \pm y) &= \cos x \cos y \pm \sin x \sin y \end{aligned}}$$

Example 1

Example 1: Find all θ satisfying $\cos\theta = \frac{\sqrt{2}}{2}$ and $\frac{3\pi}{2} < \theta < \frac{5\pi}{2}$.







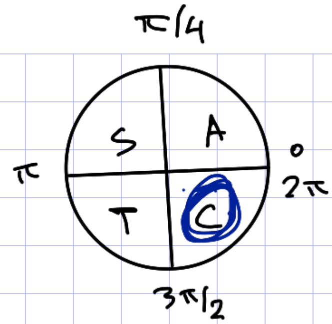
$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$\frac{\pi}{4}$ in $\frac{3\pi}{2} < \theta < \frac{5\pi}{2}$

$$\therefore \theta = \left\{ \frac{7\pi}{2}, \frac{9\pi}{2} \right\}$$

Example 2

Example 2: Given $\cos\theta = \frac{5}{6}$, $\sin\theta < 0$, and $0 < \theta < 2\pi$, find $\sin\theta$ and $\sin\frac{\theta}{2}$.



$$\cos\theta = \frac{5}{6}$$

$$\Rightarrow \cos^2\theta = \frac{25}{36}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin\theta = \pm\sqrt{1-\cos^2\theta}$$

$$= \pm\sqrt{1-\frac{25}{36}}$$

$$= \pm\frac{\sqrt{11}}{6}$$

$$\sin\theta = \frac{\sqrt{11}}{6}$$

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\frac{5}{6}}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{\frac{1}{6}}{2}}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1}{12}}$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{1}{12}}$$

$$\frac{\frac{3\pi}{2}}{2} < \frac{\theta}{2} < \frac{2\pi}{2}$$

$$\Downarrow$$

$$\frac{3\pi}{4} < \frac{\theta}{2} < \pi$$

so $\frac{\theta}{2} > 0$ ...

$$\sin\frac{\theta}{2} = +\sqrt{\frac{1}{12}}$$

Example 3

Example 3: Find all solutions to $8\sin^2\theta\cos^2\theta = 1$ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$8\sin^2\theta\cos^2\theta = 1$$

$$\sin 2\theta\,(4\sin\theta\cos\theta) = 1$$

$$(\sin 2\theta)(\sin 2\theta + 2) = 1$$

recall $\sin 2\theta = 2\sin\theta\cos\theta$

Suggested Textbook Problems

127. $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4} \approx 0.2588 \approx \frac{\sqrt{3}-1}{2\sqrt{2}}$

★ 145. $= \frac{\cos t}{\sin t} + \frac{\sin t}{1+\cos t}$

$$= \frac{\cos t}{\sin t\,\cos t} + \frac{\sin t}{(1+\cos t)\cos t}$$

$$= \frac{1}{\sin t} + \frac{\tan t}{1+\cos t}$$

$$= \sin t + \frac{1+\cos t}{\tan t}$$

A2. Inverse Functions

Sep 12, 2023

- → notion of inverse function (if a function has one or not)
- → algabraic and graphical representation
- → roles of domain restrictions
- → evaluate expressions with inverse trig functions

Inverse Functions

- Let $f(x)$ be a function defined for all x on an interval
- If there is only one value x in the function's range such that $f(x) = y$, then the inverse function is...

$f^{-1}(y)$ is the unique x in the interval I such that $y = f(x)$

Eg: $f(x) = x^2$

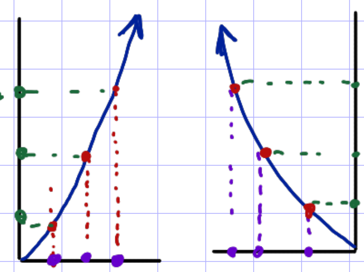
1 for interval $I = [0, \infty)$

→ function f has inverse

2 for interval $I = (-\infty, 0]$

→ function has inverse

for each value y in function's range, there is only on value x for which $f(x) = y$



for each y in the function's range ... there is only one value x for which $f(x) = y$

... in other words,

$$f(x) = x^2$$
$$y = x^2$$
$$x = \sqrt{y}$$

... so in other words

$$f(x) = x^2$$
$$-(y) = x^2$$
$$x = -\sqrt{y}$$

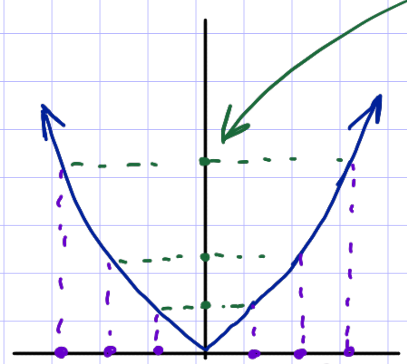
only defined if $y \geq 0$, just like range of original funtion

③ on interval $I = (-\infty, \infty)$

→ function does not have inverse

each y value in the function's range ...

... these are 2 values of x for which $f(x) = y$



- not every function has an inverse on its entire domain

Observations

- $y = f(x)$ if and only if $x = f^{-1}(y)$
- f^{-1} is the inverse of f, f is the inverse of f^{-1} } ... in other words: $(f^{-1})^{-1} = f$

  "inverse of inverse fuction is original function"

- $f^{-1}(f(x)) = x$ ← for all x in domain f

  $f(f^{-1}(y)) = y$ ← for all y in domain f^{-1}

- (a,b) is a point on f if and only if (b,a) is on f^{-1}.
- reflect f on $y=x$ to get f^{-1}

How do we know if function has inverse?

↳ function has inverse for interval where the function is increasing or decreasing

... since $f(x_1) \neq f(x_2)$ unless $x_1 = x_2$.

eg: $\sin x$

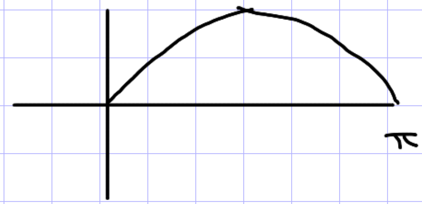
✓ ↳ inc. $[-\frac{\pi}{2}, \frac{\pi}{2}]$

✓ ↳ dec. $[\frac{\pi}{2}, \frac{3\pi}{2}]$

✗ ↳ neighter $[0, \pi]$

Inverse Trig. Functions

1. $\sin^{-1} x$ is inverse of $\sin x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$
2. $\cos^{-1} x$ is inverse of $\cos x$ on $[0, \pi]$
3. $\tan^{-1} x$ is inverse of $\tan x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$



Eg 1: Determine $\sin^{-1}\left(\sin \frac{5\pi}{6}\right)$

$= \sin^{-1}\left(\sin \frac{5\pi}{6}\right)$ ... $\sin \frac{5\pi}{6} = \frac{1}{2}$

$= \sin^{-1}\left(\frac{1}{2}\right)$ ... $\sin^{-1}\left(\frac{1}{2}\right) = \theta$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

so $\theta = \frac{\pi}{6}$ and $\therefore$ $\sin^{-1}\left(\sin \frac{5\pi}{6}\right) = \frac{\pi}{6}$

Eg 2: Determine $\tan(\sec^{-1} x)$ in terms of x, and $x < 0$

① Let $\sec^{-1} x = \theta$ ... so $\sec \theta = x$ and $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

.. since $x < 0$, $\theta \in \cup(\frac{\pi}{2}, \pi]$ bc $\sec \theta$ is $(-)$ in this interval

② Since $\sec \theta = x$, $\frac{1}{\cos \theta} = x$, so $\cos \theta = \frac{1}{x}$

... square both side

$\cos^2 \theta = \frac{1^2}{x^2}$

$\cos^2 \theta = \frac{1}{x^2}$

③ $\sin^2 \theta + \cos^2 \theta = 1$

$\sin^2 \theta = 1 - \cos^2 \theta$

$\sin^2 \theta = 1 - \frac{1}{x^2}$

$\sin \theta = \pm\sqrt{1 - \frac{1}{x^2}}$

$\sin \theta = +\sqrt{1 - \frac{1}{x^2}}$ ... in $\theta \in [\frac{\pi}{2}, \pi]$, $\sin \theta$ is $(+)$, so we choose $(+)$ root

④ $= \tan(\sec^{-1} x)$

$= \tan \theta$

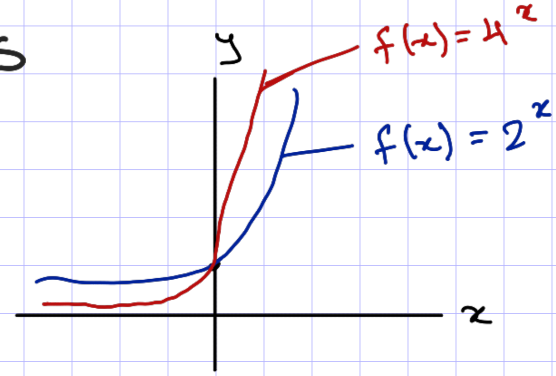
$= x\sqrt{1 - \frac{1}{x^2}}$

Sep 14, 23

A3. Expontentials & Logs.

→ solve exponential and log equations
→ Domain, range of exp. and log.
→ sketch graphs

Exponential Functions



$f(x) = a^x$ ← exponent

base

· higher base → increase sharper and drops closer to x-axis $(y=0)$

Change is Proportional

Δ value over 1 unit: $f(x+1) - f(x)$, if $f(x) = a^x$ ...

$$= f(x+1) - f(x)$$
$$= a^{x+1} - a^x$$
$$= a^x a^1 - a^x$$
$$= a^x(a^1 - 1)$$
$$= (a^1 - 1) f(x)$$

change over one unit is proportional to $f(x)$

$$f(x+1) - f(x) \propto f(x)$$

Euler's Number

compound interest: $A(t) = P\left(1 + \frac{r}{n}\right)^{tn}$

continuous compound interest: $A(t) = Pe^{rt}$

Logarithms

- for exp. function : a^x , inverse is $\log_a(a^x) = a^{\log_a(x)} = x$
- natural logarithm : $\log_e(x)$, so $\ln(e^x) = e^{\ln x} = x$

Eg: simplify ~ $\log_5(500) - 2\log_5(2) + \log_4(32) + \log_4(8)$

$$= \underbrace{\log_5(500) - 2\log_5(2)} + \underbrace{\log_4(32) + \log_4(8)}$$

$$= \log_5(500) - \log_5(2^2)$$
$$= \log_5(500) - \log_5(4)$$
$$= \log_5\left(\frac{500}{4}\right)$$
$$= \log_5(125)$$
$$= \log_5(5^3)$$
$$= 3$$

$$= \log_4(32) + \log_4(8)$$
$$= \log_4(32 \cdot 8)$$
$$= \log_4(256)$$
$$= \log_4(4^4)$$
$$= 4\log_4(4)$$
$$= 4$$

$$= 3 + 4 = 7$$

Sep 16,2

A4. Estimations, Absolute Val., Inequalit.

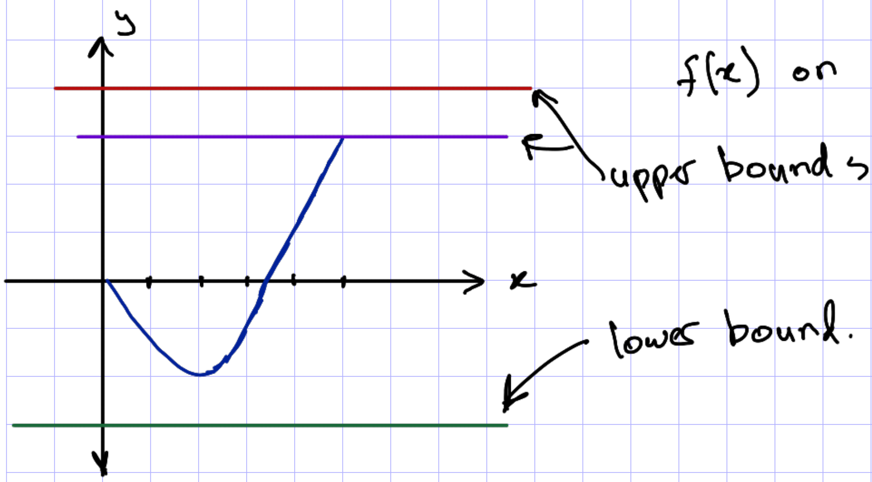
Safe Side Strategy

1. Overestimate → being too big (filling up gas for trip)
2. Underestimate → being too small (sign declaring max weight for a bridge)

Definition: Upper Bound for a function $f(x)$ on an interval I is ...
a number U such that $f(x) \leq U$ for all $x \in I$

Definition: Lower Bound for a function $f(x)$ on an interval I is ...
a number L such that $f(x) \geq L$ for all $x \in I$

$f(x)$ on interval $[0, 5]$



- if function has one upper/lower bound, then it has infinite upper/lower bounds.

Eg: You wanna attatch bulb that can handle current given by $f(t) = e^{\cos t + \sin t}$

→ upper bound for $f(t)$?

① $e < 3$, and sin/cos max is 1, so let $e = 3$ and $\cos t + \sin t = 2$

$= \cos t + \sin t$
$= 1 + 1$
$= 2$

$\therefore$ $3^2 = 9$ ← Now you can find bulb that supports 9 Amps safely

Absolute Value and Inequality Properties

→ $|ab| = |a||b|$ product rule

→ $|a+b| \leq |a| + |b|$ triangle inequality

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

* the distance that a value is from the number line
* always returns positive value

Solving Absolute Value Functions

$8 = |2x-6|$ the absolute value will $=8$ when the quantity inside is plus or minus 8 : ± 8

$8 = 2x-6$ $\quad$ $-8 = 2x-6$

$x = 7$ $\quad$ $x = -1$

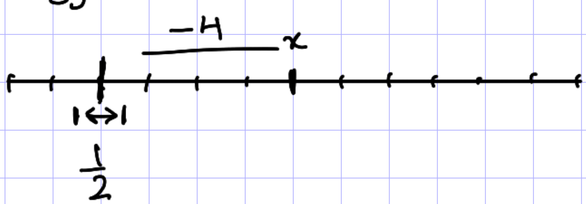
Eg: solve $0 = |4x+1| - 7$

$4x+1 = 7$ $\quad$ $4x+1 = -7$

$x = \frac{6}{4}$ $\quad$ $x = -2$

Solving Absolute Inequality

Suggested Problems: 5, 7, 8, 9, 13, 15, 17, 57, 59, 61, 63.

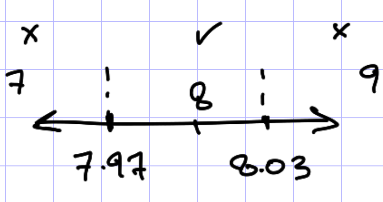
7. 

$|x-4| = \frac{1}{2}$

9. $|x-8| < 0.03$

$x-8 = 0.03$ $\quad$ $x-8 = -0.03$

$x = 8.03$ $\quad$ $x = 7.97$



$7.97 < x < 8.03$

$|f(x) - 8| < 0.03$

61. $1.5\% = |p - 8\%| \Rightarrow |p - 0.08| \leq 0.015$

63. $|x - 5| \leq 0.01$

Eg: find upper bound for $f(x) = |\cos(x) - 2\sqrt{10-x}|$ on the interval $[2, 7]$

$= |\cos x - 2\sqrt{10-x}|$

$\leq |\cos x| + |-2\sqrt{10-x}|$ triangle inequality

$= |\cos x| + |-2||\sqrt{10-x}|$ product rule

highest value for $\cos x$ is 1

$= \cos x \leq 1$

$|-2| = 2$

highest value is 2

$|\sqrt{10-x}|$

$|\sqrt{10-2}|$

$= \sqrt{8}$

$= 2\sqrt{2}$

from interval $[2, 7]$, which x gives the highest value?

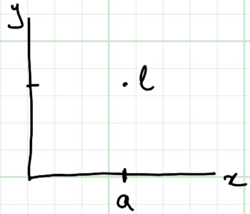
$x = 2$

$= 1 + 2(2\sqrt{2})$

$= 1 + 4\sqrt{2}$

reasonable upper bound for $f(x)$

Limits



given $f(x)$ and fixed point a

* f approaches the limit ℓ near a

$$\boxed{\lim_{x \to a} f(x) = \ell}$$

* f doesn't need to be defined at a to have a limit.

$$\left.\lim_{x \to a^+} f(x) = \ell \ \right\} \text{ right handed limit}$$

$$\left.\lim_{x \to a^-} f(x) = \ell \ \right\} \text{ left handed limit}$$

* limit exists only if left and right handed limits are equal

$$\lim_{x \to a} f(x) = \ell \quad \underline{\text{only}} \text{ if } \quad \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

Solving Limit Algebraicly

$$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \to 9} \frac{\sqrt{x} - 3}{(\sqrt{x} - 3)(\sqrt{x} + 3)}$$

$$= \lim_{x \to 9} \frac{1}{\sqrt{x} + 3} \text{, since } x \neq 9$$

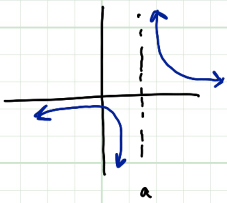
$$= \frac{1}{\sqrt{9} + 3}$$

$$= \frac{1}{3+3} = \frac{1}{6}$$

$$\therefore \lim_{x \to 9} f(x) = \frac{1}{6}$$

Limits Toward V.A.

$\lim_{x \to a^+} \frac{1}{x-a}$ D.N.E rather $\lim_{x \to a^+} \frac{1}{x-a} \to +\infty$



Infinite Limits

for any positive real numbers n, $\lim_{x \to \infty} \frac{1}{x^n} = 0$

Theorem 1

Eg 1: $\lim_{x \to +\infty} \frac{x^2 - 3x + 2}{x^2 + 1} = \lim_{x \to +\infty} \frac{\cancel{x^2}\left(1 - \frac{3}{x} + \frac{2}{x^2}\right)}{\cancel{x^2}\left(1 + \frac{1}{x^2}\right)}$

$$= \lim_{x \to +\infty} \frac{1 - \frac{3}{x} + \frac{2}{x^2}}{1 + \frac{1}{x^2}}$$

when evaluating limits for these,

$\lim_{x \to +\infty} = 0$

$$= \lim_{x \to +\infty} \frac{1}{1} = 1$$

Limit Laws

Quotient Law: $\lim_{x \to a} f(x) = L$

$\lim_{x \to a} g(x) = M$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$

Eg 2: $$\lim_{x \to -\infty} \frac{9x^3 + 5x + 7}{x^2 - 14x + 2} = \lim_{x \to -\infty} \frac{x^3\left(9 + \frac{5}{x^2} + \frac{7}{x^3}\right)}{x^2\left(1 - \frac{14}{x} + \frac{2}{x^2}\right)}$$

$$= \lim_{x \to -\infty} \frac{x\left(9 + \frac{5}{x^2} + \frac{7}{x^3}\right)}{1 - \frac{14}{x} + \frac{2}{x^2}}$$

$f(x) = x$

$$g(x) = \frac{9 + \frac{5}{x^2} + \frac{7}{x^3}}{1 - \frac{14}{x} + \frac{2}{x^2}}$$

$\lim_{x \to -\infty} f(x) \to -\infty$

$\lim_{x \to -\infty} g(x) = \frac{9}{1} = 9$

different, so $\lim_{x \to -\infty} \frac{9x^3 + 5x + 7}{x^2 - 14x + 2}$ D.N.E

$\lim_{x \to -\infty} f(x)\, g(x)$

$\lim_{x \to -\infty} g(x) = 0$

$\lim_{x \to -\infty} f(x) \to \infty$

indeterminant form

Eg 3: $\lim\limits_{x\to\infty} \dfrac{x^2+7x-1}{x^{\frac52}-17x^2+x+1} = \lim\limits_{x\to\infty} \dfrac{x^2\left(1+\frac{7}{x}-\frac{1}{x^2}\right)}{x^{\frac52}\left(1-\frac{17}{x^{1/2}}+\frac{1}{x^{3/2}}+\frac{1}{x^{5/2}}\right)}$

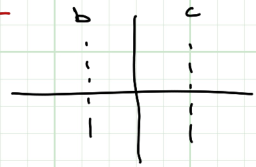
$$= \lim_{x\to\infty} \frac{1+\frac{7}{x}-\frac{1}{x^2}}{x^{\frac12}\left(1-\frac{17}{x^{1/2}}+\frac{1}{x^{3/2}}+\frac{1}{x^{5/2}}\right)}$$

$$f(x) = \frac{1}{x^{1/2}}$$

$$\lim_{x\to\infty} f(x) = 0$$

$$g(x) = \frac{1+\frac{7}{x}-\frac{1}{x^2}}{1-\frac{17}{x^{\frac12}}+\frac{1}{x^{\frac32}}+\frac{1}{x^{\frac52}}}$$

$$\lim_{x\to\infty} g(x) = \frac{1}{1} = 1$$



$$\therefore \lim_{x\to\infty} f(x)\,g(x) = 1\times 0 = 0$$

Quiz

$a < b < c$
$2 \quad 3 \quad 4$

$$f(x) = \frac{(x-a)^2}{(x-b)(x-c)} = \frac{(x-2)^2}{(x-3)(x-4)}$$

Q.1 $\lim\limits_{x\to b^-} \dfrac{(x-2)^2}{(x-3)(x-4)}$

$$= \frac{x^2-2xa+a^2}{x^2-cx-bx+cb}$$

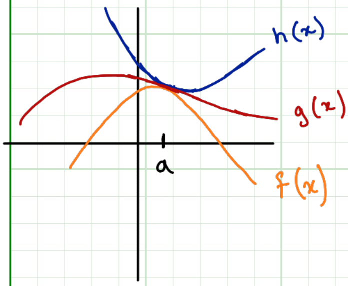
$$= \frac{\cancel{x^2}\left(1-\frac{2a}{x}+\frac{a^2}{x^2}\right)}{\cancel{x^2}\left(1-\frac{c}{x}-\frac{b}{x}+\frac{cb}{x}\right)}$$

$$= \frac{1-\frac{2a}{x}+\frac{a^2}{x}}{1-\frac{c}{x}-\frac{b}{x}+\frac{cb}{x}}$$

$$\frac{1-\frac{2a}{b}+\frac{a^2}{b}}{1-\frac{c}{b}-\frac{\cancel{b}}{\cancel{b}}+\frac{c\cancel{b}}{\cancel{b}}}$$

Squeeze Theorem

↳ let's us calculate limits of trig functions by "squeezing" a function between 2 others.



when $f(x) < g(x) < h(x)$

and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ ...

the $\lim_{x \to a} g(x) = L$

Continuity

Given $f(x)$ and input a, f is continuous at a if:

$\lim_{x \to a} f(x)$ exists and is equal to a ... in other words

f is continuous if:

1) $\lim_{x \to a^+} f(x)$ exists

2) $\lim_{x \to a^-} f(x)$ exists

3) $f(a)$ is defined

4) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

* continuous functions have no holes / gaps.

Intermediate Value Theorem

- Let f be continuous over interval $[a,b]$
- z is any real number between $f(a)$ and $f(b)$

↳ then there is a number $c \in [a,b]$ such that

$$f(c) = z$$

Quiz Time

$f(x)$ continuous at 0

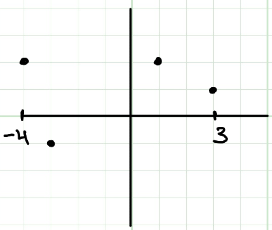
1 $$f(x) = \begin{cases} e^x \sin(2\pi x), & x < 0 \\ x^2 - cx, & x \geq 0 \end{cases}$$

$$\lim_{x \to 0^-} e^x \sin(2\pi x) = e^0 \sin(2\pi 0) = 0$$

$$\lim_{x \to 0^+} x^2 - cx = 0^2 - c(0) = 0$$

c can be any value since it'll always be zero

$$f(0) = 0^2 - c(0) = 0$$

2. 

C1: Derivative at a Point

→ Given $s(t)$, the following are equivalent :

* V_{av} from $t=a$ to $t=b$
* slope of secant through $s(t)$ at points $(a, s(a))$ and $(b, s(b))$
* $\frac{\Delta s}{\Delta t} = \frac{s(b)-s(a)}{b-a}$

Derivatives and IROC

* derivative of a function $s(t)$ at point $t=a$:

$$\boxed{s'(t) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}}$$

→ Given $s(t)$, the following are equivalent :

* instantaneous velocity at $t=a$
* m_{tan} at $(a, s(a))$
* derivative of $s(t)$ at a : $s'(t) = \lim_{h \to 0} \frac{s(a+h)-s(a)}{h}$

* tangent line to $f(x)$ at $x=a$:

↳ line passing through point $(a, f(a))$ with slope $f'(a)$

$$\boxed{y = f'(a)(x-a) + f(a)}$$

C2: The Derivative Function

- **the derivative function:** $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$
- domain of $f'(x)$ defined where $f(x)$ is differentiable
- notation: f' $\quad \dfrac{d}{dx} f(x)$ $\quad \dfrac{df(x)}{dx}$ $\quad \dfrac{df}{dx}(x)$
- differentiability implies continuity

C2: Interpretations of The Derivative

Notion of IROCs

- $s(t)$ is position (m) of object at time t (s).
- $s'(t) = v(t)$ is change in position at time t (m/s).
- $s'(5) = 3$
  ↳ at $t = 5$ s, the object's position is changing at rate of 3 metres for every second increment in time

- The derivative represents at what (instantaneous) rate the output variable would respond to changes in the input variable.

→ Eg: object has (−)ve velocity and (+)ve acceleration
  ↳ speed is decreasing

→ Eg: metal rod temp. $T(x)$ at x cm along rod length
  ↳ $T'(x)$ is the rate at which the temp is changing as we move along the rod.
  (how sharply temp changes as we move x cm along rod)

  ↳ units of $T'(x)$: $\frac{°C}{cm}$

  ↳ $T'(5) = 1.2$: At 5 cm along rod, temp will change by 1.2 °C for every cm moved

- Units of derivative: $\dfrac{\text{units of dependent variable}}{\text{units of independent variable}}$ or $\dfrac{[y]}{[x]}$

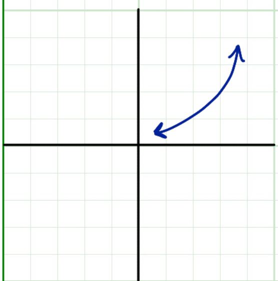
→ Eg: Dam power output $P(q)$, P [=] J/s, q [=] millions of gallons / day
  ↳ $P'(q)$: the rate at which power output is changing if we change water input volume

  units: $\dfrac{J/s}{MG/d}$ = watts / millions of gallons per day

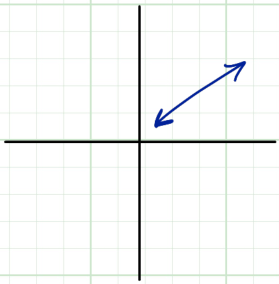
↳ $p'(500) = 300$: at 500 million gallons per day, the power output will change by 300 watts if we adjust water input volume.

Derivatives and Graph Shape

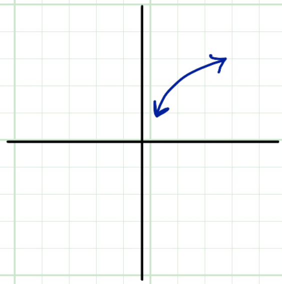
- $f'(x) > 0$, $f(x)$ increasing → vice versa
- $f'(x) < 0$, $f(x)$ decreasing → vice versa



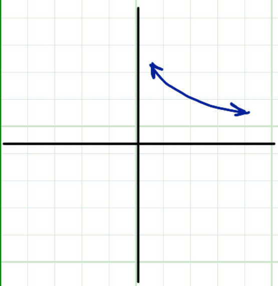
- $f''(x) > 0$
- concave up



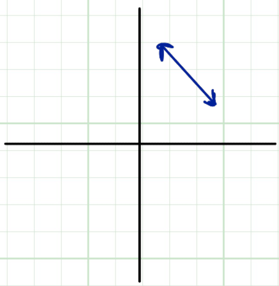
- $f''(x) = 0$
- $f'(x)$ is constant



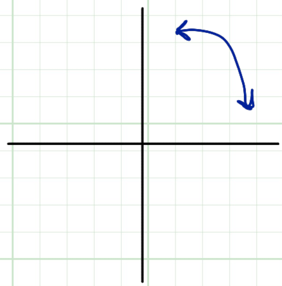
- $f''(x) < 0$
- concave down



- $f''(x) > 0$
- concave up
- large (-) to smaller (-)



- $f''(x) = 0$
- $f'(x)$ is constant



- $f''(x) < 0$
- concave down
- small (-) to large (-)

Chain Rule

- $g(x)$ differentiable at $x=a$
- $f(x)$ differentiable at $x=a$
- $f(g(x))$ differentiable at $x=a$

$$(f \circ g)'(a) = f'(g(a))\, g'(a)$$

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \times g'(x)$$

- if $y = f(u)$ and $u = g(x)$

  ↳ $y = f(g(x))$ $\quad \dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Power of a Function Rule

- if $h(x) = g(x)^n$

$$h'(x) = n\, g(x)^{n-1} \cdot g'(x)$$

→ **Eg 1:** $\dfrac{d}{dx} \sec^2(x)$

$f(x) = x^2$ $\qquad f(g(x)) = (\sec(x))^2$

$g(x) = \sec(x) = \dfrac{1}{\cos(x)}$

$\dfrac{d}{dx} f(x) = 2x$

$$\frac{d}{dx} g(x) = \frac{0(\cos x) - 1(-\sin x)}{[\cos x]^2}$$

$$= \frac{\sin x}{(\cos x)^2}$$

$$= \frac{\sin x}{\cos x} \times \frac{1}{\cos x}$$

$$= \tan x \sec x$$

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \times g'(x)$$

$$= 2(\sec x) \times \tan x \sec x$$

$$= 2 \sec^2 x \tan x$$

→ **Eg 2:** $\dfrac{d}{dx} \sin\left(1 + \cos(\sqrt{x+1})\right)$

$f(x) = \sin x$

$\dfrac{d}{dx} f(x) = \cos x$

$g(x) = 1 + \cos(\sqrt{x+1})$

$\dfrac{d}{dx} g(x) = \dfrac{d}{dx} \cos(\sqrt{x+1})$

$y = \cos u$

$\dfrac{dy}{du} = -\sin u$

$u = \sqrt{x+1} = (x+1)^{1/2}$

$$\frac{du}{dx} = \frac{1}{2}(x+1)^{-1/2} \times 1$$

$$= \frac{1}{2\sqrt{x+1}}$$

$$\frac{dy}{dx} = \frac{dy}{dx}\,\frac{du}{dx}$$

$$= -\sin u \left(\frac{1}{2\sqrt{x+1}}\right)$$

$$= -\frac{\sin u}{2\sqrt{x+1}}$$

$$= -\frac{\sin\sqrt{x+1}}{2\sqrt{x+1}}$$

$$\therefore f'(x) = x$$

$$g'(x) = -\frac{\sin\sqrt{x+1}}{2\sqrt{x+1}}$$

$$\frac{d}{dx} f\circ g(x) = f'(g(x)) \cdot g'(x)$$

$$= \cos\left(1+\cos\left(\sqrt{x+1}\right)\right) \cdot -\frac{\sin\sqrt{x+1}}{2\sqrt{x+1}}$$

$$\boxed{= \frac{-\cos\left(1+\cos\left(\sqrt{x+1}\right)\right)\sin\sqrt{x+1}}{2\sqrt{x+1}}}$$

→ Eg 3: $\frac{d}{dx}\sqrt{\frac{x}{x^2-1}}$ and prove that it's strictly negative

$$\frac{d}{dx}\underbrace{\sqrt{\frac{x}{x^2-1}}}_{f(x)} = \frac{d}{dx}\underbrace{\left(\frac{x}{x^2-1}\right)}_{h(x)}^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(\frac{x}{x^2-1}\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left(\frac{x}{x^2-1}\right)$$

$$\frac{d}{dx} h(x) = \frac{1(x^2-1)-(x)(2x)}{[x^2-1]^2}$$

$$= \frac{x^2-1-2x^2}{(x^2-1)^2} = \frac{-x^2-1}{(x^2-1)^2} = \frac{-(x^2+1)}{(x^2-1)^2}$$

$$\therefore \frac{d}{dx} f(x) = \frac{1}{2}\left(\frac{x}{x^2-1}\right)^{-\frac{1}{2}} \cdot \frac{-(x^2+1)}{(x^2-1)^2}$$

$$= \frac{-(x^2+1)}{2(x^2-1)^2} \cdot \sqrt{\frac{x^2-1}{x}}$$

$$= \frac{-(x^2+1)}{2(x^2-1)^2} \cdot \sqrt{x-1}$$

C5 · Implicit Differentiation

- think of $y = y(x)$ as function of x

→ Eg: $y^3 + x^3 = 2xy$ ... replace "y" with "$y(x)$"

$$y(x)^3 + x^3 = 2x\,y(x)$$

$$\frac{d}{dx}\left[y(x)^3 + x^3\right] = \frac{d}{dx}\left[2x\,y(x)\right]$$ ... take derivative of both sides

$$\frac{d}{dx}y(x)^3 + \frac{d}{dx}x^3 = \frac{d}{dx}2x\,y(x)$$

$$\frac{d}{dx}\left[y(x)\right] = \frac{dy}{dx}$$

$$3y(x)^2\frac{dy}{dx} + 3x^2 = 2y(x) + 2x\frac{dy}{dx}$$ ... change back from "$y(x)$" to "y"

$$3y^2y' + 3x^2 = 2y + 2xy'$$

we can 2 things

↳ isolate for y' to get explicit expression for the derivative of y

↳ find slope of tangent

eg: $(x, y(x)) = (1,1)$

plug in $x=1$, $y=1$

$$3(1)^2y' + 3(1)^2 = 2(1) + 2(1)y'$$

$$y' = 1$$

∴ slope of tangent of $y^3 + x^3 = 2xy$ at $(1,1)$ is 1

Inverse Function Differentiation

$\frac{d}{dx}e^x = e^x$, find derivative formula for $\ln x$

Let $y = \ln x$

then $e^y = e^{\ln x}$

$e^y = x$

differentiate both sides, remember y is a function of x

① $\frac{d}{dx}e^y = \frac{d}{dx}x$

② $e^y\frac{dy}{dx} = 1$

③ $e^y y' = 1$

$$e^y y' = 1$$

$$y' = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

$$\therefore \frac{d}{dx}\ln x = \frac{1}{x}$$

→ Eg: find derivative of $y = \sin^{-1} x$

$$[y = \sin^{-1} x]$$

$$\sin y = x \quad \text{... consider } y \text{ as function of } x$$

$$\frac{d}{dx} \sin y = \frac{d}{dx} x$$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \sec y = \sec(\sin^{-1} x)$$

Meaning of Inverse Derivatives

$s(n)$ represents salinity (pp thousand) at n metres depth

$s^{-1}(n)$ represents at what depth we have salinity n

↳ Eg: $s^{-1}(35)$ represents what depth we have salinity 35 pp thousand

$(s^{-1})'(n)$ represents rate of change of depth at n salinity if we change salinity any further

↳ Eg: $(s^{-1})'(35)$ represents instantaneous rate of change of depth at 35 pp thousand salinity for any further change in salinity.

↳ units $\frac{\text{meters}}{\text{parts per thousand}}$

C6. Related Rates — Ch. 4.1

- **related rates:** 2 or more time varying quantities that can be related by same equation

    ↳ you can find ROC of one quantity given info about the other quantity.

eg: filling balloon with air: both volume and radius increase

→ ## Eg: Ladder Sliding

- let velocity at top be $\frac{dx}{dt}$
- let velocity at bottom be $\frac{dy}{dx}$

* let that specific moment be t_0.

- at some specific moment*, bottom end is 0.6 m from wall and is moving at 3 m/s, how fast is top end moving?

Step 1: Draw pic, label, and assign variables

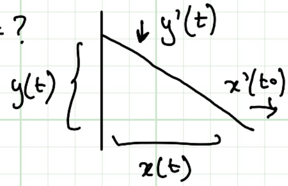
let $x = x(t)$ be distance from wall to bottom end

$y = y(t)$ be distance from ground to top end

Given: $x(t_0) = 0.6$ m

$x'(t_0) = 3$ m/s

Required: $y'(t_0) = ?$



Step 2: Find equation relating the functions of interest.

we know $a^2+b^2=c^2$, so $a^2+b^2=1$ is always true

∴ our equation is $x^2+y^2=1$

$$[x(t)]^2 + [y(t)]^2 = 1$$

Step 3: Differentiate both sides w/r to time

$$\frac{d}{dt}\left(x(t)\right)^2 + \frac{d}{dt}\left(y(t)\right)^2 = \frac{d}{dt}(1)$$

$$2x(t)\cdot\frac{dx}{dt} + 2y(t)\frac{dy}{dt} = 0$$

$$2x(t)\cdot x'(t) + 2y(t)\cdot y'(t) = 0$$

Step 4: Solve for what you need

we know $x(t_0) = 0.6$, $x'(t_0) = 3$, we want $y'(t_0) = ?$

$$\frac{\cancel{2}x(t_0)\,x'(t_0) + \cancel{2}y(t_0)\,y'(t_0)}{\cancel{2}} = \frac{0}{2}$$

$$(x(t))^2 + (y(t))^2 = 1$$

$$y(t) = \sqrt{1-(x(t))^2}$$

$$= \sqrt{1-0.6^2}$$

$$= 0.8$$

$$\therefore y(t_0) = 0.8$$

$$x(t_0)x'(t_0) + y(t_0) + y'(t_0) = 0$$

$$0.6(3) + y(t_0)y'(t_0) = 0$$

$$0.6(3) + 0.8\,y'(t_0) = 0$$

$$y'(t_0) = \frac{-0.6(3)}{0.8} = -2.25 \text{ m/s}$$

$\therefore$ top of ladder velocity is -2.25 m/s

Eg 2: Balloon Filling

Balloon filling with water rate $1\ cm^3/s$

How fast radius changing when it contains $100\ cm^3$ water?

Step 1: volume $v(t)$ at t secs. $\quad v = v(t_0)$

radius $r(t)$ at t secs $\quad r = r(t_0)$

Given: $\frac{dv}{dt} = 1\ cm^3/s$ always

at t_0, $V(t_0) = 100\ cm^3$

Required: $\frac{dr}{dt}$ at t_0 or basically $r'(t_0)$

Step 2: $V = \frac{4}{3}\pi r^3 \quad \dots \quad v(t) = \frac{4}{3}\pi[r(t_0)]^3$

Step 3: $v(t_0) = \frac{4}{3}\pi[r(t_0)]^3$

$$\frac{d}{dt}v(t_0) = \frac{4}{3}\pi\frac{d}{dt}[r(t_0)]^3$$

$$\frac{dv}{dt} = \frac{4}{\cancel{3}}\pi\cdot\cancel{3}r(t_0)^2\cdot\frac{d}{dt}r(t_0)$$

differentiate both sides and use chain rule

$$v'(t_0) = 4\pi r(t_0)^2 r'(t_0)$$

$V'(t_0) = 1$

$V(t_0) = 100$

$$V(t_0) = \frac{4}{3}\pi[r(t_0)]^3$$

$$100 = \frac{4\pi}{3}[r(t_0)]^3$$

$$r(t_0) = \sqrt[3]{100 \div \frac{4\pi}{3}}$$

$$= \left(\frac{100\cdot 3}{4\pi}\right)^{\frac{1}{3}} = \left(\frac{300}{4\pi}\right)^{1/3}$$

$\therefore r(t_0) = \left(\frac{300}{4\pi}\right)^{1/3}$

$$v'(t_0) = 4\pi r(t_0)^2 r'(t_0)$$

$$1 = 4\pi \left[\left(\frac{300}{4\pi}\right)^{1/3}\right]^2 r'(t_0)$$

$$r'(t_0) = \frac{1}{4\pi \cdot \left(\frac{300}{4\pi}\right)^{2/3}}$$

$\therefore$ this is how fast radius is changing when volume is 100 cm^3

- linear approximation $L(x)$: used to approximate the values of a differential function $f(x)$.
- Definition: given · a differentiable func. $f(x)$
  · a point $x = a$ in dom f

→ the linear approximation to f at a or (linearization of f at a) is given by function: $L(x) = f(a) + f'(a)(x-a)$

- if $a \approx x$, then $L(x) \approx f(x)$
- same as formula for tangent line to f at a

Derivation From tangent line formula

$$y - y_1 = m_{tan}(x - x_1) \qquad y_1 = f(x)$$
$$x = a$$
$$y = f'(x)(x - x_1) + y_1$$
$$y = y + f'(x)(x - x_1)$$
$$y = f(a) + f'(a)(x-a)$$

→ Eg: linearization of $f(x) = \sqrt{x}$ at $x = 4$ is ... $f'(x) = \frac{1}{2\sqrt{x}}$

$$L(x) = f(a) + f'(a)(x-a)$$
$$= f(4) + f'(4)(x-4)$$
$$= \sqrt{4} + \frac{1}{2\sqrt{4}}(x-4)$$
$$= 2 + \frac{1}{4}(x-4)$$

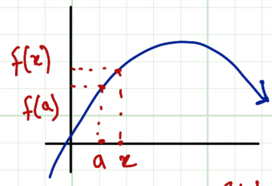
Evaluating $L(x)$ at $x = 4.02$ ...

$$L(x) = 2 + \frac{1}{4}(x-4)$$
$$L(4.02) = 2 + \frac{1}{4}(4.02 - 4)$$
$$= 2 + \frac{1}{4}(0.02)$$
$$= 2.005$$

$L(2.02) = 2.005$ is our approx.

actual value: $f(2.02) = \sqrt{2.02} \approx 2.00499$

Approximating Change:



$f(x) \approx L(x)$

$f(x) - f(a) \approx L(x) - f(a)$

$f(x) - f(a) \approx \left(\cancel{f(a)} + f'(a)(x-a) \right) - \cancel{f(a)}$

$\Delta y = f(x) - f(a)$

$\Delta x = x - a$

$f(x) - f(a) \approx f'(a)(x-a)$ for all x near a

$\Delta y \approx f'(a)\,\Delta x$ for all x near a

$\Delta x = (x-a)$: displacement of x from a

$\Delta y = f(x) - f(a)$: the resulting change in y values.

- above formulation tells us approx. how much y changes when we have small changes in $x = a$
- basically what derivative tells us.

Pre-class Question

$f(x) = \ln x$ what is $\ln 3$? $x = 3$

$f'(x) = \frac{1}{x}$

$\ln e = 1$

let $a = e$

$$L(x) = f(a) + f'(a)(x-a)$$
$$= f(e) + f'(e)(x-e)$$
$$= 1 + \frac{1}{e}(x-e)$$
$$L(3) = 1 + \frac{1}{e}(3-e)$$
$$= 1 + \frac{3-e}{e}$$
$$\approx 1.103638\ldots$$

$\ln 3 = 1.0986$

Ordinary Differential Equations

- **ordinary differential equation**

an equation involving an unknown function (eg: $y(t)$) and one or more of its derivatives.

- **solution to ODE:** a function $y(t)$ that satisfies the differential equation when it and its derivaties are substituted into the equation.

Examples of ODE's

* radioactive decay: $y' = -ky$
* Newton 2nd Law: $F(t) = m\,x''(t)$
* charge $Q(t)$ in RLC circuit: $LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = 0$

circuit with resistor (R), inductor (L) and capacitor (C)

* given ODE, check if a function is a solution

→ Eg: $\frac{dy}{dt} = t - y$

① Is $y_1(t) = e^{-t} + t - 1$ a solution? $\boxed{\text{Yes}}$

show that $\frac{dy_1}{dt} = t - y_1$

$$\frac{dy_1}{dt} = (e^{-t}\ln e)(-1) + 1$$
$$= -e^{-t} + 1$$

$$= t - y_1$$
$$= t - (e^{-t} + t - 1)$$
$$= \cancel{t} - e^{-t} - \cancel{t} + 1$$
$$= -e^{-t} + 1$$

Since they're equal, $y_1(t)$ is solution to ODE

② Is $y_2(t) = e^t - t$ a solution ? $\boxed{\text{No}}$

$$\frac{dy}{dt} = t - y \implies \frac{dy_2}{dt} = t - y_2$$

$$y_2(t) = e^t - t$$

$$\frac{dy_2}{dt} = e^t \ln e - 1$$
$$= e^t - 1$$

$$= t - y_2$$
$$= t - (e^t - t)$$
$$= t - e^t + t$$
$$= -e^t + 2t$$

not equal

so $y_2(t)$ not solution to ODE

* both sides need to be same function for these to be a solution to the ODE.

→ Eg: $y' = -0.00013y$

- $y(t)$: models the number of C^{14} atoms present in organic material t years after its death.

* this differential equation says that the change in the number of C^{14} atoms present (y') is proportional to the number of C^{14} atoms present (y) with constant of proportionality being -0.00013

* is $y(t) = 1000e^{-0.00013t}$ a solution to ODE?

$$y'(t) = (1000)e^{-0.00013t}(-0.00013)$$
$$= -0.13e^{-0.00013t} \leftarrow \text{L.S.}$$

$= -0.00013\, y(t)$

$= -0.00013(1000e^{-0.00013t})$

$= -0.13e^{-0.00013t} \leftarrow$ R.S

Since R.S. = L.S., we know $y(t) = 1000e^{-0.00013t}$ is solution to ODE.

* Since $y(t)$ represents the numbes of C^{14} atoms present at time t, we know that there will be $1000e^{-0.00013t}$ C^{14} atoms t time afer something dies. We can also ans. following:
* how many C^{14} did we start with ? : $y(0)$
* does $y(t)$ inc. or dec. over time? : dec bc. $y'(t) < 0$
* how many C^{14} atoms as $t \to \infty$? : $\lim_{t\to\infty} y(t) = 0$

* ODE can have many solutions

eg: for ODE $\frac{dy}{dt} = t - y$

$y_1(t) = e^{-t} + t - 1$ solution

$y_3(t) = t - 1$ solution

Equilibrium Solutions

* equilibrium solution to ODE is a sol. $y(t)$ to ODE that also happens to be a constant function.

* solution of form: $y(t) = c \leftarrow$ "c" is a constant

→ Eg: find any equilibrium sol. to ODE: $\underbrace{v' = 9.8 - 0.5v}$

t is [s] and $v(t)$ is [m/s]

models downward velocity $v(t)$ of body in free fall with air resistance

* suppose sol. $v(t) = c$

$v(t) = c$

$v'(t) = 0$

$v'(t) = 9.8 - 0.5v(t)$

$0 = 9.8 - 0.5c$

$\boxed{\therefore\ c = 19.6\ m/s}$

* this means body falls at a constant 19.6 m/s.
* for this case, all other solutions will approach this value:

terminal velocity

$v_1(t) = 19.6 + e^{-0.5t}$ and $v_2(t) = 19.6 - 5e^{-0.5t}$

$v'(t) = -0.5e^{-0.5t}$

$v' = 9.8 - 0.5v$

$-0.5e^{-0.5t} = 9.8 - 0.5(19.6 + e^{-0.5t})$

$= \cancel{9.8} - \cancel{9.8} - 0.5e^{-0.5t}$

$-0.5e^{-0.5t} = -0.5e^{-0.5t}$

Yes solution

$v_2'(t) = 2.5e^{-0.5t}$

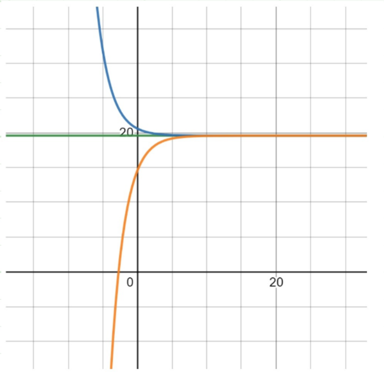
$v' = 9.8 - 0.5v$

$2.5e^{-0.5t} = 9.8 - 0.5(19.6 - 5e^{-0.5t})$

$= \cancel{9.8} - \cancel{9.8} + 2.5e^{-0.5t}$

$2.5e^{-0.5t} = 2.5e^{-0.5t}$

Yes solution



$v_1(t) = 19.6 + e^{-0.5t}$

$v_2(t) = 19.6 - 5e^{-0.5t}$

Notes

* $y(t) = t$ ← not equilibrium sol. bc. not a constant
* $t = 3$ ← not equilibrium sol. bc. not a function $y(t)$
* $y(t) = 2$ ← yes equilib. sol.

Phase Lines

- we want to learn more about the solution $y(t)$ to the ODE $y'(t) = f(y)$ which satisfies initial data $y(0) = a$.
- by looking at the sign of $f(y)$ at various values of y, we can know where our sol. is inc., dec., or constant.
- phase line is a graphical representation which depicts the regions where $y(t)$ (a sol. to ODE) is inc., dec., or constant.

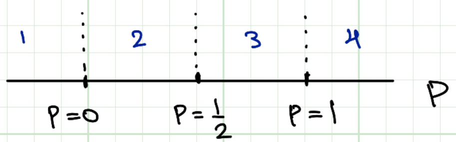
→ **Eg:** sketch phase lines for ODE: $P'(t) = P(1-P)\left(\frac{1}{2}-P\right)^2$

- **step 1:** find equilibrium points of ODE

$$P'(t) = P(1-P)\left(\tfrac{1}{2}-P\right)^2 \implies f(P) = P(1-P)\left(\tfrac{1}{2}-P\right)^2$$

$$\underbrace{P=0 \qquad P=1 \qquad P=\tfrac{1}{2}}_{\text{3 equilibrium solutions}}$$

- **step 2:** draw foundation of phase line:



- **step 3:** test intervals to see where P is inc or dec.

↳ $f(P^*) > 0$ P is increasing

↳ $f(P^*) < 0$ P is decreasing

P^* is any point in the interval

interval 1: $P < 0$

$$f(-1) = (-1)(1-(-1))\left(\tfrac{1}{2}-(-1)\right)^2 = -\frac{9}{2} < 0 \quad \therefore P \text{ is dec.}$$

interval 2: $0 < P < \frac{1}{2}$

$f(\frac{1}{4}) = (\frac{1}{4})(1-\frac{1}{4})(\frac{1}{2}-\frac{1}{4})^2 > 0$ $\therefore$ P is inc.

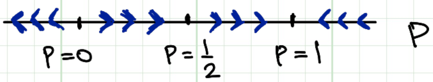
interval 3: $\frac{1}{2} < P < 1$

$f(\frac{3}{4}) = (\frac{3}{4})(1-\frac{3}{4})(\frac{1}{2}-\frac{3}{4}) > 0$ $\therefore$ P is inc.

interval 4: $P > 1$

$f(2) = (2)(1-2)(\frac{1}{2}-2)^2 < 0$ $\therefore$ P is dec.

- Step 4 : sketch arrows



Equilibria Types : ODE of form $y'(t) = f(y)$ have 3:

1. stable equilibria :

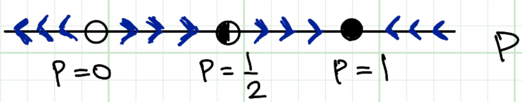
- solutions in neighboring intervals approach equilibrium.
- arrows point toward equilibrium on both sides.
- filled-in circle

2. unstable equilibria:

- solutions in neighboring intervals repell equilibrium.
- arrows point away from equilibrium on both sides.

3. semi-stable equilibria:

- solutions in one interval approach and in another interval repel
- arrows point same directions both sides.
- filled half on side where solutions are approaching.



Theorem

- initial value problem $\begin{cases} y'(t) = f(y) \\ f(t_0) = y_0 \end{cases}$

- if $f(y)$ and $f'(y)$ are continuous functions on int. $I = (a, b)$ and $a < y_0 < b$, then $\exists$ exactly one sol. to initial value problem on interval I.

→ **Eg:** explain why $\lim_{t \to \infty} P(t) = \frac{1}{2}$ where $P(t)$ solves the initial value problem

$$\begin{cases} P'(t) = P(1-P)\left(\frac{1}{2} - P\right)^2 \\ P(0) = \frac{2}{5} \end{cases}$$

← since $0 < P(0) < \frac{1}{2}$, from phase line we can see that $P(t)$ will continue to increase closer and closer to $\frac{1}{2}$.

- does it ever reach $\frac{1}{2}$? No, because that would mean we would have 2 solutions to our initial value problem.

Autonomous & Non-Autonomous ODE's

1. autonomous: $y'(t) = y + 3$

   contains only dependent variable $y(t)$

2. non-autonomous: $y'(t) = y + 2t$

   contains both dependent variable $y(t)$ and independent variable t

Differences

- suppose we want to know slope of sol. at $y=1$
  - ↳ autonomous ODE: slope is 4
  - ↳ non autonomous ODE: slope is $1+2t$ ← slope of solution depends on t
- non-autonomous ODE's describe
  - ↳ slope of $y(t)$ at specific point y and specific value of t.
- non-autonomous ODE must be written in the form

  $y'(t) = f(t,y)$ ← function of 2 variables $\boxed{f: \mathbb{R}^2 \to \mathbb{R}}$
- can't draw phase line, so we do slope fields.

Slope Fields

- consider non-autonomous ODE $y'(t) = f(t,y)$
- we construct a slope field by evaluating $f(t,y)$ at various points in the t-y plane.
- we then estimate the trajectory of the solutions by matching the slopes of the solutions with the slope field.

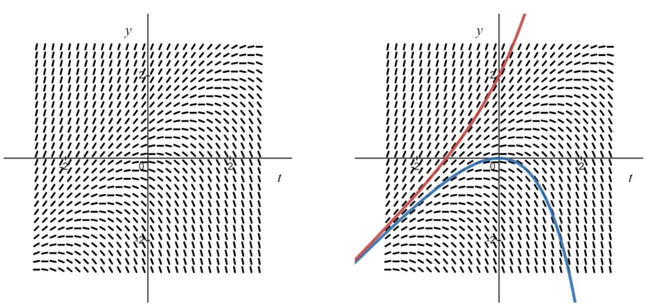


Figure 1: Slope field and sample solution curves for $y'(t) = y - t$.

Euler's Method Example

$y'(t) = y - t$ ... differential equation

$y(0) = 2$ ... initial value / condition

- want to find sequence of points (t_j, y_j) s.t. $y_j \approx y(t_j)$

t	y	y'
0	2	2
0.2	2.4	2.2
0.4	2.84	2.44
0.6	3.328	2.728
0.8	3.8736	3.0736
1.0	4.48832	

step size $\Delta t = 0.2$

$y'(0) = y(0) - 0$
$= 2 - 0 = 2$

slope of linear approx. for this iteration

$\therefore y(t) \approx L(t) = y'(t)(t-0) + y(t)$

$\boxed{L(t) = 2t + 2}$

$y(0.2) \approx L(0.2) = 2(0.2) + 2 = 2.4$

$y'(0.2) = y(0.2) - 0.2$
$= 2.4 - 0.2 = 2.2$

$\therefore y(t) \approx L(t) = y'(t)(t - 0.2) + y(t)$

$\boxed{L(t) = 2.2(t - 0.2) + 2.4}$

$y(0.4) \approx L(0.4) = 2.2(0.4 - 0.2) + 2.4 = 2.84$

$y'(0.4) = y(0.4) - 0.4$
$= 2.84 - 0.4 = 2.44$

$\boxed{y(t) \approx L(t) = 2.44(t - 0.4) + 2.84}$

$y(0.6) \approx L(0.6) = 2.44(0.6 - 0.4) + 2.84 = 3.328$

$y'(0.6) = 3.328 - 0.6 = 2.728$

$\boxed{y(t) \approx L(t) = 2.728(t - 0.6) + 3.328}$

$y(0.8) \approx L(0.8) = 2.728(0.8 - 0.6) + 3.328$
$= 2.728(0.2) + 3.328$
$= 3.8736$

$y'(0.8) = 3.8736 - 0.8 = 3.0736$

$L(t) = 3.0736(t - 0.8) + 3.8736$

$y(1) = 3.0736(0.2) + 3.8736$
$= 4.48832$

$\therefore (t_5, y_5) = (1, 4.48832)$

$y' = y - t \quad y' = f(t, y), \ \Delta t = 0.2$

$t_n = t_{n-1} + h$

$y_n = y_{n-1} + \Delta t f(t_{n-1}, y_{n-1})$

start

$t_0 = 0 \qquad y_0 = 2$

$t_1 = 0.2 \qquad y_1 = 2 + 0.2(2-0) = 2.4$

$t_2 = 0.4 \qquad y_2 = 2.4 + 0.2(2.4 - 0.2) = 2.84$

$t_3 = 0.6 \qquad y_3 = 2.84 + 0.2(2.84 - 0.4) = 3.328$

$t_4 = 0.8 \qquad y_4 = 3.328 + 0.2(3.328 - 0.6) = 3.8736$

$t_5 = 1.00 \qquad y_5 = 3.8736 + 0.2(3.8736 - 0.8) = 4.48832$

$\therefore (t_5, y_0) = (1, 4.48832)$

Slope Fields

- using Euler's method, it follows the slope field for a step of Δt before updating and using the new slope
- the smaller the step, the more accurate the approximation because the slope is being updated more frequently.
- the error is proportional to the step size Δt
  - ↳ if step size is halved, the error is halved
- which means there is some constant c s.t.

$$\text{actual value at some } t^* - \text{approximate value at that } t^* = c\,\Delta t$$

L'Hopital's Rule

consider $\lim_{x \to 1} \frac{\ln(3x-2)}{x-1}$

$\lim_{x \to 1} \ln(3x-2) = 0$

$\lim_{x \to 1} x-1 = 0$

$\frac{0}{0}$ indeterminate

- $f(x) = \ln(3x-2) \implies f'(x) = \frac{3}{3x-2}$

↳ linear approx for x near a : $L_f(x) = f'(a)(x-a) + f(a)$

for x near 1, $a=1$: $L_f(x) = f'(1)(x-1) + \cancel{f(1)}^{\to 0}$

$$L_f(x) = f'(1)(x-1)$$

- $g(x) = x-1 \implies g'(x) = 1$

↳ linear approx. for x near a : $L_g(x) = g'(a)(x-a) + g(a)$

for x near 1, $a=1$: $L_g(x) = g'(1)(x-1) + \cancel{g(1)}^{\to 0}$

$$L_g(x) = g'(1)(x-1)$$

$$\frac{\ln(3x-2)}{x-1} = \frac{f(x)}{g(x)} \approx \frac{L_f(x)}{L_g(x)} = \frac{f'(1)(x-1) + \cancel{f(1)}^{0}}{g'(1)(x-1) + \cancel{g(1)}^{0}} = \frac{f'(1)(x-1)}{g'(1)(x-1)}$$

$$\lim_{x \to 1} \frac{\ln(3x-2)}{x-1} = \lim_{x \to 1} \frac{f'(1)\cancel{(x-1)}}{g'(1)\cancel{(x-1)}}$$

$$= \frac{f'(1)}{g'(1)} = \frac{\frac{3}{3-2}}{1} = \frac{\frac{3}{1}}{1} = \frac{3}{1} = 3$$

$$\boxed{\therefore \lim_{x \to 1} \frac{\ln(3x-2)}{x-1} = 3}$$

- **L'Hôpital's Rule:** if f and g are differentiable and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then...

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$ provided $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists

- can also apply to one-sided limits

→ Eg: $\lim_{x\to 0} \frac{\tan x - x}{x - \sin x}$

1. direct evaluation

$$\lim_{x\to 0} \frac{\tan x - x}{x - \sin x} = \frac{\tan 0 - 0}{0 - \sin 0} = \frac{0}{0}$$ indeterminate

2. L'Hopital's rule try #1

$$\lim_{x\to 0} \frac{(\tan x - x)'}{(x - \sin x)'} = \lim_{x\to 0} \frac{\sec^2 x - 1}{1 - \cos x} = \frac{0}{0}$$ indeterminate

3. L'Hopital's rule try #2

$$\lim_{x\to 0} \frac{(\sec^2 x - 1)'}{(1 - \cos x)'} = \lim_{x\to 0} \frac{2\sec x \tan x}{\sin x} = \frac{0}{0}$$ indeterminate

4. L'Hopital's rule try #3

$$\lim_{x\to 0} \frac{(2\sec x \tan x)'}{(\sin x)'} = \frac{4\sec^2 x \tan x + 2\sec^4 x}{\cos x} = 2$$ exists

$$\boxed{\therefore \lim_{x\to 0} \frac{\tan x - x}{x - \sin x} = 2}$$

L'Hôpital's Rule (Infinity)

- f and g are differentiable and a is $\mathbb{R}$ or $\pm\infty$.
- If...

↳ $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$, or...

↳ $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ then...

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists

→ Eg: $\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x$ ... has form $(1+0)^\infty = 1^\infty$

$e^{\ln x} = x$

$$\left(1 + \frac{2}{x}\right)^{x} = e^{\ln\left(1+\frac{2}{x}\right)^x} = e^{x\ln\left(1+\frac{2}{x}\right)}$$

Let's find the limit of this as $x\to\infty$

$$\lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right) = \infty \cdot \ln(1+0)$$

$$= \infty \cdot 0 \leftarrow \text{indeterminate}$$

$$\ln\left(1+\frac{2}{x}\right) \times x = \ln\left(1+\frac{2}{x}\right) \times \frac{x}{1} = \ln\left(1+\frac{2}{x}\right) \div \frac{1}{x}$$

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$$

indeterminate form, so we can use L'Hôpital's rule

$$\lim_{x\to\infty} \frac{\left(\ln\left(1+\frac{2}{x}\right)\right)'}{\left(\frac{1}{x}\right)'} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}} \cdot -2x^{-2}}{-x^{-2}}$$

$$= \lim_{x\to\infty} \frac{-\frac{2}{\left(1+\frac{2}{x}\right)x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} -\frac{2}{\left(1+\frac{2}{x}\right)\cancel{x^2}} \cdot -\frac{\cancel{x^2}}{1} = \frac{2}{1+0} = \boxed{2}$$

since we have $\lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right) = 2$

$$\boxed{\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{2}{x}\right)} = e^2}$$

Intervals: $a, b \in \mathbb{R}$ and $a < b$

- open interval if $I_o = (a, b)$
- closed interval if $I_o = [a, b]$
- neither closed or open if $I_o = [a, b)$ or $I_o = (a, b]$

Absolute Extrema: given $f(x)$, function has...

- absolute max at $x = c$ if $f(c) \geq f(x)$ for all $x \in \text{dom } f$
- absolute min at $x = c$ if $f(c) \leq f(x)$ for all $x \in \text{dom } f$
- extrema → maxima, minima

absolute = global

Function Not Attaining Global Extrema

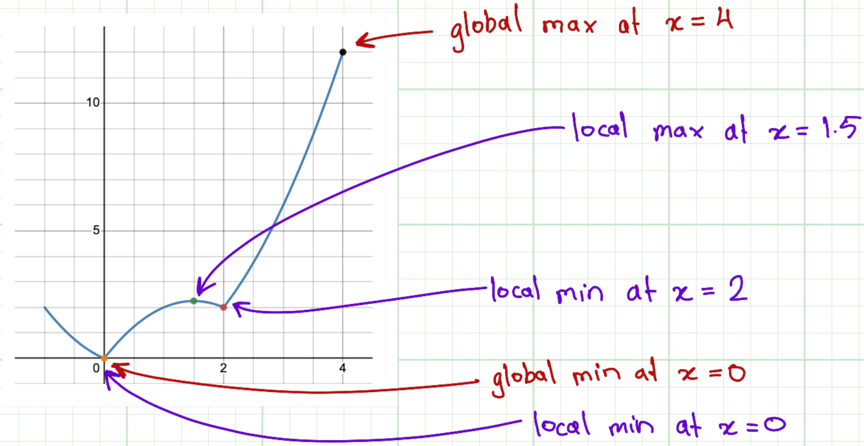
- f would have reached global extrema but the x-value is not in the domain / interval:
  - ↳ eg: $f(x) = \frac{1}{x^2+1}$ on $I = (0, 1)$
  - does not occur for closed intervals
- function grows boundlessly to $\pm\infty$ on the interval
  - ↳ eg: $f(x) = \tan x$ on $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Local Extrema: given $f(x)$, function has...

- local max if at $x = c$, $\exists$ a small open interval I_o around c contained in dom f s.t. $f(c) \geq f(x)$ for all $x \in I_o$
- local min if at $x = c$, $\exists$ a small open interval I_o around c contained in dom f s.t. $f(c) \leq f(x)$ for all $x \in I_o$

Important Notes

- global extrema can occur at endpoints of a closed interval, but local extrem cannot.
- every global extrema (that is not at endpoints) is also a local extrema, but not the other way around.



Critical Points

- critical point at $x = c$ if for a small open interval $I_0 \subset \text{dom} f$ where $f'(c) = 0$ or $f'(0)$ is undefined
- if c is local extrema, then c is also critical point

First Derivative Test : f cont. on I with crit. pt. c

- if $f'(x)$ changes sign from $(+) \to (-)$, $f(c)$ is local max.
- if $f'(x)$ changes sign from $(-) \to (+)$, $f(c)$ is local min.
- if $f'(x)$ doesn't change sign, the $f(c)$ is neither

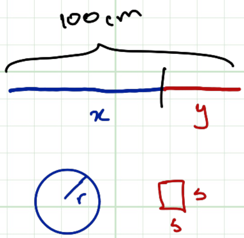
Closed Interval Method

- $f(x)$ is cont. on $[a,b]$, to find absolute max/min of $[a,b]$

1. find critical points (x,y) in (a,b)
2. find value of $f(x)$ at endpoints a and b.
3. highest of those values ← global max

   lowest of those values ← global min

→ Eg: **Example**: You have a 100cm piece of wire that you will cut into two pieces. One piece will be bent into a circle, the other will be bent into a square. No wire is wasted. Where would you cut the wire to make the sum of the two areas as small as possible?



$\boxed{1}$ Quantity to be optimizd : sum of areas circle and square to be smallest

$\boxed{2}$ get formula for area sum

$$\boxed{A = \pi r^2 + s^2}$$

$x = 2\pi r \qquad y = 4s$

$x + y = 100$

$2\pi r + 4s = 100$

$y = 100 - x$

$4s = 100 - x$

$s = \frac{100 - x}{4}$

$$\boxed{s = \frac{100-x}{4}} \qquad \boxed{r = \frac{x}{2\pi}}$$

$$r = \frac{100 - 4s}{2\pi} = \frac{50 - 2\left(\frac{100-x}{4}\right)}{\pi} = \frac{50 \frac{-200+2x}{4}}{\pi} = \frac{\cancel{50} - \cancel{50} + \frac{1}{2}x}{\pi} = \frac{x}{2\pi}$$

$$A = \pi r^2 + s^2$$

$$A = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{100-x}{4}\right)^2$$

$$= \frac{\cancel{\pi} x^2}{4\pi^{\cancel{2}}} + \frac{(100-x)^2}{16}$$

$$\boxed{A = \frac{x^2}{4\pi} + \frac{(100-x)^2}{16}}$$ ← objective function

$\boxed{3}$ Decide on dom and find min value of A on $x \in [0, 100]$

↳ critical point

↳ endpoints

$$A = \frac{x^2}{4\pi} + \frac{(100-x)^2}{16}$$

$$A' = \frac{2}{4\pi}x + \frac{2(100-x)(-1)}{16}$$

$$= \frac{1}{16}(100-x)^2$$

$$= \frac{2}{16}(100-x)(-1)$$

$$= \frac{-1(100-x)}{8} = \frac{-100+x}{8}$$

$$= \frac{x-100}{8}$$

$$A' = \frac{x}{2\pi} + \frac{x-100}{8}$$

$$0 = \frac{1}{2\pi}x + \frac{x-100}{8}$$

$$\frac{-x}{2\pi} = \frac{x-100}{8}$$

$$-8x = 2\pi x - 200\pi$$

$$-8x - 2\pi x = -200\pi$$

$$x(-8-2\pi) = -200\pi$$

$$x = \frac{-200\pi}{-8-2\pi} = \frac{100\pi}{4+\pi}$$

$$A(0) = \frac{0^2}{4\pi} + \frac{(100-0)^2}{16} = 625$$

$$A(100) = \frac{100^2}{4\pi} + \frac{(100-100)^2}{16} \approx 795.8$$

$$A\left(\frac{100\pi}{4+\pi}\right) \approx 350.1$$ ← that value of x gives least area

$\boxed{\therefore \text{ we cut the rope so one of them is } \frac{100\pi}{4+\pi} \text{ long}}$

Checklist

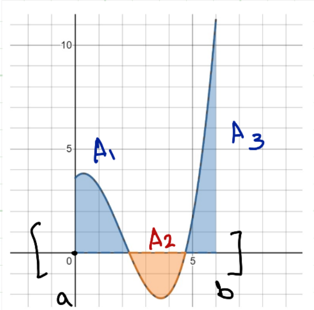
The applied optimization guideline/check-list:

1. Introduce and name all variables. If possible, draw a picture and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time)
3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.
4. Write any equations relating the independent variables ("constraints") in the formula from step 3. Use these equations to write the quantity to be maximized or minimized as a function of one variable.
5. Identify the domain for the function in step 4. based on the physical problem to be solved. Do not skip this step.
6. Apply the Closed Interval Method to find the global maximum or minimum value.
7. Interpret the real-world significance in terms of the question being asked. Do not skip this step.

A well-written answer will explain the steps of modelling the problem so that the work can be followed easily, and justify using mathematical principles that the quantity you found is indeed the desired quantity.

Signed Area Under Curve for f(x) on [a,b]

$$\boxed{\text{signed area underneath a curve}} = \boxed{\text{area enclosed by positive part of function and } x\text{-axis}} - \boxed{\text{area enclosed by negative part of function and } x\text{-axis}}$$



net signed area under curve

$= A_1 - A_2 + A_3$

same as

- the definite integral

$$\boxed{\int_a^b f(x)\,dx}$$

Integral Notation

$$\int_a^b f(x)\,dx$$

upper limit

limits of integration

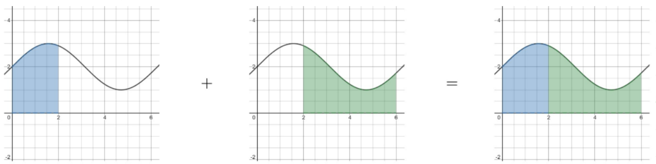
lower limit of integration

"dx" is variable of integration meaning "x" is the independent var. of $f(x)$

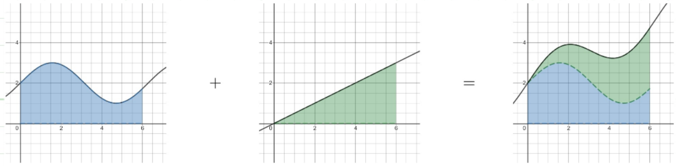
function "$f(x)$" we are taking the integral of

Properties of Integration

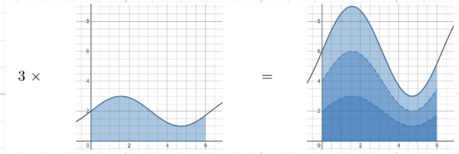
1. $\int_a^b f(x)\,dx + \int_c^d f(x)\,dx = \int_a^d f(x)\,dx$



2. $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b (f(x) + g(x))\,dx$

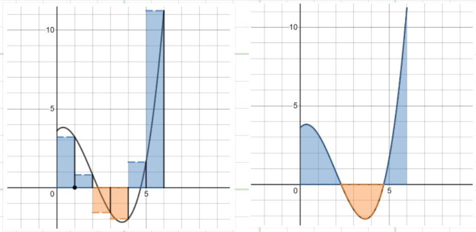


3. $c\int_a^b f(x)\,dx = \int_a^b c f(x)\,dx$ ... for given constant c



Riemann Sums • to approx a definite integral numerically

• the signed area of rectangles approximates signed area under curve.



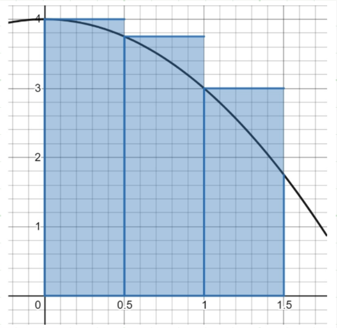
2 things to consider

• the number of rectangles
• the height of the rectangles

Left-Endpoint Rieman Sum

- approx. $\int_0^{1.5} (4-x^2)\,dx$ with 3 rectangles ← "L_3"

1. divide dom $[0,1.5]$ into 3 "$\Delta x = 0.5$"
2. draw 3 rectangles with height of $f($left Δx endpoint$)$
   ↳ $f(0)$, $f(0.5)$, $f(1)$



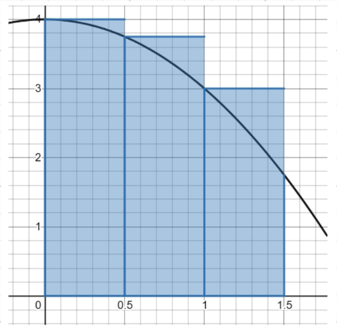
We have ...

$$
\begin{aligned}
L_3 &= f(0)\Delta x + f(0.5)\Delta x + f(1)\Delta x \\
&= 4(0.5) + 3.75(0.5) + 3(0.5) \\
&= 5.375
\end{aligned}
$$

Right-Endpoint Rieman Sum

- approx. $\int_0^{1.5} (4-x^2)\,dx$ with 3 rectangles ← "R_3"

1. divide dom $[0,1.5]$ into 3 "$\Delta x = 0.5$"
2. draw 3 rectangles with height of $f($right Δx endpoint$)$
   ↳ $f(0.5)$, $f(1)$, $f(1.5)$



We have ...

$$
\begin{aligned}
R_3 &= f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x \\
&= 3.75(0.5) + 3(0.5) + 1.75(0.5) \\
&= 4.25
\end{aligned}
$$

Riemann Sum

- given $f(x)$ defined on $[a,b]$, a Riemann Sum is of form:

$$\boxed{\sum_{k=1}^{n} f(x_k^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x}$$

... where "x_k^*" are representative points chosen from each subinterval of width Δx.

the approximate signed areas underneath the curve between $x=a$ and $x=b$ $\equiv$ A Riemann Sum $\sum_{k=1}^{n} f(x_k^*)\Delta x$

Sigma Notation

- notation for long sums, of the form: $\sum_{k=n}^{N} a_k$, $n < N$

"add up the terms a_k as k ranges over the integers from lower index $k=n$ to upper index $k=N$."

$$\sum_{k=n}^{N} a_k = a_n + a_{n+1} + \cdots + a_{N-1} + a_N$$

can be any expression that varies as k changes

→ Eg: $\sum_{k=2}^{7} k^3 = 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$

index "k" is only a placeholder ... can use any letter for k

$$\sum_{k=2}^{7} k^3 = \sum_{j=2}^{7} j^3 = \sum_{l=2}^{7} l^3$$

Properties of Sigma Notation

1. For constant c,

$$\sum_{k=n}^{N} c a_k = c \sum_{k=n}^{N} a_k$$

...distributive property

2. $$\sum_{k=n}^{N} (a_k + b_k) = \sum_{k=n}^{N} a_k + \sum_{k=n}^{N} b_k$$

... commutative property

3. For any whole number M between n and N,

$$\sum_{k=n}^{N} a_k = \sum_{k=n}^{M} a_k + \sum_{k=M+1}^{N} a_k$$

... can break up sum into sum of sums

$$(a_n + \cdots a_M) + (a_{M+1} + \cdots a_N)$$

4. Reindexing: can write sum in infinitely many ways by performing a change of variables on index ...

$$\sum_{k=2}^{7} k^2 \quad \text{... set } j = k-2 \text{ and write ...} \quad \sum_{j=0}^{5} (j+2)^3$$

$$\sum_{k=2}^{7} k^2 = 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 = 139$$

$$\sum_{j=0}^{5} (j+2)^2 = (0+2)^2 + (1+2)^2 + (2+2)^2 + (3+2)^2 + (4+2)^2 + (5+2)^2 = 139$$

Subinterval Notation

→ approximate definite integral $\int_a^b f(x)\,dx$ with n subintervals

- approximate definite integral $\int_2^5 (x^2-1)\,dx$ with $n=6$ subintervals

1. What is the entire interval $[a,b]$?

↳ finding area under x^2-1 from $x=2$ to $x=5$

so interval is $[2,5]$ and it's $b-a = 5-2 = 3$ units long

2. What are widths Δx of the six subintervals?

↳ dividing 3-unit-long interval into 6 intervals, so

$$\Delta x = \frac{b-a}{n} = \frac{5-2}{6} = \frac{3}{6} = \frac{1}{2}$$

3. What are the endpoints $x_0, x_1, x_2 \cdots x_6$ of the six subintervals?

↳ if six subintervals, seven endpoints

x_0	$x_1 = x_0 + \Delta x$	$x_2 = x_0 + 2\Delta x$	$x_3 = x_0 + 3\Delta x$	$x_4 = x_0 + 4\Delta x$	$x_5 = x_0 + 5\Delta x$	$x_6 = x_0 + 6\Delta x$
2	2.5	3	3.5	4	4.5	5

n subintervals, $n+1$ endpoints

endpoints given by: $x_k = x_0 + k\Delta x$, $k = 1, 2 \cdots n$

$\Big\{$ k'th subinterval is $[x_{k-1}, x_k]$, $k = 1, 2 \cdots n$

4. What are our representative points $x_1^*, x_2^*, \cdots, x_n^*$ we are choosing from each of our six subintervals?

intervals are $[x_{k-1}, x_k]$ for $k = 1, 2, \cdots, n$

↳ right endpt. Riemann

choose "x_k", so $x_k^* = x_k$, $k = 1, 2, \cdots, n$

$$x^*_k = x_0 + k\Delta x$$

↳ left endpt. Riemann

choose "x_{k-1}", so $x_k^* = x_{k-1}$, $k = 1, 2, \cdots, n$

$$x^*_k = x_0 + (k-1)\Delta x$$

5. What are the areas of n rectangles $A_1, A_2, \cdots A_n$?

↳ area of k'th rectangle in each subint. $[x_{k-1}, x_k]$ is

$f(x^*_k)\Delta x$, $k = 1, 2, \cdots, n$

right end pt. Riemann

area is $f(x_0 + k\Delta x)\Delta x$ for $k = 1, 2, \cdots, n$

left end pt Riemann

area is $f(x_0 + (k-1)\Delta x)\Delta x$ for $k = 1, 2, \cdots, n$

6. How to write Riemann Sum?

↳ Riemann sum ≡ sum of areas of rectangle, $\sum_{k=1}^{N} f(x^*_k)\Delta x$

Eg: right endpt. Riemann

$$R_6 = \sum_{k=1}^{6} f(x_0 + k\Delta x)\Delta x = \sum_{k=1}^{6} f\left(2 + k\frac{1}{2}\right)\frac{1}{2} = \sum_{k=1}^{6} \frac{1}{2}\left((2+k)^2 - 1\right)$$

Eg: Left endpt. Riemann

$$L_6 = \sum_{k=1}^{6} f(x_0 + (k-1)\Delta x)\Delta x$$

$$= \sum_{k=1}^{6} f\left(2 + (k-1)\tfrac{1}{2}\right)\tfrac{1}{2}$$

$$= \sum_{k=1}^{6} \frac{1}{2}\left(\left(2+(k-1)\tfrac{1}{2}\right)^2 - 1\right)$$

$$= \sum_{k=1}^{6} \frac{1}{2}\left(\left(2 + \frac{k-1}{2}\right)^2 - 1\right) \quad \cdots \text{reindex, rewrite } j = k-1$$

$$= \sum_{j=0}^{6} \frac{1}{2}\left(\left(2 + \frac{j}{2}\right)^2 - 1\right)$$

Summary; Given an interval $[a, b]$, a function $f(x)$, a number of subintervals n, and a rule (left or right endpoint) for selecting representative points x_k^*, we can compute the widths Δx and endpoints x_k of the subintervals, and using the representative points x_k^*, compute the areas A_k of rectangles, then add them up to create a Riemann sum.

Limit Definition of the Definite Integral

- the more rectangles we use under a curve, the better the approximation.
- given $f(x)$ defined from $x=a$ to $x=b$, **definite integral** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} f(x_k^*)\,\Delta x}_{\text{Riemann sum with } n \text{ intervals}}$$

} if exists, $f(x)$ is **integrable** on $[a,b]$

→ Eg: find $\int_2^5 (x^2-1)\,dx$

1. $\Delta x = \dfrac{b-a}{n} = \dfrac{5-2}{n} = \dfrac{3}{n}$ $[x_{k-1}, x_k]$

2. right endpt. $x_k^* = x_0 + k\Delta x$
$= 2 + k\dfrac{3}{n}$

left endpt. $x_k^* = x_{k-1}^*$ let $k = k-1$
$= x_0 + (k-1)\Delta x$
$= 2 + (k-1)\dfrac{3}{n}$

3. area of rectangles
↳ right endpt. $A_n = f(x_k^*)\,\Delta x$
$= f\left(2 + k\dfrac{3}{n}\right)\dfrac{3}{n}$

↳ left endpts $A_n = f(x_{k-1}^*)\,\Delta x$
$= f\left(2 + (k-1)\dfrac{3}{n}\right)\dfrac{3}{n}$

4. Write the Riemann sum

↳ $R_n = \sum_{k=1}^{n} f\left(2 + k\frac{3}{n}\right)\frac{3}{n} = \sum_{k=1}^{n}\left(\left(2+k\frac{3}{n}\right)^2 - 1\right)\frac{3}{n}$

$L_n = \sum_{k=1}^{n} f\left(2 + (k-1)\frac{3}{n}\right)\frac{3}{n} = \sum_{k=1}^{n}\left(\left(2+(k-1)\frac{3}{n}\right)^2 - 1\right)\frac{3}{n}$

Applying Limit Definition

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}} \qquad \boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$\rightarrow \int_2^5 (x^2-1)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= \lim_{n\to\infty} R_n$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\left(2+k\frac{3}{n}\right)^2-1\right)\frac{3}{n}$$

$$= \frac{3}{n}\left(2+\frac{3k}{n}\right)^2 - \frac{3}{n}$$

$$= \frac{3}{n}\left(4+\frac{12k}{n}+\frac{9k^2}{n^2}\right) - \frac{3}{n}$$

$$= \frac{12}{n}+\frac{36k}{n^2}+\frac{27k^2}{n^3}-\frac{3}{n}$$

$$= \frac{9}{n}+\frac{36k}{n^2}+\frac{27k^2}{n^3}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{9}{n}+\frac{36k}{n^2}+\frac{27k^2}{n^3}\right)$$

$$= \lim_{n\to\infty} \left(\frac{9}{n}\sum_{k=1}^{n} 1 + \frac{36}{n^2}\sum_{k=1}^{n} k + \frac{27}{n^3}\sum_{k=1}^{n} k^2\right)$$

$$= \lim_{n\to\infty} \left(\frac{9}{\cancel{n}}\cdot\cancel{n} + \frac{36}{n^2}\cdot\frac{n(n+1)}{2} + \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty} \left(9 + \left(18+\frac{18}{n}\right) + \left(9+\frac{27}{2n}+\frac{9}{2n^2}\right)\right)$$

$$= \lim_{n\to\infty} (9+18+9) = \boxed{36}$$

$$= \frac{36}{n^2} \times \frac{n(n+1)}{2}$$

$$= \frac{36n(n+1)}{2n^2}$$

$$= \frac{36n^2+36n}{2n^2}$$

$$= \frac{n^2\left(36+\frac{36}{n}\right)}{2n^2}$$

$$= \frac{36+\frac{36}{n}}{2}$$

$$= 18 + \frac{18}{n}$$

$$= \frac{27}{n^3} \times \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{27n(n+1)(2n+1)}{6}$$

$$= \frac{27n(2n^2+3n+1)}{6}$$

$$= \frac{54n^3+81n^2+27n}{6}$$

$$= 9n^3 + \frac{27n^2}{2} + \frac{9n}{2}$$

$$☆ = n^3\left(9+\frac{27}{2n}+\frac{9}{2n^2}\right)$$

$$\boxed{\therefore \int_2^5 (x^2-1)\,dx = 36}$$

General Steps for Modelling

1. divide the problem into small parts
2. approximate the physical quantity of each part
3. add up the approximations to approximate the physical quatity as a whole
4. take the limit of the # of divisions goes to infinity.

Total Displacement

- total displacement of object with $v(t)$ velocity is: $\int_a^b v(t)\,dt$

1. $\Delta t = \frac{b-a}{n}$ [units of time]   divide time interval $[a, b]$ into n equal subint. of lenght n.

↳ n subint. are ...

$[t_0, t_1], \dots, [t_{n-1}, t_n]$ ... and $\boxed{\Delta t = t_k - t_{k-1}}$

($t_0 = a$, $t_n = b$)

2. choose representative time t^*_k from int. $[t_{k-1}, t_k]$ and so the total displacement of object in int. $[t_{k-1}, t_k]$ is $v(t^*_k)\,\Delta t$ [metres]

we take a representative time t^*_k and take the object's velocity $v(t^*_k)$ over the time int, so that's why it's "$v(t^*_k)\,\underline{\Delta t}$"

3. Add up approximations: $\sum_{k=1}^{n} v(t^*_k)\,\Delta t$

4. Take the limit as # of subint. n goes to infinity

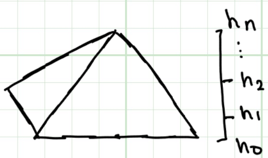
$$\text{Total Displacement} = \lim_{n\to\infty} \sum_{k=1}^{n} v(t^*_k)\,\Delta t$$

$$= \int_a^b v(t)\,dt$$

Volume by Slicing

- find volume of pyramid with height "a" m and side length "b" m

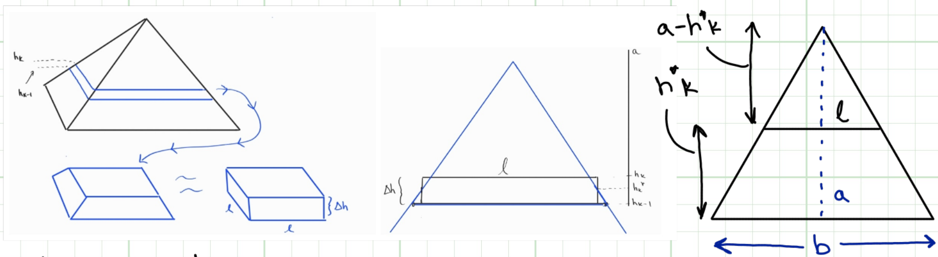
1. $\Delta h = \frac{a}{n}$ divide the height into n equal parts of height Δh



- n subint. are: $[h_0, h_1], \dots, [h_{n-1}, h_n]$ where $h_0 = 0$ and $h_n = a$

... and $\boxed{\Delta h = h_k - h_{k-1}}$

2. choose representative height h^*_k from subint. $[h_{k-1}, h_k]$ and the thickness as Δh to find volume of each slice ... what is base ℓ ?



with similar triangles...

$$\frac{a - h^*_k}{\ell} = \frac{a}{b} \Rightarrow \ell = a - h^*_k \left(\frac{b}{a}\right) \Rightarrow \ell = \frac{ba - bh^*_k}{a}$$

$$\ell = b - \frac{b}{a} h^*_k$$

Volume of slice $= \ell \times \ell \times \Delta h = \left(b - \frac{b}{a} h^*_k\right)^2 \Delta h$

3. Add up all the n slice volumes: $\sum_{k=1}^{n} \left(b - \frac{b}{a} h^*_k\right)^2 \Delta h$

4. Take the limit as # of height subint. n goes to infinity

$$\text{Volume} = \lim_{n \to \infty} \sum_{k=1}^{n} \left(b - \frac{b}{a} h^*_k\right)^2 \Delta h$$

$$= \int_0^a \left(b - \frac{b}{a} h\right)^2 dh$$

(2) $= \frac{125}{n^2} \cdot \frac{n(n+1)}{2} - 25$

$= \frac{125n^2 + 125n}{2n^2} - 25$

$= \frac{\cancel{n^2}\left(125 + \frac{125}{n}\right)}{\cancel{n^2}(2)} - 25$ $\longrightarrow$ $= \frac{125}{2} + \frac{125}{2n} - 20$

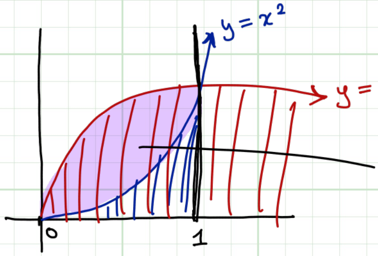
$$\int_0^5 v(t)\,dt = \lim_{n \to \infty}\left(-\frac{125}{3} - \cancelto{0}{\frac{375}{6n}} - \cancelto{0}{\frac{125}{6n^2}} + \frac{125}{2} + \cancelto{0}{\frac{125}{2n}} - 20\right)$$

$$= -\frac{125}{3} + \frac{125}{2} - 20$$

$$\boxed{= \frac{5}{6} \approx 0.833 \text{ m}}$$ ✓

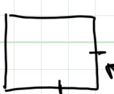
- $\int_0^5 \left|v(t)\right| dt = 8.1667 = 49/6$
- $-\int_0^1 v(t)dt + \int_1^4 v(t)dt - \int_4^5 v(t)d(t) = 8.1663 = 49/6$

2.



$\int_0^1 (\sqrt{x} - x^2)\,dx$

$$\boxed{\int_0^1 (\sqrt{x} - x^2)^2\,dx}$$



$(\sqrt{x}) - (x^2)$

1. $[0,1]$ $\quad \Delta x = \frac{1}{n}$ $\qquad$ height $= \sqrt{\frac{k}{n}} - \left(\frac{k}{n}\right)^2$

$x_k^* = x_0 + k\Delta x$ $\qquad$ Volume $= (\text{height})^2$

$= 0 + k\frac{1}{n}$ $\qquad$ $= \left(\left(\frac{k}{n}\right)^{1/2} - \left(\frac{k}{n}\right)^2\right)^2 \Delta x$

$= \frac{k}{n}$

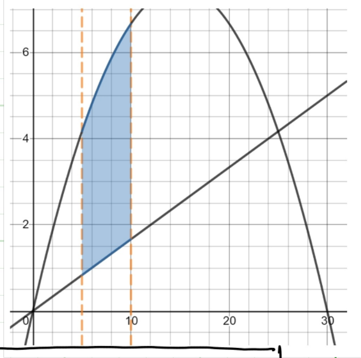
$$\text{Volume} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt{\frac{k}{n}} - \left(\frac{k}{n}\right)^2 \right)^2 \Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt{x_k^*} - (x_k^*)^2 \right)^2 \Delta x$$

$$\boxed{= \int_0^1 \left(\sqrt{x} - x^2\right)^2 dx}$$

Area Between 2 Curves

→ Find area of region btwn. $f(x) = \frac{1}{6}x$ and $g(x) = x - \frac{1}{30}x^2$ from $x = 5$ to $x = 10$.



1. n subintervals of length $\Delta x = \frac{10-5}{n} = \frac{5}{n}$

↳ n subint. are $[x_0, x_1], \dots, [x_{n-1}, x_n]$ ($x_0 = 5$, $x_n = 10$)

↳ $\boxed{\Delta x = x_k - x_{k-1}}$

2. how to approximate area of each subint?

↳ take x^*_k from $[x_{k-1}, x_k]$

↳ height will be $g(x^*_k) - f(x^*_k) \Rightarrow \boxed{\left(x^*_k - \frac{(x^*_k)^2}{30}\right) - \frac{x^*_k}{6}}$

↳ width will be $\Delta x = \frac{5}{n}$

∴ area of subint $= \left(x^*_k - \frac{(x^*_k)^2}{30} - \frac{x^*_k}{6}\right)\frac{5}{n}$

3. Add up areas: $\sum_{k=1}^{n}\left(x^*_k - \frac{(x^*_k)^2}{30} - \frac{x^*_k}{6}\right)\frac{5}{n}$

4. Take the limit as # of subint. n goes to infinity

$$\text{Area} = \lim_{n\to\infty}\sum_{k=1}^{n}\left(x^*_k - \frac{(x^*_k)^2}{30} - \frac{x^*_k}{6}\right)\frac{5}{n}$$

$$= \int_5^{10}\left(x - \frac{x}{30} - \frac{x}{6}\right)dx$$

→ Eg: find area of region bounded by curves $x=y$ and $x=6-y^2$

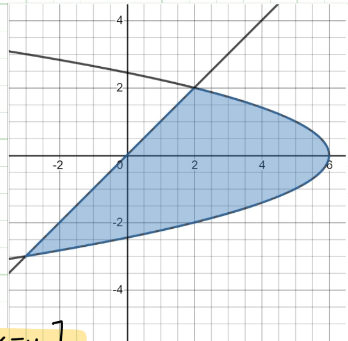
- note that curves intersect at $(-3,-3)$ and $(2,2)$
- we divide along y-axis instead of x-axis like before.

1. divide region btwn. $y=-3$ and $y=2$ into n subint.

$$\Delta y = \frac{2-(-3)}{n} = \frac{2+3}{n} = \frac{5}{n}$$

↳ subints: $[y_0, y_1], \dots, [y_{n-1}, y_n]$ ($y_0 = -3$, $y_n = 2$)

↳ $\boxed{\Delta y = y_k - y_{k-1}}$



2. Find area of each subint.

↳ height: [curve $x=6-y^2$] − [curve $x=y$]

choosing rep. pt. $y^*_k \in [y_{k-1}, y_k]$, height $= (6-(y^*_k)^2) - y^*_k$

↳ width is $\Delta y = \frac{5}{n}$

↳ area is height x width $= \left(6-(y^*_k)^2 - y^*_k\right)\frac{5}{n}$

3. Sum up all subint. on y-axis from $y=-3$ to $y=2$ $[-3,2]$ as...

$$\sum_{k=1}^{n} \left(6-(y^*_k)^2 - y^*_k\right)\frac{5}{n}$$

4. Take the limit as # of subint. n goes to infinity

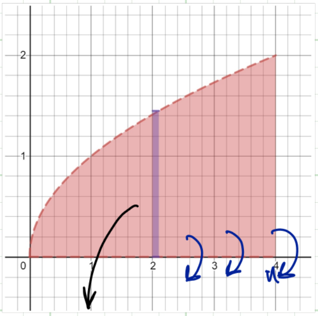
$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(6-(y^*_k)^2 - y^*_k\right)\frac{5}{n}$$

$$= \int_{-3}^{2} (6-y^2-y)\,dy$$

Volumes of Solids of Revolutions

- obtained by rotating an area on the xy-plane about an axis or line

→ Find volume of solid obtained by rotating the area btwn. $f(x) = \sqrt{x}$ and x-axis btwn. $x=0$ and $x=4$ about x-axis



1. n subint. of length $\Delta x = \frac{4-0}{n} = \frac{4}{n}$

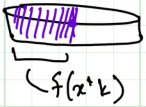
2. Find volume of disk, choose $x^*_k \in [x_{k-1}, x_k]$

↳ find height of rectangle : $f(x^*_k)$

"radius" thickness of disc

↳ find volume of disc : $\pi r^2 \Delta x$

area of circle "height" of disc



$\therefore$ Volume $= \pi (f(x^*_k))^2 \Delta x$

$$= \pi (\sqrt{x^*_k})^2 \frac{4}{n} = \pi x^*_k \frac{4}{n}$$

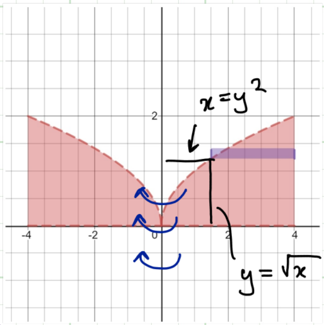
3. Sum up volumes: $\sum_{k=1}^{n} \pi x^*_k \frac{4}{n}$

4. Take the limit as # of subint. n goes to infinity

$$\text{Volume} = \lim_{n\to\infty} \sum_{k=1}^{n} \pi x^*_k \frac{4}{n}$$

$$= \int_0^4 \pi x \, dx$$

→ Eg: Find Volume of solid obtained by rotating the area btwn. $f(x) = \sqrt{x}$ and x-axis btwn. $x=0$ and $x=4$ about y-axis.



1. divide y-axis from $y=0$ to $y=2$ into n

$\Delta y = \frac{2-0}{n} = \frac{2}{n}$

↳ n subint: $[\overset{0}{y_0}, y_1], \dots, [y_{n-1}, \overset{2}{y_n}]$

↳ $\Delta y = y_k - y_{k-1}$

2. Find volume of washer ⟶ rotating rectangle about the y-axis and choose $y^*_k \in [y_{k-1}, y_k]$

outer radius: $x = 4$
inner radius: $x = y^2$

$$\therefore \text{volume} = \pi(4)^2 \Delta y - \pi\left((y^*_k)^2\right)^2 \Delta y$$

$$= \underbrace{\pi(16 - (y^*_k)^4)}_{\text{area of washer cross section}} \Delta y$$

area of washer cross section

"height" or "thickness" of washer

3. Sum up volumes: $\sum_{k=1}^{n} \pi(16 - (y^*_k)^4) \frac{2}{n}$

4. Take the limit as # of subint. n goes to infinity

$$\text{Volume} = \lim_{n \to \infty} \sum_{k=1}^{n} \pi(16 - (y^*_k)^4) \frac{2}{n}$$

$$= \int_0^4 \pi(16 - y^4)\, dy$$

$$= \int_0^4 \pi (4x - x^2)\,dx$$

$$V = \int_0^4 \pi (2\sqrt{x})^2\,dx - \int_0^4 \pi (x)^2\,dx$$

$$= \left[\int_0^4 \pi (2\sqrt{x})^2 - \int_0^4 \pi (x)^2 \right] dx$$

$$= \int_0^4 \left[\pi (2\sqrt{x})^2 - \pi (x)^2 \right] dx$$

$$= \int_0^4 \pi \left[(2\sqrt{x})^2 - (x)^2 \right] dx$$

$a^2 - b^2$

$(a+b)(a-b)$

$a = 2\sqrt{x} \quad b = x$

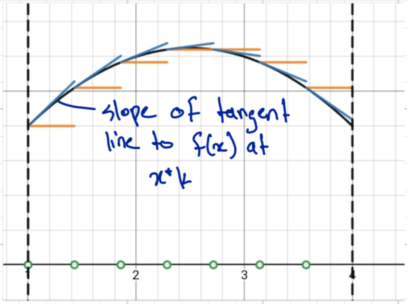
$$= \int_0^4 \pi \left[(2\sqrt{x} + x)(2\sqrt{x} - x) \right] dx$$

$$= \int_0^4 \pi \left(4x - x^2 \right) dx$$

E4-2: Arc Length and S.A. of a Solid of Revolution (Ch. 6.4)

Arc Length

- Find length "s" of the curve $f(x) = 5x - 2x^2$ on $[1, 4]$



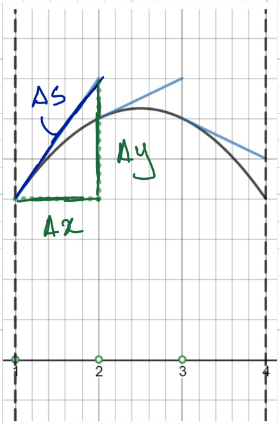
- orange lengths will always sum to 3, the length of int. [1, 4]
- blue lengths will give us the length of the arc "s"

1. n subintervals of length $\Delta x = \frac{4-1}{n} = \frac{3}{n}$

   ↳ subints: $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ ($x_0 = 1$, $x_n = 4$)

   ↳ $\boxed{\Delta x = x_k - x_{k-1}}$

2. To get length of arc Δs, think of it like hypotenuse of triangle



$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Δx → length of each subint.

- we know $\frac{\Delta y}{\Delta x}$ is the slope of the tangent line to $f(x) = 5x - 2x^2$ at the left endpt (x_0)

$$\frac{\Delta y}{\Delta x} = f'(x_0) \implies \Delta y = f'(x_0)\,\Delta x$$

$$\Delta s = \sqrt{(\Delta x)^2 + (f'(x_0)\,\Delta x)^2}$$

$$\Delta s = \sqrt{1 + (f'(x_0))^2}\;\Delta x$$

... since left endpt, $x^*_k = x_{k-1}$

$$\Delta s = \sqrt{1 + (f'(x^*_k))^2}\;\Delta x$$

approximates length of f(x) on the subint.

$$\boxed{\Delta s = \sqrt{1 + (5 - 2x_{k-1})^2}\;\frac{3}{n}}$$

3. Sum up all lengths (using left endpt. Riemann sum) as...

$$\sum_{k=1}^{n} \sqrt{1+(5+2x_{k-1})^2}\ \frac{3}{n}$$

4. Take the limit as # of subint. $\to \infty$

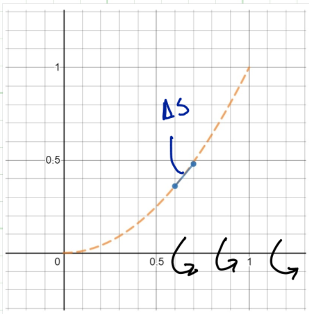
$$\text{Arc Length} = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1+(5+2x_{k-1})^2}\ \frac{3}{n}$$

$$\boxed{= \int_1^4 \sqrt{1+(5+2x)^2}\, dx}$$

$\leftarrow$ definite integral representing arc length.

Surface Area

* Consider $f(x) = x^2$ on $[0,1]$. Find S.A. of shape once we revolve $f(x)$.



1. n subint. of length $\Delta x = \frac{1-0}{n} = \frac{1}{n}$

↳ subints: $[\overset{0}{x_0}, x_1], \ldots, [x_{n-1}, \overset{1}{x_n}]$

↳ $\boxed{\Delta x = x_k - x_{k-1}}$

2. On each subint, choose left endpt.

$x^*_k = x_{k-1}$ ... $SA =$ circumference of circle $\times\ \Delta S$

↳ $f(x_{k-1})$

$$S.A = 2\pi r\,\Delta S = 2\pi f(x_{k-1})\,\Delta S = 2\pi f(x_{k-1}) \sqrt{1+(f'(x_{k-1}))^2}\ \Delta x$$

$$\boxed{= 2\pi (x_{k-1})^2 \sqrt{1+(2x_{k-1})^2}\ \frac{1}{n}}$$

3. Sum up all inter. $\sum_{k=1}^{n} 2\pi (x_{k-1})^2 \sqrt{1+(2x_{k-1})^2}\ \frac{1}{n}$

4. Take the limit as # of subint. $\to \infty$

$$\text{Surface Area} = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi (x_{k-1})^2 \sqrt{1+(2x_{k-1})^2}\ \frac{1}{n}$$

$$\boxed{= \int_0^1 2\pi x^2 \sqrt{1+(2x)^2}\, dx}$$

$\leftarrow$ definite integral representing S.A.

$$= \int_0^1 2\pi \sqrt{y} \sqrt{1 + \left(\tfrac{1}{2} y^{-1/2}\right)^2}\, dy$$

$$= \left(\tfrac{1}{2}\right)^2 \left(y^{-1/2}\right)^2 = \frac{1}{4}\,\frac{1}{y} = \frac{1}{4y}$$

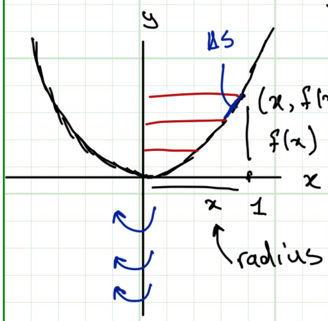
$$= \int_0^1 2\pi \sqrt{y} \sqrt{1 + \frac{1}{4y}}\, dy$$

$y = x^2$

$$= \int_0^1 2\pi \sqrt{x^2} \sqrt{1 + \frac{1}{4x^2}}\, dx^2$$

$$= \int_0^1 2\pi\, x \sqrt{1 + (4x^2)^{-1}}\, dx^2$$

$f(x) = x^2$



$$= \int_0^1 2\pi r \underbrace{\sqrt{1 + (f'(x))^2}\, dx}_{\Delta s}$$

$$= \int_0^1 2\pi x \sqrt{1 + (2x)^2}\, dx$$

$$\boxed{= \int_0^1 2\pi x \sqrt{1 + 4x^2}\, dx}$$

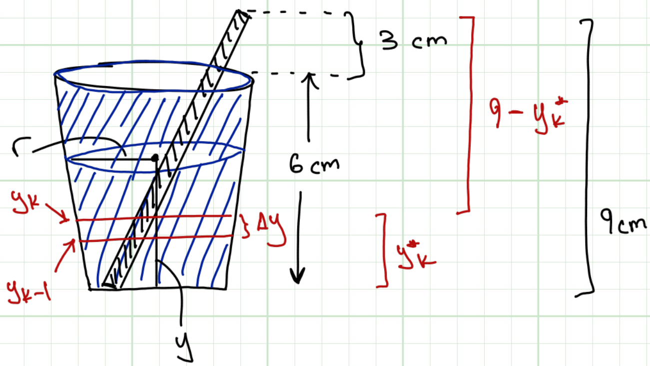
Work Done By Filling up Container

Work

$W = Fd$ ... work done by constant force over distance d

$W = mgh$ ... gravitational pot. energy inc. by mgh J.

→ Eg: Find work done to bring liquid to top of straw



$r = \frac{y}{6} + 3$

density: $\rho \frac{kg}{m^3}$

acc. due. grav: $g \frac{m}{s^2}$

1. $\Delta y = \frac{6-0}{n} = \frac{6}{n}$ ... divide region btwn. y=0 and 6 into n

2. choose left endpt. $y^*_k = y_0 + (k-1)\Delta y$

↳ all fluid has to travel "$9 - y^*_k$" up ← travel distance for liquids in this slice (in m)

$$(9 - y^*_k)\ \cancel{cm} \times \frac{1\ m}{100\ \cancel{cm}} = \boxed{0.01(9 - y^*_k)}$$

↳ Volume of slice is $= \pi r^2 h$

$$= \pi\left(\frac{y}{6} + 3\right)^2 \Delta y\ \cancel{cm}^3 \times \left(\frac{1\ m}{100\ \cancel{cm}}\right)^3$$

$$= \pi\left(\frac{y^*_k}{6} + 3\right)^2 \Delta y \qquad (0.01)^3$$

↳ Mass of slice

$\rho = \frac{m}{V} \Rightarrow m = \rho V$ $\qquad \therefore m = \rho (0.01)^3 \pi \left(\frac{y^*_k}{6} + 3\right)^2 \Delta y$

↳ Work done : $W = mgh$

$$W = P(0.01)^3 \pi \left(\frac{y^*k}{6} + 3\right)^2 \Delta y \quad \times \quad g \quad \times \quad 0.01(9 - y^*k)$$

$$= \left(0.01(9 - y^*k)\right) g \left(P(0.01)^3 \pi \left(\frac{y^*k}{6} + 3\right)^2 \Delta y\right)$$

3. Add up ...

$$\sum_{k=1}^{n} 0.01(9 - y^*k)\, g\, P(0.01)^3 \pi \left(\frac{y^*k}{6} + 3\right)^2 \Delta y$$

4. Take limit of Riemann sum...

$$W = \lim_{n \to \infty} \sum_{k=1}^{n} 0.01(9 - y^*k)\, g\, P(0.01)^3 \pi \left(\frac{y^*k}{6} + 3\right)^2 \Delta y$$

$$= \int_0^6 \left(0.01(9 - y)\right) g \left(P(0.01)^3 \pi \left(\frac{y}{6} + 3\right)^2\right) dy$$

Antiderivatives

- given $f(x)$, an antiderivative of $f(x)$ is a function $F(x)$ s.t.

$$\boxed{F'(x) = f(x)}$$

1. if $F(x)$ is antiderivative of $f(x)$ on I, then $F(x)+C$ is also an antiderivative for any constant C

   basically means adding constant gives you another antiderivative

2. if $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$ on I, then $\exists$ a constant C s.t. $F(x) = G(x) + C$

   basically means any 2 antiderivatives differ by a const. "C."

Eg 1

$F(x) = x^2$ and $G(x) = x^2+1$ are both antiderivatives of $f(x) = 2x$ b.c. $F'(x) = 2x = G'(x)$

Eg 2

5 antiderivatives of $f(x) = 2x$

$$\left.\begin{aligned} F_1(x) &= x^2-1 \\ F_2(x) &= x^2 \\ F_3(x) &= x^2+1 \\ F_4(x) &= x^2+2 \\ F_5(x) &= x^2+3 \end{aligned}\right\}$$

all antiderivatives are part of one family of function that differ by a const.

Indefinite Integral

- given $f(x)$, indefinite integral of $f(x)$ is:

$$\int f(x)\,dx = F(x) + C$$

C ← arbitrary constant

$F(x)$ ← antiderivative of $f(x)$

Eg 3: $\int 2x\,dx = x^2 + C$ constants

Eg 4: $\int \sin x\,dx = -\cos x + C$

Eg 5:

- acceleration of object given by $a(t) = 3t^2$
- find all functions that describe displacement $s(t)$

→ velocity $v(t) = \int a(t)\,dt$

$$= \int 3t^2\,dt$$

$$= t^3 + C_1$$ constant

→ displacement $s(t) = \int v(t)\,dt$

$$= \int t^3 + C_1\,dt$$

$$= \int \frac{1}{4}t^4 + C_1 t + C_2$$ another constant

Quiz Time

1. $\frac{d}{dx}\sin^2 x = \boxed{2\sin x\cos x}$ $\qquad f(x) = 2\sin x\cos x$

$\frac{d}{dx}\cos^2 x = 2\cos x - \sin x = -2\sin x\cos x$

$\frac{d}{dx}\sin^2 x + \frac{\pi}{2} = \boxed{2\sin x\cos x}$

$\frac{d}{dx} -\frac{1}{2}\cos 2x = -\frac{1}{2}(-\sin 2x)2 = \sin 2x = \boxed{2\sin x\cos x}$

$\frac{d}{dx} -\cos^2 x + 1 = -2\cos x(-\sin x) = \boxed{2\cos x\sin x}$

2. $F(1) = G(1) + 2$

$F(x) = G(x) + 2 \implies F(x) - G(x) = 2$

$F(3) - G(3) = 2$ $\qquad \boxed{\therefore 2}$

Initial Value Problems

Eg: height $y(t)$ must obey ODE $y'(t) = -9.8t$

- sol. to ODE: antiderivative of $y'(t) \to \boxed{y(t) = -4.9t^2 + \underset{\text{constant}}{C}}$
- if given initial cond., we can find particular solution.
  - given: $y(0) = 6$
  - sol. is $\boxed{y(t) = -4.9t^2 + 6}$

Terminology	Example	Interpretation
Differential Equation	$y'(t) = -9.8t$	A physical law about the velocity of a dropped object
General Solution	$y(t) = -4.9t^2 + C$	A general expression for the height of the dropped object
Initial Condition	$y(0) = 6$	Specific information about the initial height of the object
IVP Solution	$y(t) = -4.9t^2 + 6$	Models the height of an object dropped from an initial height of 6 metres

Eg 2: Given that $y(t) = 186 + Ce^t$ is gen. sol. to ODE $y' = y - 186$

find sol. to IVP: $y'(t) = y - 186$, $y(0) = 7$

$y(t) = 186 + Ce^t$

$y(0) = 186 + Ce^0$

$7 = 186 + C$

$C = 7 - 186$

$C = -179$

$\boxed{\therefore \text{sol. is } y(t) = 186 - 179e^t}$

Eg 3: acceler. at t: $a(t) = 8e^{2t}\ m/s^2$, at $t=0$, $s(0) = 0$, $v(0) = 0$

Determine $s(t)$ at t. displacement velocity

- $v(t)$ antider. of $a(t)$: $v(t) = 4e^{2t} + C_1$
  - since $v(0) = 0$ … $0 = 4 + C_1$, $C_1 = -4$

$\boxed{\therefore v(t) = 4e^{2t} - 4}$

- $s(t)$ antider. of $v(t)$: $s(t) = 2e^{2t} - 4t + C_2$
  - since $s(0) = 0$ … $0 = 2 + C_2$, $C_2 = -2$

$\boxed{\therefore s(t) = 2e^{2t} - 4t - 2}$

PCE : F3 - FTC PT. I | | C.V.I : 5.3

Fundamental Theorem of Calculus

Theorem: If $f(x)$ cont. on int. I, the function $F(x)$ ...

$F(x) = \int_{x_0}^{x} f(t)\,d(t)$ is antiderivative of $f(x)$ for all $x_0 \in I$

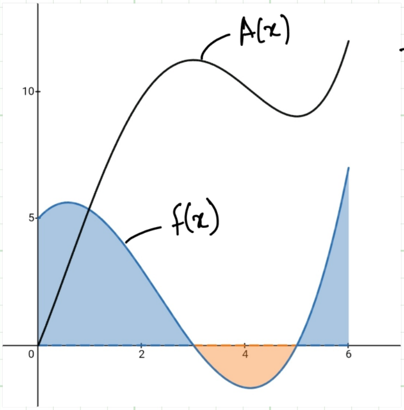
$$\frac{d}{dt}\int_{x_0}^{x} f(t)\,d(t) = f(x)$$

Area Function

function of x

"t" is just placeholder

* Given a func. $f(x)$, the area function is: $A(x) = \int_0^x f(t)\,d(t)$
* $F(x)$ is the net signed area under $f(t)$ from $t=0$ to $t=x$



→ $A(x) = \int_0^x f(t)\,d(t)$

1. $A(x)$ inc. when $f(x)$ is (+)
2. $A(x)$ dec. when $f(x)$ is (−)
3. $A(x)$ inc. fastest when $f(x)$ is most positive.

... as if $f(x)$ is derivative of $A(x)$

Eg: IVP $\begin{cases} \frac{dy}{dx} = e^{-x^2} \\ y(1) = 0 \end{cases}$

By FTC, $\int_{x_0}^{x} e^{-t^2}dt$ is an antiderivative that solves IVP

↳ $y(x) = \int_{x_0}^{x} e^{-t^2}dt$ solves $\frac{dy}{dx} = e^{-x^2}$

↳ we require $y(1) = 0 = \int_{x_0}^{1} e^{-t^2}dt$

* basically, this has to be true: $0 = \int_{x_0}^{1} e^{-t^2}dt$
* which means area under e^{-t^2} must be 0 ... possible interval is 1 to 1

$\therefore x_0 = 1$ and $\boxed{y(x) = \int_1^x e^{-t^2}dt \text{ solves IVP}}$

$$A(x) = \int_{-10}^{x} e^{-t^2}(t^2-1)\,dt$$

$$\frac{d}{dx}A(x) = \frac{d}{dx}\int_{-10}^{x} e^{-t^2}(t^2-1)\,dt$$

$$= \frac{d}{dx}\int_{-10}^{x} f(t)\,d(t) \quad , f(t) = e^{-t^2}(t^2-1)$$

$$= \frac{d}{dx} F(x)\Big|_{-10}^{x} \quad , F'(x) = f(t)$$

$$= \frac{d}{dx}F(x) - \frac{d}{dx}F(-10)$$

$$= F'(x)(1) - \cancel{F'(-10)}(0)$$

$$A'(x) = e^{-x^2}(x^2-1)$$

$$A''(x) = e^{-x^2}(-2x)(x^2-1) + e^{-x^2}(2x)$$

$$= -2x\left(e^{-x^2}(x^2-1) - e^{-x^2}\right)$$

$$0 = -2xe^{-x^2}(x^2-2)$$

DNE

$$x^2-2=0$$

$$x = \pm\sqrt{2}$$

	-2	$-\sqrt{2}$ | -1.2	-1 |	0 |	1 | 1.2	$\sqrt{2}$ | 2
$A'(x)$	(+)	$\boxed{(+)}$	(−)	(−)	(+)	$\boxed{(+)}$
$A''(x)$	(+)	$\boxed{(-)}$	(−)	(+)	(+)	$\boxed{(-)}$
$A(x)$	↗	$\boxed{↗}$	↘	↘	↗	$\boxed{↗}$

$(\sqrt{2}, \infty)$ ... both inc. & cc.d.

Fundamental Theorem of Calculus Pt 2

- Let $F(x)$ be antiderivative of $f(x)$ ... $F'(x) = f(x)$

Then...
$$\boxed{\begin{aligned}\int_a^b f(x)\,dx &= F(b) - F(a) \\ \int_a^b F'(x)\,dx &= F(b) - F(a)\end{aligned}}$$
... definite integral of a rate of change gives total net change

Eg 1: $v(t) = \int_0^4 t^2 - 4\,dt$. Find total displacement & distance

- an antiderivative of $v(t) = t^2 - 4$ is ... $s(t) = \frac{1}{3}t^3 - 4t$
- Total displacement $= \int_0^4 t^2 - 4\,dt$ ... area under $v(t)$ graph

$$= s(4) - \cancel{s(0)}^{\,0}$$
$$= \frac{1}{3}(4)^3 - 4(4) = \boxed{\frac{16}{3}\ m}$$

- Distance $= \int_0^4 |v(t)|\,dt$

$$= \int_0^2 -v(t)\,dt + \int_2^4 v(t)\,dt$$
... find t-int, make region below x-axis (+) by multiplying by (−)
$$= \int_0^2 -t^2 + 4\,dt + \int_2^4 t^2 - 4\,dt$$
$$= \left[\left(-\frac{1}{3}(2)^3 + 4(2)\right) - \cancel{\left(-\frac{1}{3}(0)^3 + 4(0)\right)}^{\,0}\right] + \left[\left(\frac{1}{3}(4)^3 - 4(4)\right) - \left(\frac{1}{3}(2)^3 - 4(2)\right)\right]$$
$$\boxed{= 16 \text{ metres}}$$

Eg 2: Solve IVP $\begin{cases} y' = 3t^2 + 1 \\ y(0) = 2 \end{cases}$

sol. $y(t)$ is equal to $y(0)$ plus the total change $y(t) - y(0)$

$$y(t) = y(0) + y(t) - y(0)$$
$$= 2 + \int_0^t y'(t)\,dt$$
$$= 2 + (t^2 + t)$$

$$\boxed{\therefore \text{sol. } y(t) = t^2 + t + 2}$$

Indefinite Integrals

- reverse chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$

→ Eg: evaluate $\int (2x+3) \cos(x^2+3x)\, dx$

derivative of $\sin x$ is $\cos x$

inside function (x^2+3x) has its derivative as a factor in the integrand

$$\Rightarrow \frac{d}{dx} \sin(x^2+3x) = \cos(x^2+3x)(2x+3)$$

$$\therefore \int (2x+3)\cos(x^2+3x) = \sin(x^2+3x) + C$$

→ Theorem: if g' cont. and f' cont. on range of g, then

$$\boxed{\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C}$$

substitution:

$$\boxed{u = g(x)}$$

☆ $\frac{du}{dx} = g'(x)$

$$\boxed{du = g'(x)\, dx}$$

$$\int f'(g(x))\, g'(x)\, dx = \int f'(u) \frac{du}{\cancel{dx}} \cancel{dx}$$

$$= \int f'(u)\, du$$

$$= f(u) + C$$

$$= f(g(x)) + C$$

→ Eg: $\int 4x(x^2+5)^3\, dx$

$u = x^2+5$

$du = 2x\, dx \Rightarrow dx = \frac{du}{2x}$

$$= \int 4x(u)^3 \frac{du}{2x}$$

$$= 2\int (u)^3\, du$$

$$= 2 \cdot \frac{1}{4}(u)^4 + C$$

$$= \frac{1}{2}(u)^4 + C$$

$$= \frac{1}{2}(x^2+5)^4 + C$$

$$\boxed{\therefore \int 4x(x^2+5)^3 dx = \frac{1}{2}(x^2+5)^4 + C}$$

→ Eg: $\int 8\cos(4x)\,dx$

$u = 4x$

$du = 4\,dx \Rightarrow dx = \frac{du}{4}$

$= \int 8\cos(u)\,\frac{du}{4}$

$= 2\int \cos(u)\,du$

$= 2\sin(u) + C$

$= 2\sin(4x) + C$

$$\boxed{\therefore \int 8\cos(4x)\,dx = 2\sin(4x) + C}$$

Eg: $\int \frac{x}{1+x^2}\,dx$

$u = 1 + x^2$

$du = 2x\,dx \Longrightarrow dx = \frac{du}{2x}$

$= \int \frac{x}{u}\,dx$

$= \int \frac{x}{u}\,\frac{du}{2x} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|x| + C = \frac{1}{2}\ln(1+x^2) + C$

$$\boxed{\therefore \int \frac{1}{1+x^2}\,dx = \frac{1}{2}\ln(1+x^2) + C}$$

<u>Definite Integrals</u>: if g' cont. and f' cont. on range of g, then using u-substitution $u = g(x)$

$$\boxed{\int_a^b f'(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du}$$

<u>L.S.</u> since $f'(g(x))\,g'(x)$ has antiderivative $f(g(x))$

$\int_a^b f'(g(x))\,g'(x)\,dx = f(g(b)) - f(g(a))$

$\Updownarrow$

<u>R.S.</u>

$\int_{g(a)}^{g(b)} f'(u)\,du = f(g(b)) - f(g(a))$

since $f(x)$ is antiderivative of $f'(x)$

Eg: $\int_1^4 x\sqrt{x+3}\,dx$

$u = x+3 \implies x = u-3$

$du = 1dx \implies dx = du$

$$= \int_{1+3}^{4+3} (u-3)\sqrt{u}\,du$$

antiderivative of $u^{\frac{3}{2}} - 3u^{\frac{1}{2}}$ is...

$$= \int_4^7 u\sqrt{u} - 3\sqrt{u}\,du$$

$$F(x) = \frac{2}{5}u^{5/2} - \frac{6}{3}u^{3/2}$$

$$= \int_4^7 u^{3/2} - 3u^{1/2}\,du$$

$$= F(x)\Big|_4^7 = F(7) - F(4)$$

$$= \left(\frac{2}{5}(7)^{5/2} - \frac{6}{2}(7)^{3/2}\right) - \left(\frac{2}{5}(4)^{5/2} - \frac{6}{2}(4)^{3/2}\right)$$