ECE311 Notes from Fall 2025

Please Note: the first 75 pages or so contain notes copied from lectures (i.e., straight up copy of what the prof wrote during lecture). That is fine, because the first 5 weeks mainly cover content from earlier courses done in first and second year ECE.

In my opinion, more useful notes are found in the **weekly recaps written for weeks 5-10**. These are summaries that explain the key concepts instead of fussing over specific examples.

Week 1 Lec 1 Sep 2, 2025

Systems

- A system transforms input signals to output signals
- " " is a function mapping input signals to output signals

can be linear, non-linear, first order, higher orders

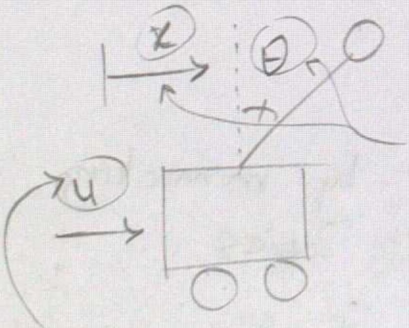
$$\xrightarrow{u} \boxed{f(x)} \xrightarrow{y}$$

input output

- SISO System: systems with one input and one output
  ↳ single-input single-output

- Examples:

Cart-Pendulum Model



output

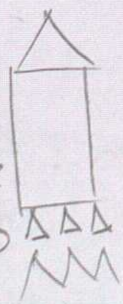
input multiple output

SIMO system

- input is force u,
  output ① can be x
  output ② can be θ

Rocket

rocket can be MIMO



$$\begin{matrix} u_1 \rightarrow \\ u_n \rightarrow \end{matrix} \boxed{\text{MIMO}} \begin{matrix} \rightarrow y_1 \\ \rightarrow y_n \end{matrix}$$

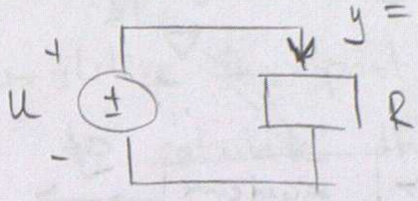
$y_1 = f(u_1, \cdots, u_n)$

$y_n = f(\underbrace{u_1, \cdots u_n}_{\text{can be only part of inputs}})$

complexity: interrelated input-output relations

System Control

- Open Loop Control:



$y = \text{current } I = \dfrac{u}{R}$

reference

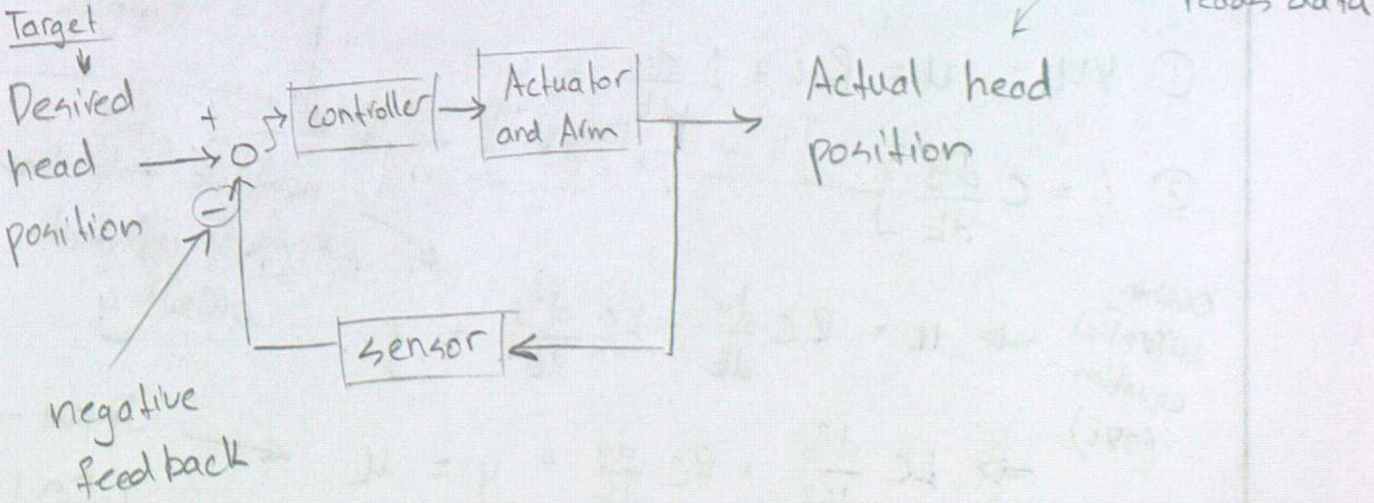
① open loop control : $I^* \cdot R = u$ ← input

② $R = 10\,\Omega \longrightarrow 12.5\,\Omega$ (say due to temp. or ageing)

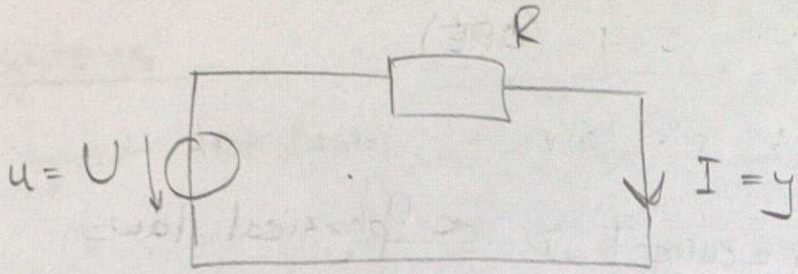
if you apply the input now, you wouldn't get the expected value anymore

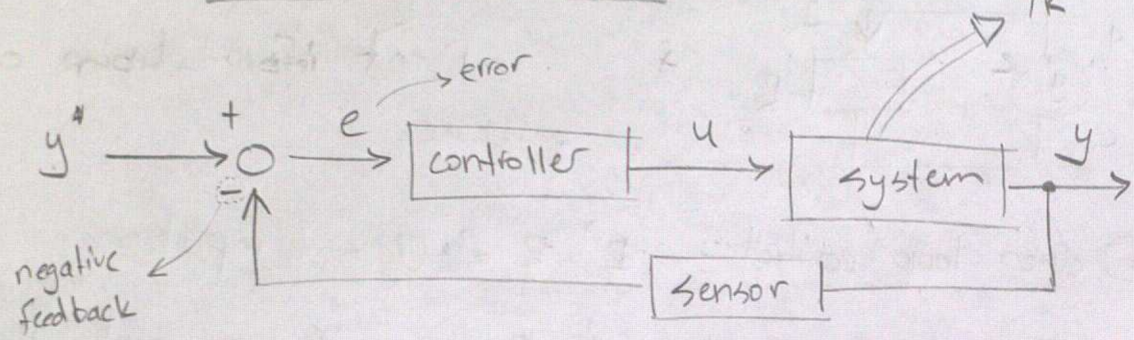
⇒ Closed loop control : measure I to check the difference if I < what I want, and if it is, then I'll have to increase u

Hard Disk Example



Week 1 Lec 2 Sep 4, 2025

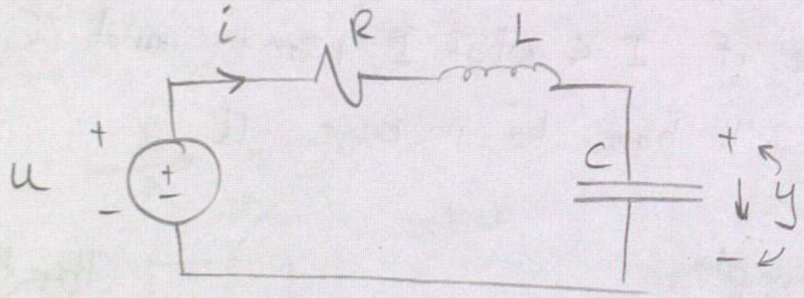




Define : in steady state, $e = 0$

RLC Series Ckt : Input - Output Model ⇒

shows the relationship b/w the input applied to a system and the output driven by the input



① KVL : $u = R \cdot i + L \frac{di}{dt} + y$

② $i = C \frac{dy}{dt}$

now we only have input u and output y

Ordinary differential equation (ODE)

$$\Rightarrow u = RC \frac{dy}{dt} + LC \frac{d^2y}{dt^2} + y$$

$$\Rightarrow LC \frac{d^2y}{dt^2} + RC \frac{dy}{dt} + y = u$$

Input - Output (IO) model

Ordinary Differential Equ. (ODE)

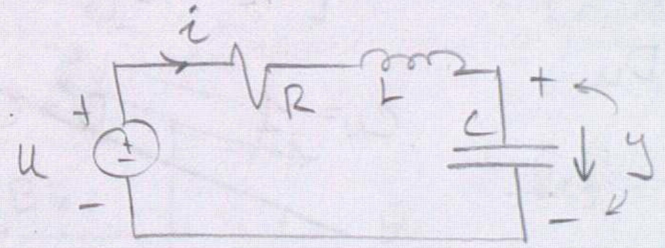
* The ODE's describe the dynamic performance of a physical system by utilizing the physical laws

  ↳ utilize the past and the current information to calculate the future

RLC series State-Space model

* Apply a set of first order ODE's to describe dynamic systems

  ↳ account for internal conditions (states).



select: $\begin{cases} x_1 = y \\ x_2 = i \end{cases}$

$$\Rightarrow \begin{cases} u = x_2 \cdot R + L \cdot \dot{x}_2 + x_1 \\ C \cdot \dot{x}_1 = x_2 \end{cases}$$

$$\begin{cases} \dot{x}_2 = -\frac{1}{L} x_1 - \frac{R}{L} x_2 + \frac{1}{L} u \\ \dot{x}_1 = \frac{1}{C} x_2 \end{cases}$$

- if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and since $\begin{cases} \dot{x}_2 = -\frac{1}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}u \\ \dot{x}_1 = \frac{1}{C}x_2 \end{cases}$

$$\begin{cases} \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

→ State-Space Model (SS)

$$\Rightarrow \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad A = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}$$

$$C = 1 \qquad D = 0$$

Linear System

- A linear system satisfies the properties of superpos. and homogeneity (scaling):

$$f(a_1 x_1 + a_2 x_2) = a_1 f(x_1) + a_2 f(x_2)$$

Ex: $f(x) = x^2 \Rightarrow$ not linear superposition $\quad f(x_1 + x_2) = (x_1 + x_2)^2 \neq f(x_1) + f(x_2) = x_1^2 + x_2^2$

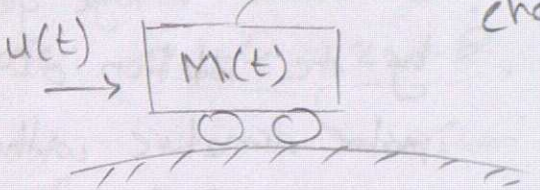
$f(x) = mx + b$ ← linear function but not linears system

Time Invariant System

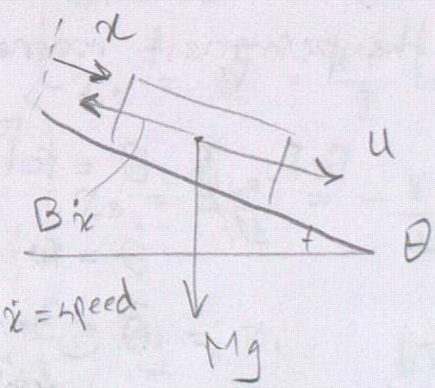
- A system whose properties do not change w/n time

→ systems that are Linear and time invariant, often called LTI system

Time-Varying System



mass of cart alway changes with time (rocket too)



Define: u = input y = output, speed

$y = \dot{x}$

$$M\ddot{x} = u - B\dot{x} + Mg\sin\theta$$

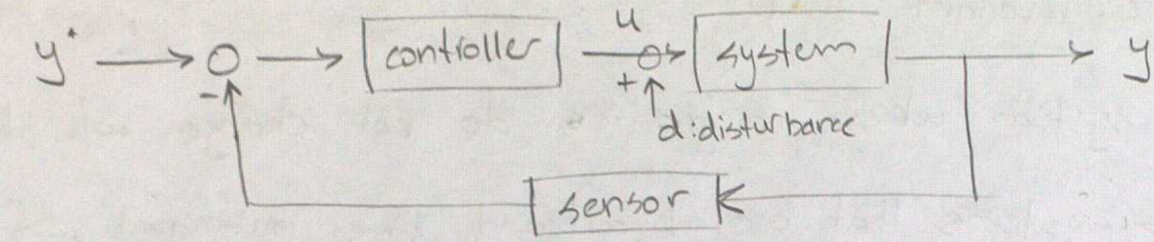
$$y = \dot{x}$$

nonlinear

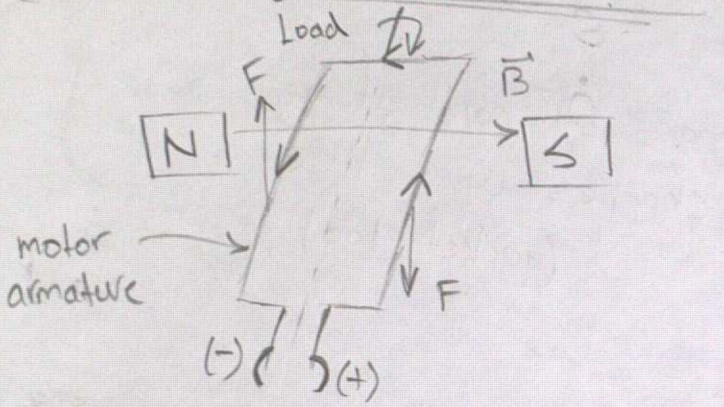
$$\therefore M\dot{y} = u - By + \overbrace{Mg\sin\theta}^{\text{nonlinear}}$$

$$\dot{y} = -\frac{B}{M}y + \frac{1}{M}\left(u + Mg\underbrace{\sin\theta}_{\text{"disturbance"}}\right)$$

Week 1 Lec 3 Sep 5, 2025

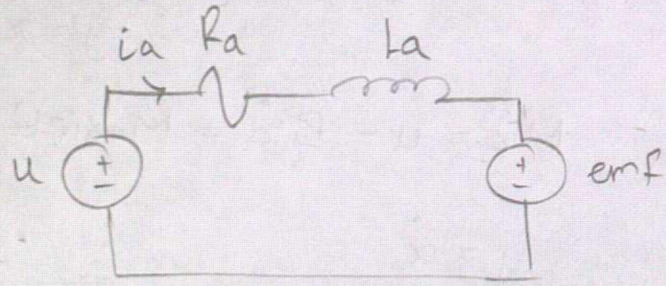


Permanent Magnet DC Motor



Back electromotif force (emf)

- an induced voltage generated by the rotation of the motor armature within the magnetic field produced by the permanent magnet



Def: θ = rot. angle

$\dot{\theta}$ = rot. speed

$\ddot{\theta}$ = angular acceleration

Back emf: $emf = k_e \cdot \dot{\theta}$ ← "back emf constant" → proportional to speed

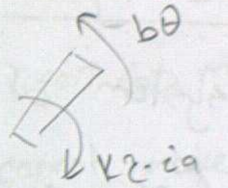
driving torque: $\tau = k_\tau \cdot i_a$ → "torque constant" → proportional to current

load torque: $b \cdot \dot{\theta}$ → constant from datasheet

Electric : $u = R_a \cdot i_a + L_a \dfrac{di_a}{dt} + K_e \cdot \dot{\theta}$

Mechanical : $I \cdot \ddot{\theta} = K_\tau \cdot i_a - b\dot{\theta}$

rotational inertia



State Space Model

$$\begin{cases} \text{① position control: } & y = \theta \\ \text{② speed control: } & y = \dot{\theta} \end{cases}$$

State: $\underbrace{x_1 = \theta\ ,\ x_2 = \dot{\theta}}$, $x_3 = i_a \implies x = \begin{bmatrix} \theta \\ \dot{\theta} \\ i_a \end{bmatrix}$

→ from these: $\dot{x}_1 = x_2$

$$\begin{cases} \dot{x}_1 = x_2 & \leftarrow \text{definition} \\ \dot{x}_2 = \ddot{\theta} = \dfrac{K_\tau}{I} i_a - \dfrac{b}{I}\dot{\theta} = -\dfrac{b}{I}x_2 + \dfrac{K_\tau}{I}x_3 & \text{mechanical} \\ \dot{x}_3 = \dfrac{di_a}{dt} = -\dfrac{K_e}{L_a}\dot{\theta} - \dfrac{R_a}{L_a}i_a + \dfrac{1}{L_a}u & \leftarrow \text{electric} \end{cases}$$

(ss)

$$\begin{cases} \text{①} & y = x_1 \\ \text{②} & y = x_2 \end{cases}$$

LTI system in state-space matrix form

Matrix Form

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -b/I & K_\tau/I \\ 0 & -K_e/L_a & \dfrac{-R_a}{L_a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} \cdot u$$

① $y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ ② $y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

LTI System (SS Form)

- A system of ODE of the form:

$$\dot{x} = a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n + b_i u \qquad (i = 1, 2, \cdots, n)$$

$$y = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n + du$$

recall RLC motor

Matrix Form:

$$\dot{X} = Ax + Bu$$
$$y = Cx + Du$$

(SISO system)

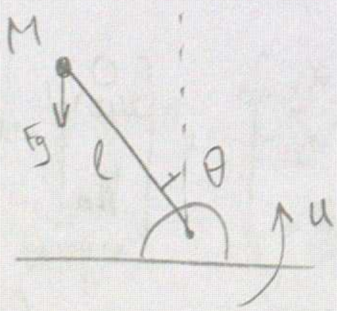
Input - Output Form:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{du^m}{dt^m} + \cdots + b_0 u \qquad (m \leq n)$$

LC ckt eg: $LC \dfrac{d^2 y}{dt^2} + RC \dfrac{dy}{dt} + y = u$

Non-Linear System (SS Form)

→ inverted pendullum, define: M - mass, l - length



θ - angle, $\dot{\theta}$ - speed

$\ddot{\theta}$ - acceleration

I - rotational inertia

input: u: motor torque

output: θ: angle

$I\ddot{\theta} = u + Mg\sin\theta \cdot l$

choose $x_1 = \theta \quad x_2 = \dot{\theta}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{\theta} = \frac{1}{I}u + \frac{mgl\sin\theta}{I} \\ y = \theta \end{cases}$$

non-linear SS model

→ cannot make it as

$\dot{x} = Ax + Bu$

$y = Cx + Du$

→ instead it is

$\dot{x} = f(x, u)$

$y = h(x, u)$

→ Non-linear system in state space form

$\dot{x}_i = f_i(x_1, x_2, \cdots x_n, u)$

$y = h(x_1, x_2, \cdots x_n, u)$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

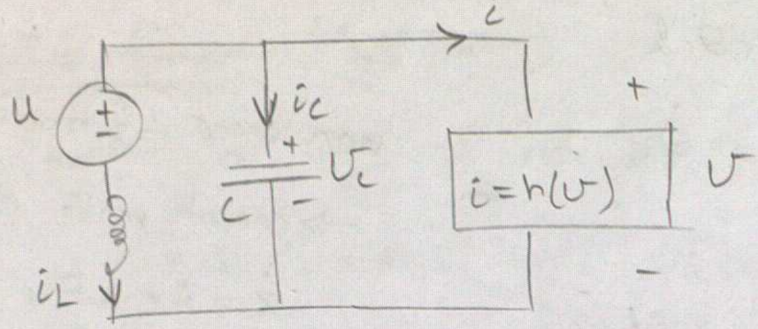
$$f(x, u) = \begin{bmatrix} f_1(\dots) \\ \vdots \\ f_n(\dots) \end{bmatrix} \rightarrow \begin{matrix} \dot{x} = f(x, u) \\ y = (x, u) \end{matrix}$$

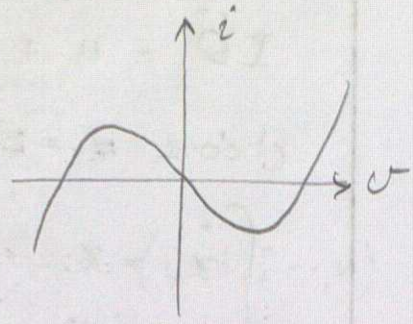
time-invariant : f_i don't depend on t

non-linear : f_i or h_i are not linear

Week 2 Tutorial 1 Sep 9, 2025

Q1.1





$h(v) = -v + \frac{1}{3}v^3$

Output is voltage accross capacitor

i) write model in terms of V_c, i_L, u

ii) state space $x_1 = i_L$, $x_2 = V_c$

$i_c = C\frac{dV_c}{dt}$

$V_L = L\frac{di_L}{dt}$

i) KCL @ top node : $i_L + i_c + i = 0$

$\rightarrow i_L + C\frac{dV_c}{dt} + i = 0$ $\dot{x}_2$

$i_L + C\frac{dV_c}{dt} - v + \frac{1}{3}v^3 = 0$ (eq I)

KVL @ left loop: $V_c = u + V_L$

$\rightarrow V_c = u + L\frac{di_L}{dt}$ (eq. II) $\dot{x}_1$

ii) This is only for LTI (key = linear):

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \text{LTI}$$

"the derivatives of the states in terms of the states and the input and the output likewise"

$x_1 = i_L \quad x_2 = v_C \quad (= v)$ ← order of states matters

non-linear state space model
$$\begin{cases} \dot{x}_1 = \left(\frac{1}{L}\right)x_1 - \left(\frac{1}{L}\right)u \\ \dot{x}_2 = -\left(\frac{1}{C}\right)x_1 + (1/C)\,x_2 - (1/3C)\,x_2^3 \\ y = x_2 \end{cases}$$

Second way: $x_1 = v_C \quad x_2 = \dot{v}_C$

$$i_L + C\frac{dv_C}{dt} + h(v) = 0$$ ↓ derivative (chain rule for $h(v)$)

$$\frac{di_L}{dt} + C\frac{d^2v_C}{dt^2} + h'(v)\cdot\frac{dv_C}{dt} = 0 \quad \text{(multiply by } L)$$

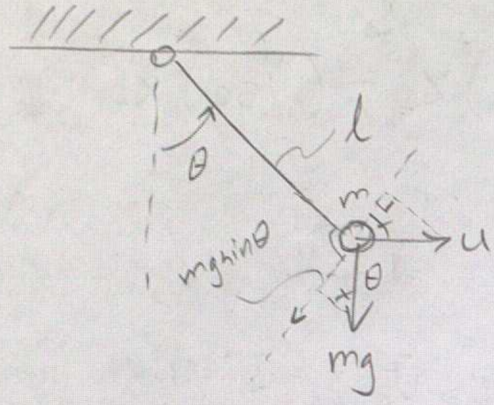
$$\underbrace{L\frac{di_L}{dt}}_{v_L = v_C - u} + LC\frac{d^2v_C}{dt^2} + L\underbrace{h'(v)}_{= -1 + v_C^2}\frac{dv_C}{dt} = 0$$

$$(v_C - u) + LC\,\ddot{v}_C + L(-1 + v_C^2)\,\dot{v}_C \quad (\text{since } \dot{x}_1 = x_2)$$

$$\underbrace{\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -(1/LC)\,x_1 + \frac{1}{C}(1 - x_1^2)\,x_2 + \frac{1}{LC}\,u \\ y = x_1 \end{cases}}_{\text{nonlinear s.s.}}$$

Week 2 Lecture 1 (Lec. 2.4) Sep 9, 2025

Non Linear System: Crane Model



Define: I - rotational inertia $= ml^2$

θ - angle

Input: u Output: $y = \theta$

$\Rightarrow$ recall $I\ddot{\theta} = \sum \text{Moments}$

$$I \cdot \ddot{\theta} = -mg \cdot \sin\theta\, l + u \cdot \cos\theta\, l$$

$$ml^2 \cdot \ddot{\theta} = -mg \cdot \sin\theta\, l + u \cdot \cos\theta\, l$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{u}{ml}\cos\theta \longleftarrow \text{non-linear}$$

Choose $x_1 = \theta$ $x_2 = \dot{\theta}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\left(g/l\right)\sin x_1 + \left(u/ml\right)\cos x_1 \\ y = x_1 \end{cases}$$

we have to linearize the sin and cos at operating point

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -(g/l)\sin(x_1) + (u/ml)\cos(x_1) \end{bmatrix} = f(x, u)$$

Equilibrium Condition

Define: consider a (NL) system and fix a constant control input $\bar{u}$. Then, a state $\bar{x}$ is said to be an equilibrium of this (NL) system if $f(\bar{x}, \bar{u}) = 0$

$\Rightarrow$ Explain: set $u(t) \equiv \bar{u}$ and initiate

and initiate (NL) at $\bar{x}$: set $x(0) = \bar{x}$

Then if $\bar{x}$ is an equilibrium in the sense of the definition above, the solution of $x(t) \equiv \bar{x}$ because

1) $x(t)$ satisfies the initial condition, $x(0) = \bar{x}$

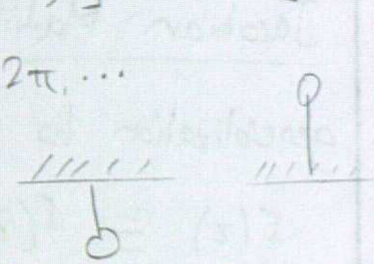
2) $x(t)$ is a solution of (NL) $\frac{dx}{dt} = 0 = f(\bar{x}, \bar{u}) = 0$

Example: set $\bar{u} = 0$, find all equilibria of the crane

$$f(\bar{x}, \bar{u}) = 0 \Rightarrow f(\bar{x}, 0) = \begin{bmatrix} \bar{x}_2 \\ -\frac{g}{l}\sin(\bar{x}_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{matrix} -1 \\ -2 \end{matrix}$$

2: $\sin(\bar{x}_1) = 0 \Rightarrow \bar{x}_1 = k\pi = 0, \pm\pi, \pm 2\pi, \cdots$

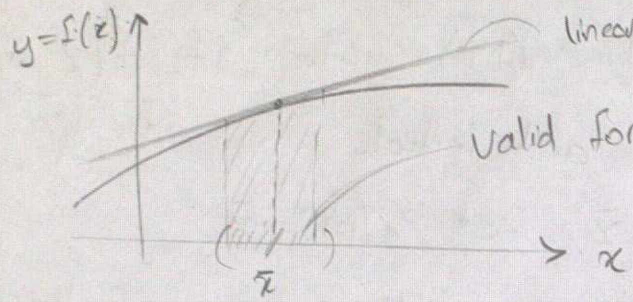
$\Rightarrow$ All equilibria are $\bar{x} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}$



1. set $u = 0$ and plug into eq^{n}s
2. set all eq^{n}s $= 0$
3. solve for what x's get you that 0

Linearization of Non-linear Fun^n

- $f: \quad f(x) \cong f(\bar{x}) + f'(\bar{x})(x-\bar{x})$ → valid for x near $\bar{x}$



linearized function of $f(x)$

valid for x near $\bar{x}$

Taylor Series

$$f(x) = \sum_{0}^{\infty} C_n (x-x_0)^n \quad \text{for } C_n = \frac{f^{(n)}(x_0)}{n!}$$

$$= \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{\text{keep}} + \underbrace{\frac{f''(x_0)}{2}(x-x_0)^2 + \cdots}_{\text{discard from approx.}}$$

Jacobian Matrix

generalization to vector functions:

$$\underset{n\times 1}{f(x)} \cong \underset{n\times 1}{f(\bar{x})} + \underset{n\times m}{\left[\frac{\partial f}{\partial x}(\bar{x})\right]} \cdot \underset{m\times 1}{(x-\bar{x})}$$

size of matrix in general case

where $\frac{\partial f}{\partial x}(\bar{x})$ is the Jacobian Matrix of function $f(x)$ at $\bar{x}$

☆ fill this out

$$\left[\frac{\partial f}{\partial x}(\bar{x})\right]_{ij} = \frac{\partial f_i}{\partial x_j}(\bar{x}) \qquad \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix}$$

→ further generalization.

→ Linearize at $(\bar{x}, \bar{u})$

$$\dot{x} = f(x,u) \cong \underbrace{f(\bar{x},\bar{u})}_{=0 \text{ since that is def}^n \text{ of equilibrium pnt}} + \underbrace{\left[\frac{\partial f}{\partial x}(\bar{x},\bar{u})\right]}_{A}(x-\bar{x}) + \underbrace{\left[\frac{\partial f}{\partial x}(\bar{x},\bar{u})\right]}_{B}\cdot(u-\bar{u})$$

$$y = h(x,u) \cong h(\bar{x},\bar{u}) + \underbrace{\left[\frac{\partial h}{\partial x}(\bar{x},\bar{u})\right]}_{C}(x-\bar{x}) + \underbrace{\left[\frac{\partial h}{\partial x}(\bar{x},\bar{u})\right]}_{D}(u-\bar{u})$$

Define:

$$\tilde{x} = x - \bar{x}, \quad \tilde{u} = u - \bar{u}, \quad y = y - h(\bar{x},\bar{u})$$

$$\begin{cases} \dot{\tilde{x}} = A\tilde{x} + B\tilde{u} \\ \tilde{y} = C\tilde{x} + D\tilde{u} \end{cases}$$

Example: (from earlier $\frac{11}{6}$)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}\sin(x_1) + \frac{1}{l}\cos(x_1)\cdot u \end{cases}$$

equilibrium point:

$$\bar{x} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}$$

Linearize at $\bar{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$, $\bar{u} = 0$

Define: $\tilde{x} = x - \bar{x} = \begin{bmatrix} x_1 - \pi \\ x_2 \end{bmatrix}$

$\tilde{u} = u - \bar{u} = u$

$\tilde{y} = y - h(\bar{x}, \bar{u}) = x_1 - \bar{x}_1 = x_1 - \pi$

$$f(x,u) = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) + \frac{1}{l}\cos(x_1)\cdot u \end{bmatrix} \qquad \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$\therefore A = \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(x_1) - \frac{1}{l}\sin(x_1)\cdot u & 0 \end{bmatrix}\Bigg|_{\substack{x=\bar{x} \\ u=\bar{u}}}$$

$$= \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}(\bar{x}, \bar{u}) = \begin{bmatrix} 0 \\ \frac{1}{l}\cos x_1 \end{bmatrix}\Bigg|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \begin{bmatrix} 0 \\ -1/l \end{bmatrix}$$

~~fings~~ sign

$C = [1 \;\; 0]$, $D = 0$

$$\Longrightarrow \begin{cases} \dot{\tilde{x}} = \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix}\tilde{x} + \begin{bmatrix} 0 \\ -1/l \end{bmatrix}\tilde{u} \\ \tilde{y} = [1 \;\; 0]\,\tilde{x} \end{cases}$$

Week 2 Lec 2 Sep 11, 2025

Linearized at $(\bar{x}, \bar{u})$

$$\dot{x} = f(x,u) \cong \overset{0}{f(\bar{x},\bar{u})} + \underbrace{\left[\frac{\partial f}{\partial x}(\bar{x},\bar{u})\right]}_{A}(x-\bar{x})$$

$$+ \overbrace{\left[\frac{\partial f}{\partial u}(\bar{x},\bar{u})\right]}^{B}(u-\bar{u})$$

$$y = h(x,u) \cong h(\bar{x},\bar{u}) + \overbrace{\left[\frac{\partial h}{\partial x}(\bar{x},\bar{u})\right]}^{C}(x-\bar{x})$$

$$+ \underbrace{\left[\frac{\partial h}{\partial u}(\bar{x},\bar{u})\right]}_{D}(u-\bar{u})$$

Define: $\tilde{x} = x - \bar{x}$, $\tilde{u} = u - \bar{u}$, $\tilde{y} = y - h(\bar{x},\bar{u})$

$$\Rightarrow \begin{cases} \dot{\tilde{x}} = A\tilde{x} + B\tilde{u} \\ \tilde{y} = C\tilde{x} + D\tilde{u} \end{cases} //$$

Laplace Transform

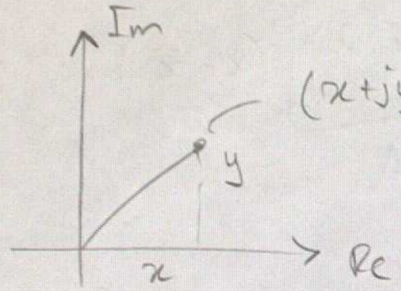
the $\mathcal{L}\{\}$ of a signal (function f) is the function

$F(s) = \mathcal{L}\{f\}$ defined by

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$ where "s" is a complex number

Note: only sensible if this integral exist

s plane: $s = Re + j\,Im$

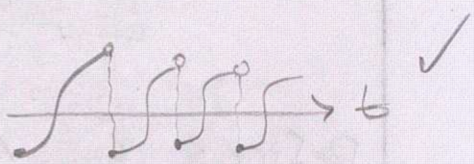


complex exponential:

$$e^{z} = e^{x+jy} = e^{x}(\cos y + j\sin y)$$

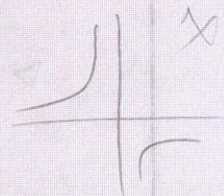
The Laplace transforms integrals and ODE's into relatively easily-solvable algebraic equations

Existence of the Integral



1) Piecewise Continuous (PWC)

↳ $f(x)$ is defined at every point except for a finite number of points where it might be discontinuous.



☆ exam might ask verif

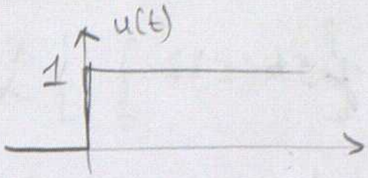
↳ at these discontinuous, $f(x)$ has finite right hand and left hand limits.

2) Exponentially Bounded

↳ $\exists$ an $M \geq 0$, $a \in \mathbb{R}$ s.t.

$$|f(t)| \leq M \cdot e^{at}, \quad \text{for } t \geq 0$$

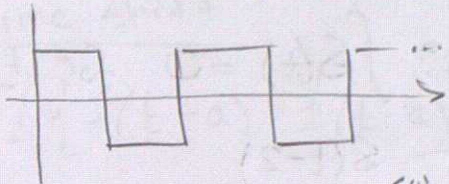
1) Example: unit step $u(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases}$



pwc ✓

$|f(t)| \leq 1 \leq \underset{M}{1} \cdot e^{\underset{a}{0} t}$

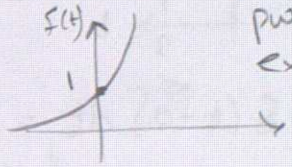
2) Example: square wave:



pwc ✓

exponentially bounded ✓

<u>Ex:</u> $f(t) = e^t$

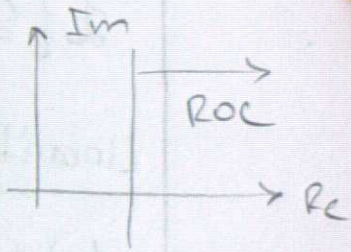


pwc ✓

exponentially bounded ✓

$$\mathcal{L}\{e^t\} = \int_0^\infty e^t e^{-st}\, dt = \int_0^\infty e^{(1-s)t}\, dt = \left[\frac{1}{1-s} e^{(1-s)t}\right]_0^\infty$$

$$= 0 - \frac{1}{1-s} e^0 = -\frac{1}{1-s} = \boxed{\frac{1}{s-1}}$$



For $Re\{s\} > 1$, the integral converges

<u>Ex:</u> $\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}$

<u>Ex:</u> $\mathcal{L}\{\cos\omega t\}$ $\Rightarrow$ know: $\cos\omega t = \frac{1}{2} e^{j\omega t} + \frac{1}{2} e^{-j\omega t}$

yadda yadda

$$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2} \quad //$$

Week 2 Lecture 3 Sep 12, 2025

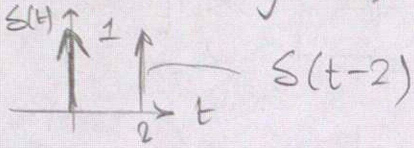
Recap:

$$F(s) = \int_0^{\infty} f(t) e^{-st} \, dt \quad \text{where } s \in \mathbb{C}$$

$\Rightarrow$ only sensible if integral exists

* Define: $\delta(t)$ Dirac function $= \int \delta(t) = 0$ for $t \neq 0$

and $\int_{-\infty}^{\infty} \delta(t) = 1$



* Sifting property: $\int_{-\infty}^{\infty} f(t)\, \delta(t-a)\, dt = f(a)$

$$\mathcal{L}(\delta(t)) = \int_0^{\infty} \delta(t) e^{-st} \, dt = e^{-s(0)} = 1$$

$$\mathcal{L}\{\delta(t-2)\} = \int_0^{\infty} \delta(t-2) e^{-st} \, dt = e^{-2s}$$

Linearity

$$\mathcal{L}\{C_1 f(t) + C_2 f(t)\} = C_1 \mathcal{L}\{f(t)\} + C_2 \mathcal{L}\{f(t)\}, \quad C_1, C_2 \in \mathbb{R}$$

Differentiation:

$$\mathcal{L}\left\{\frac{d}{dt} f(t)\right\} = sF(s) - \lim_{t \to 0^-} f(t)$$

$$\mathcal{L}\left\{\frac{d^k}{dt} f(t)\right\} = s^k F(s) - s^{k-1} f(0^-) - s^{k-2} f'(0^-) - \cdots$$

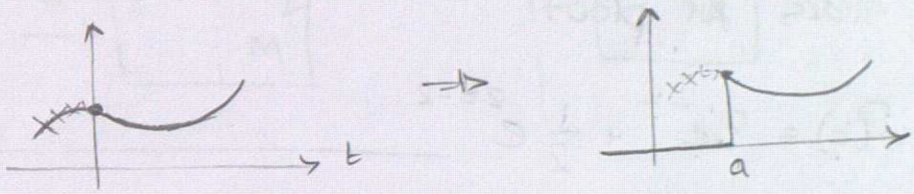
Integration:

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s) \quad \text{and since} \quad \mathcal{L}\{\delta(t)\} = 1$$

$$\mathcal{L}\{u(t)\} = \mathcal{L}\{1 \cdot t\} = \frac{1}{s} \quad \text{and since} \quad \frac{d}{dt}u(t) = \delta(t)$$

Time shift:

$$\mathcal{L}\{f(t-a)\cdot 1(t-a)\} = e^{-as}F(s) \quad \text{for } t \geq 0$$



Multiplication by t

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$$

Shift in s-dom

☆ fill it up...

$$\mathcal{L}\{e^{at}f(t)\} =$$

Convolution

- convolution of a signal $f(t)$ and $g(t)$:

$$h(t) = f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$\Rightarrow h(t) = f(t) * g(t) = g(t) * f(t)$

$$\mathcal{L}\{h(t)\} = \mathcal{L}\{f(t) * g(t)\} = F(s)G(s)$$

$\Rightarrow$ see lec slides for proof

Example: $f(t) = 3e^{-t} + \frac{1}{2}e^{2t-2}$

$$\mathcal{L}\{3e^{-t}\} = 3\,\mathcal{L}\{e^{-t}\} = \frac{3}{s+1}$$

$$\mathcal{L}\{\tfrac{1}{2}e^{2t-2}\} = \tfrac{1}{2}e^{-2}\mathcal{L}\{e^{2t}\} = \tfrac{1}{2}e^{-2}\cdot\frac{1}{s-2}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{3}{s+1} + \frac{1}{2e^2(s-2)} \quad //$$

Example: $f(t) = te^{-2t} + \frac{1}{2}t\sin(t)$

$$\mathcal{L}\{te^{-2t}\} = -\frac{d}{ds}\left[\frac{1}{s+2}\right] = \frac{1}{(s+2)^2}$$

$$\mathcal{L}\{\tfrac{1}{2}t\sin t\} = -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = \frac{1}{2}\cdot\frac{1}{(s^2+1)^2}\cdot 2s = \frac{s}{(s^2+1)^2}$$

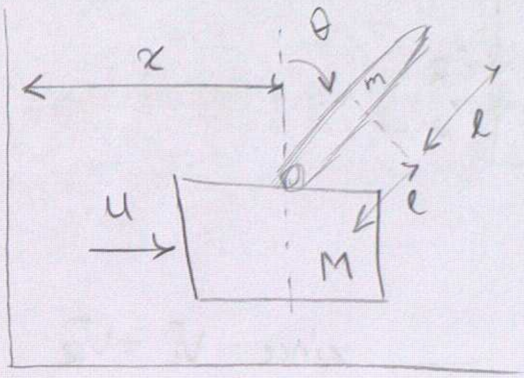
$$\mathcal{L}\{f(t)\} = \frac{1}{(s+2)^2} + \frac{s}{(s^2+1)^2} \quad //$$

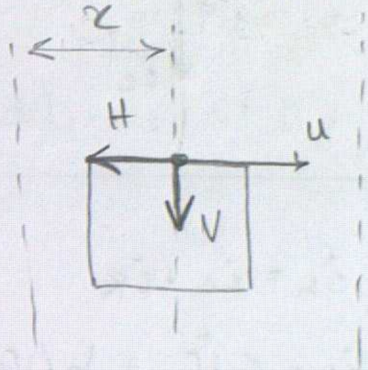
Week 3 Tutorial 2

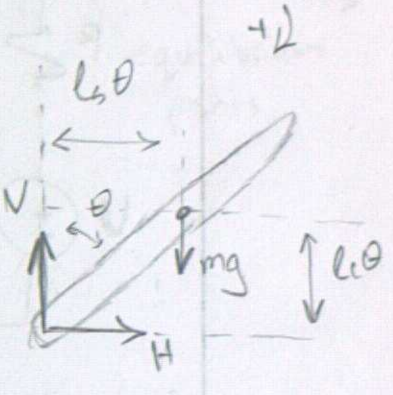
Problem 1.3

$l s\theta =$ " $l \sin\theta$ "

- write E.O.M of the system assuming:
  - the c.o.g. of pendulum @ centre
  - neglect moment of inertia of wheels







1) translation of cart

$$M\ddot{x} = u - H$$

2) rotation of pend.

$$I\ddot{\theta} = V l s\theta - H l c\theta$$

3) trans. of pend.

$$H = m\frac{d^2}{dt^2}\{x + l s\theta\}$$

$$H = m\ddot{x} + \frac{d}{dt}\{ml\dot{\theta}c\theta\}$$

$$= m\ddot{x} + ml[\ddot{\theta}c\theta - \dot{\theta}^2 s\theta]$$

$$V - mg = m\frac{d^2}{dt^2}\{l c\theta\}$$

$$V = ml(-c\theta\dot{\theta}^2 - s\theta\ddot{\theta}) + mg$$

$$M\ddot{x} = -m\ddot{x} - ml\theta\ddot{\theta} + mls\theta\dot{\theta}^2 + u \quad \text{— (eq I)}$$

$$I\ddot{\theta} = ml^2[-s\theta c\theta\cancel{\dot{\theta}^2} - s^2\theta\cdot\ddot{\theta} + s\theta c\cancel{\theta\cdot\dot{\theta}^2}$$

$$-c^2\theta\ddot{\theta}] - ml\ddot{x}c\theta + mgls\theta$$

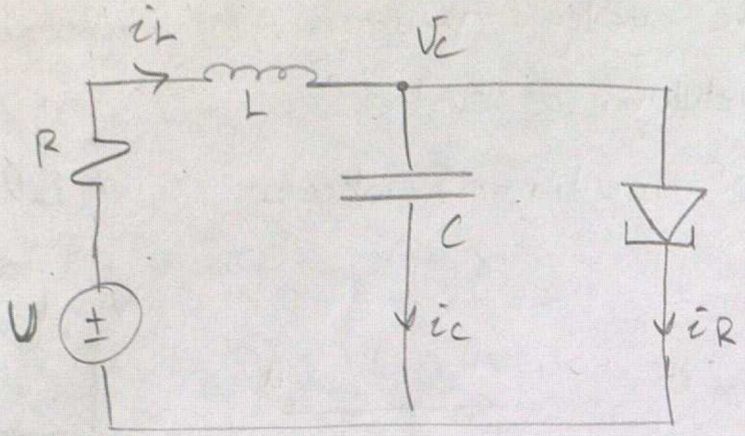
$$I\ddot{\theta} = ml^2\ddot{\theta} - ml\ddot{x}c\theta + mgls\theta \quad \text{— (eq II)}$$

$I\ddot{\theta}$ means we are assuming rot. abo. centre

potential next steps: pick states

Problem 1.4

a)



$i_C = C\frac{dv_C}{dt}$

$v_L = L\frac{di_L}{dt}$

KCL @ top: $i_L = i_C + i_R$

$i_L = C\dot{v}_C + h(v_R)$ — since $v_C = v_R$

$i_L = C\dot{v}_C + h(v_C)$ — (eq. I)

KVL on left loop: $u - Ri_L - L\dot{i}_L = v_C$

$u = v_C + i_L R + \dot{i}_L L$ — (eq. II)

b) $x_1 = v_C$ and $x_2 = i_L$ $\Rightarrow$ $\dot{x}_1 = \dot{v}_C$ and $\dot{x}_2 = \dot{i}_L$

$\dot{x}_1 = \left(\frac{1}{C}\right)x_2 - \left(\frac{1}{C}\right)h(x_1)$

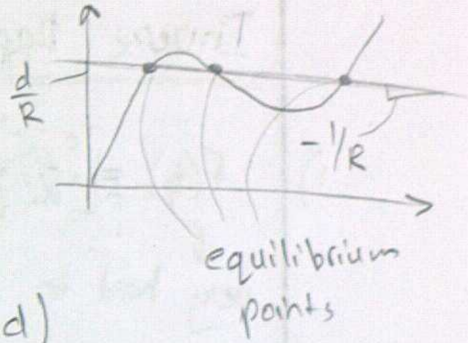
$\dot{x}_2 = -\left(\frac{1}{L}\right)x_1 - \left(\frac{R}{L}\right)x_2 + \left(\frac{1}{L}\right)u$

c) find equilibria: (1) set derivatives equal to zero
(add bar when you shift away from original eq. and start talking about equilibrium pnts)

$u = d \in \mathbb{R}$

$$0 = \left(\tfrac{1}{C}\right)\bar{x}_2 - \left(\tfrac{1}{C}\right)h(\bar{x}_1) \quad \text{—} \ (\mathrm{I})$$

$$0 = -\left(\tfrac{1}{L}\right)\bar{x}_1 - \left(\tfrac{R}{L}\right)\bar{x}_2 + \left(\tfrac{1}{L}\right)d \ \text{—} \ (\mathrm{II})$$



(I): $\bar{x}_2 = h(\bar{x}_1)$ (II): $\bar{x}_2 = \frac{1}{R}(-\bar{x}_1 + d)$

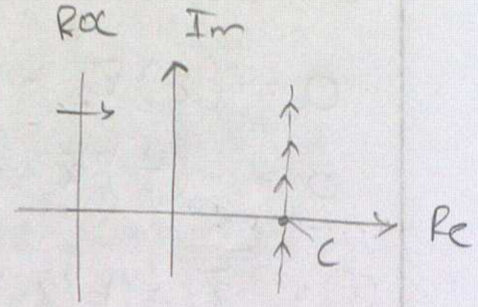
$$\Rightarrow h(\bar{x}_1) = \frac{1}{R}(-\bar{x}_1 + d)$$

Week 3 Lecture 1 — Sep 16, 2025

Inverse Laplace Transform

1) $f(t) = \mathcal{L}^{-1}\{F(s)\} = \int_{c-j\infty}^{c+j\infty} F(s)\, e^{st}\, ds$

$\downarrow$
very hard in practice



2) Residue Formula:

if $F(s)$ is analytic everywhere except for a finite number of poles $\{P_1, P_2, \dots, P_N\}$, then $\mathcal{L}^{-1}$ can be computed by Residue Formula

$$f(t) = \sum_{i=1}^{N} \text{Res}\big(\underbrace{F(s)\, e^{st}, s = P_i}_{\text{the residue of } F(s)\cdot e^{st} \text{ at } s = P_i}\big)$$

- An analytic func^{n}: a func^{n} that is differentiable at every point and its derivative is continuous

Residue Calculation

Let $G(s) = F(s)\cdot e^{st}$ be an analytic function with one pole at $s = p$ of multiplicity of $r \geq 1$

Then the $\boxed{\text{Res}(G(s), s=p) = \dfrac{1}{(r-1)!}\cdot\left[\dfrac{d^{r-1}}{ds^{r-1}}\left(G(s)\cdot(s-p)^{r}\right)\right]_{s=p}}$

- if $r=1$, $Res(G(s), s=p) = \left[G(s)\cdot(s-p)\right]_{s=p}$
- if $r=2$, $Res(G(s), s=p) = \frac{1}{1!}\cdot\frac{d}{ds}\left[G(s)\cdot(s-p)^2\right]_{s=p}$

Example:

$$F(s) = \frac{3}{(s+3)(s+9)} \quad \begin{matrix}\nearrow 1)\ \text{PFD} \\ \searrow 2)\ \text{residue thm}\end{matrix}$$

1) $\frac{3}{(s+3)(s+9)} = \frac{A}{s+3} + \frac{B}{s+9} \Rightarrow f(t) = Ae^{-3t} + Be^{-9t}$

2) poles: $p_1 = -3$ and $p_2 = -9$ with multiplicity of 1

$$f(t) = \underbrace{Res\left(\frac{3}{(s+3)(s+9)}\cdot e^{st}, s=-3\right)} + \underbrace{Res\left(\frac{3}{(s+3)(s+9)}\cdot e^{st}, s=-9\right)}$$

$$= \left[\frac{3e^{st}}{s+9}\right]_{s=-3} = \frac{3e^{-3t}}{-3+9} = \frac{1}{2}e^{-3t} \qquad = \left[\frac{3e^{st}}{(s+3)}\right]_{s=-9} = -\frac{1}{2}e^{-9t}$$

$$\Rightarrow f(t) = \frac{1}{2}e^{-3t} - \frac{1}{2}e^{-9t}$$

Example 2: $F(s) = \frac{3}{(s+3)(s+9)^2}$

1) $F(s) = \frac{A}{s+3} + \frac{Bs+C}{(s+9)^2} + \frac{D}{s+9} + \dots$

2) $f(t) = \underbrace{Res\left(\frac{3e^{st}}{(s+3)(s+9)^2}, s=-3\right)}_{A} + \underbrace{Res\left(\frac{3e^{st}}{(s+3)(s+9)^2}, s=-9\right)}_{B}$

$$A: = \frac{3e^{-3t}}{(-3+9)^2}$$

$$B: = \frac{d}{ds}\left[\frac{3e^{st}}{s+3}\right]_{s=-9} = \left[\frac{t\cdot 3e^{st}(s+3) - 3e^{st}}{(s+3)^2}\right]_{s=-9}$$

$$\Rightarrow f(t) = \frac{1}{12}e^{-3t} - \frac{1}{2}te^{-9t} - \frac{1}{12}e^{-9t} \quad //$$

Example 3: $F(s) = \frac{1}{(s+1)(s^2+2s+2)}$ $\quad P_1 = -1$, $P_2 = -1 \pm j$

1) $F(s) = \frac{A}{s+1} + \frac{Bs+C}{(s+1)^2+1}$ $\rightarrow$ Finds $A=1$, $B=-1$, $C=-1$

$(s+1)^2+1 = s^2+2s+2$

$$= \frac{1}{s+1} - \frac{s+1}{(s+1)^2+1} \xrightarrow{\mathcal{L}^{-1}} f(t) = e^{-t} - e^{-t}\cos t \quad //$$

2) $f(t) = \text{Res}\left(\underbrace{\frac{e^{st}}{(s+1)(s^2+2s+2)}}_{=G(s)}, s=-1\right) + \text{Res}(G(s), s=-1+j)$
$+ \text{Res}(G(s), s=-1-j)$

$$= \left[\frac{e^{st}}{(s+1)^2+1}\right]_{s=-1} + \left[\frac{e^{st}}{(s+1)(s+1+j)}\right]_{s=-1-j} + \left[\frac{e^{st}}{(s+1)(s+1-j)}\right]_{s=-1+j}$$

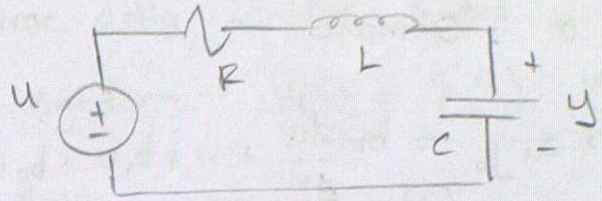
$$= \frac{e^{-t}}{1} + \frac{e^{-t}e^{jt}}{j\cdot 2j} + \frac{e^{-t}\cdot e^{-jt}}{-j(-2j)} = e^{-t}\left(1 - \frac{1}{2}e^{jt} + \frac{1}{2}e^{jt}\right)$$

$$= e^{-t}(1-\cos t) \quad //$$

RLC Revesit

set: $u(t) = u(t) = 1(t)$

$y(0) = \dot{y}(0) = 0$



RLC model:

$$LC \frac{d^2y}{dt^2} + RC \frac{dy}{dt} + y = u$$ — transfer to t-dom

$$LC(s^2 Y(s) - \cancel{s y(0)} - \cancel{y''(0)}) + RC(sY(s) - \cancel{y(0)}) + Y(s) = \frac{1}{s}$$

$$s^2 LC\, Y(s) + sRC\, Y(s) + Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s(s^2 LC + sRC + 1)}$$

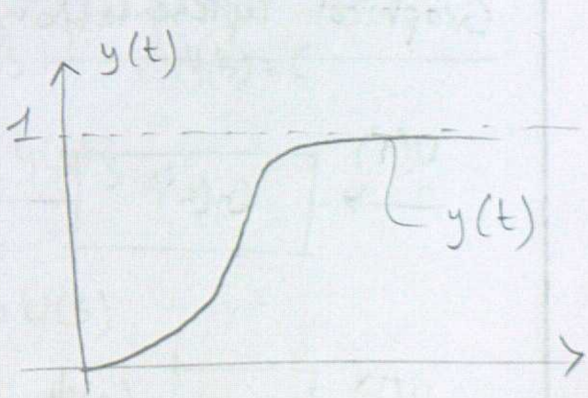
pick $LC = \frac{1}{3}$, $\frac{R}{L} = 4$

$$Y(s) = \frac{1/LC}{s(s^2 + \frac{R}{L} + \frac{1}{LC})} = \frac{3}{s(s^2+4s+3)} = \frac{3}{s \cdot (s+1)(s+3)}$$

back to t-dom

$$y(t) = \text{Res}\left(\frac{3e^{st}}{s(s+1)(s+3)}, s=0\right) + \text{Res}\left(\frac{3e^{st}}{s(s+1)(s+3)}, s=-1\right) + \cdots$$

$$= 1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}$$



If you have non-zero intial conditions, then you'll have extra terms after taking $\mathcal{L}\{\frac{d^2y}{dt^2}\}$ and $\mathcal{L}\{\frac{dy}{dt}\}$ ⟶ $Y(s) = \left(\text{zero state response}\right) + \left(\text{zero input response}\right)$

have same poles

Week 3 Lecture 2 Sep 18, 2025

Definition of Transfer Function

Consider a generic LTI system in I/O model

$$\frac{d^n y}{dt^n} + (a_{n-1}) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \dot{y}_1 + a_0 y_0 = b_m \frac{d^m u}{dt^m} + \cdots + b_1 \dot{u} + b_0 u$$

$$\left.\begin{aligned} y(0) = \dot{y}(0) = \cdots = \frac{d^{n-1} y}{dt^{n-1}}(0) = 0 \\ u(0^-) = \dot{u}(0^-) = \cdots = \frac{d^{n-1} u}{dt^{n-1}}(0) = 0 \end{aligned}\right\} \text{zero state (no initial cond}^{\text{n}}\text{s)}$$

Apply the $\mathcal{L}\{\}$ to both sides of this I/O model

$$Y(s) \cdot \left[s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0\right] = U(s)\left[b_m s^m + \cdots + b_1 s + b_0\right]$$

$$Y(s) = \underbrace{\left[\frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}\right]}_{G(s),\ \text{transfer func}^{\text{n}}} U(s)$$

$$Y(s) = G(s) U(s) \quad \text{(TF model)}$$

Graphical Representation

$$U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

$$u(t) \longrightarrow \boxed{G(s)} \longrightarrow y(t)$$

Careful: whenever you use the transfer fun$^{\text{n}}$ model, you are assuming zero initial cond$^{\text{n}}$s

For non-zero initial cond$^{\text{n}}$s:

$$Y(s) = G(s)\,U(s) + \text{Polynomial}$$

depending on: $\dfrac{y(0)\ \dot{y}(0)}{s^n + a_{n-1}s^{n-1} \ldots}$

zero input response

Impulse Response:

Dirac delta func^n: $\delta(t)$

a system: $Y(s) = G(s) \cdot U(s)$ in s-dom $\longrightarrow \delta(t) = u(t)$

$$\delta(t) \longrightarrow \boxed{G(s)} \xrightarrow{y(t)}$$

we have $\mathcal{L}\{g(t) * u(t)\} = G(s) U(s)$

$$Y(s) = \mathcal{L}\{g(t) * \delta(t)\} = G(s) \cdot 1 = G(s)$$

$$y(t) = g(t) = \mathcal{L}^{-1}\{G(s)\} \text{ if input is } \delta(t)$$

$\Rightarrow$ The impulse response can be measured experimentally to identify the system's transfer function

Transfer Fun^c State Space

A: $n \times n$

B: $n \times 1$

C: $1 \times n$

D: 1×1

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

assume $x(0) = 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$X(s) = \mathcal{L}\{x(t)\} = \begin{bmatrix} \mathcal{L}\{x_1(t)\} \\ \vdots \\ \mathcal{L}\{x_n(t)\} \end{bmatrix}$$

$Y(s) = C X(s) + D U(s)$

$Y(s) = C$

$$U(s) = \mathcal{L}\{u(t)\} \qquad Y(s) = \mathcal{L}\{y(t)\}$$

$$\mathcal{L}\{s.s.\} = sX(s) = AX(s) + BU(s)$$

$$(s \cdot \underbrace{I_n}_{\substack{n \times n \text{ identity} \\ \text{matrix}}} - A) X(s) = B \cdot U(s)$$

$$X(s) = (sI_n - A)^{-1} \cdot B \cdot U(s)$$

$$Y(s) = CX(s) + DU(s)$$

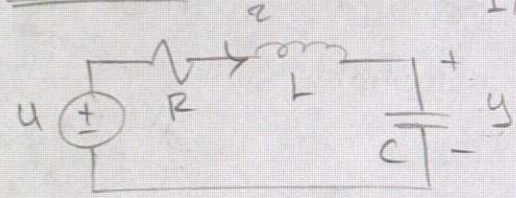
$$= \underbrace{\left[C\cdot(sI_n - A)^{-1}B + D\right]}_{G(s)}\cdot U(s)$$

★ review dimensions

the transfer function associated with this s.s. model is:

$$G(s) = \underbrace{C}_{1\times n}\cdot\underbrace{(sI_n - A)^{-1}}_{n\times n}\cdot\underbrace{B}_{n\times 1} + \underbrace{D}_{1\times 1} \implies G(s) \text{ is } n\times n$$

RLC ckt



$$\text{I/O}: LC\ddot{y} + RC\dot{y} + y = u$$

$$\mathcal{L}\{\}$$

$$G(s) = \frac{1}{LCs^2 + RC\cdot s + 1}$$

from Lec 2:

if set $x_1 = y$ and $x_2 = i$

$$(ss) = \begin{cases} \dot{x} = \overset{A}{\begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix}}\cdot x + \overset{B}{\begin{bmatrix} 0 \\ 1/L \end{bmatrix}}\cdot u \\ y = \overset{C}{\begin{bmatrix} 1 & 0 \end{bmatrix}} x \qquad D = 0 \end{cases}$$

$$G(s) = C(sI_n - A)^{-1}\cdot B + D$$

★

$$= \frac{1}{L}\begin{bmatrix} 1 & 0 \end{bmatrix}\cdot\begin{bmatrix} s & -1/C \\ 1/L & s+\frac{R}{L} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{L}\,\frac{\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s+\frac{R}{L} & 1/C \\ -1/L & s \end{bmatrix}^{*}}{\det[\quad]}\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

pick up ... of $[\ ]^{-1}$

$$G(s) = \frac{1}{L} \cdot \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \times \frac{1}{C}$$

same as I/O

$$\Rightarrow G(s) = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \quad ✓$$

$G(s) \Rightarrow$ review second order matrix

<u>Lec 3:</u> PMDC Model, set $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = i_a$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$y = \dot{\theta}$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{b}{I} & \frac{K_\tau}{I} \\ 0 & -\frac{K_e}{L_a} & -\frac{R_a}{L_a} \end{bmatrix}}_{A} \cdot x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{1}{L_a} \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{C} x \qquad D = 0$$

$$G(s) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} s & -1 & 0 \\ 0 & s + b/I & -K_\tau/I \\ 0 & K_e/L_a & s + R_a/L_a \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{L_a}$$

element (row 2, col 3) of $[\ ]^{-1}$ you pick up

↓

cofactor (3,2) of $[\ ]^{-1} \Rightarrow -\det \begin{bmatrix} s & 0 \\ 0 & -\frac{K_\tau}{I} \end{bmatrix}$

$$= \frac{s \cdot K_\tau}{I}$$

$$G(s) = \frac{\frac{s \cdot K_\tau}{I}}{s \cdot \det \begin{bmatrix} s + b/I & -K_\tau/I \\ \frac{K_e}{L_a} & s + \frac{R_a}{L_a} \end{bmatrix}} \times \frac{1}{L_a}$$

$$= \frac{K_\tau}{\left(s^2 + \left(\frac{b}{I} + \frac{R_a}{L_a}\right)s + \frac{K_\tau K_e + bR_a}{I L_a}\right)} \cdot \frac{1}{L_a I}$$

$$G(s) = \frac{K_t/(L_a \cdot I)}{s^2 + \left(\frac{b}{I} + \frac{R_a}{L_a}\right)s + \frac{K_t K_e + b R_a}{I \cdot L_a}}$$

$$\underset{\text{voltage}}{\xrightarrow{U(s)}} \boxed{G(s)} \xrightarrow{Y(s)} \text{speed}$$

---

system modelling

$\boxed{\text{IO Model}}$ $\xleftrightarrow{\mathcal{L}}$ $\boxed{\text{TF model}}$ $\rightarrow$ $\underline{\underline{C \cdot (sI_n - A)^{-1} \cdot B + D}}$

$\boxed{\text{SS Model}}$ $\rightleftarrows$ $\boxed{\text{TF model}}$

$Y(s) = G(s)\, U(s)$

not unique

Time - Domain | Laplace domain (s-domain)

- going from TF model to SS model not unque since SS model wasn't unique to begin with
- now, we will learn this

Week 3 Lec 3 — Sep 19, 25

- Given a TF $G(s)$ and therefore, a system : $Y(s) = G(s)U(s)$

$\Rightarrow$ Find A, B, C, D s.t. $C\cdot(sI_n - A)^{-1}\cdot B + D = G(s)$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$\Rightarrow$ since this state space model can only produce rational proper TF's, we have

$$G(s) = \frac{b_m s^m + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}, \quad m \leq n$$

---

* Following, we assume $m < n$ $\leftarrow$ strictly proper

$$U(s) \longrightarrow \boxed{\frac{b_m s^m + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}} \longrightarrow Y(s)$$

$$U(s) \longrightarrow \underset{①}{\boxed{\frac{1}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}}} \xrightarrow{V(s)} \underset{②}{\boxed{b_m s^m + \dots + b_0}} \longrightarrow Y(s)$$

* Address the blocks separately

1) $U(s) \longrightarrow \boxed{\dfrac{1}{s^n + a_{n-1}s^{n-1} + \dots + a_0}} \longrightarrow V(s)$

if we choose: (*)

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ as } \begin{cases} x_1 = V \\ x_2 = \dot{V} \\ \vdots \\ x_n = \dfrac{d^n V}{dt^n} \end{cases}$$

$\Downarrow$ save as

$$x_2 = \dot{x}_1$$
$$x_3 = \dot{x}_2$$
$$\vdots$$
$$x_n = \dot{x}_{n-1}$$

$$U(s) = (s^n + a_{n-1}s^{n-1} + \dots + a_0)\, V(s)$$

$\mathcal{L}^{-1}\{\}$

$$u = \frac{d^n V}{dt^n} + a_{n-1}\frac{d^{n-1} V}{dt^{n-1}} + \dots + a_0 V$$

$\downarrow$ next pg.

$$\dot{x}_n = -a_0 v - a_1 \dot{v} - \cdots - a_{n-1} \frac{d^{n-1} v}{dt^{n-1}} + u$$

now we replaced all derivatives with our state variables

(**)

$$= -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n + u$$

(*)

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & & \\ 0 & & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & & & -a_{n-1} \end{bmatrix}}_{A} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{B} u$$

(**)

---

2) $\xrightarrow{V(s)} \boxed{b_m s^m + \cdots + b_1 s + b_0} \xrightarrow{Y(s)}$

$$Y(s) = V(s)\left[b_m s^m + \cdots + b_1 s + b_0\right] \quad \mathcal{L}^{-1}\{\}$$

$$y(t) = b_m \underbrace{\frac{d^m v}{dt^m}}_{x_{m+1}} + \cdots + b_1 \underbrace{\dot{v}}_{x_2} + b_0 \underbrace{v}_{x_1} \qquad m < n$$

you stop at m+1, there are n-(m+1) which you keep empty at zero

$$\rightarrow y(t) = \begin{bmatrix} b_0 & b_1 & \cdots & b_m & 0 & \cdots & 0 \end{bmatrix} \cdot x$$

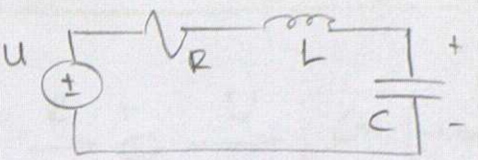
→ <u>Conclusion:</u> a SS representation of $G(s)$ is

$$\dot{x} = \overbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \\ 0 & & & & & 1 \\ -a_0 & -a_1 & \cdots & & & -a_{n-1} \end{bmatrix}}^{A} \cdot x + \overbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}^{B} \cdot u$$

$$D = 0$$

$$y = \overbrace{\begin{bmatrix} b_0 & b_1 & \cdots & b_m & 0 & \cdots & 0 \end{bmatrix}}^{C} \cdot x$$

RLC Example



$$LC\frac{d^2y}{dt^2} + RC\frac{dy}{dt} + y = u \qquad x_1 = LCy$$

$$\xrightarrow{n=2} \ddot{y} + \underbrace{\frac{R}{L}}_{a_1}\dot{y} + \underbrace{\frac{1}{LC}}_{a_0}y = \underbrace{\frac{1}{LC}}_{b_0}u \qquad x_2 = LC\dot{y}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ \underbrace{-\frac{1}{LC}}_{-a_0} & \underbrace{-\frac{R}{L}}_{-a_1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

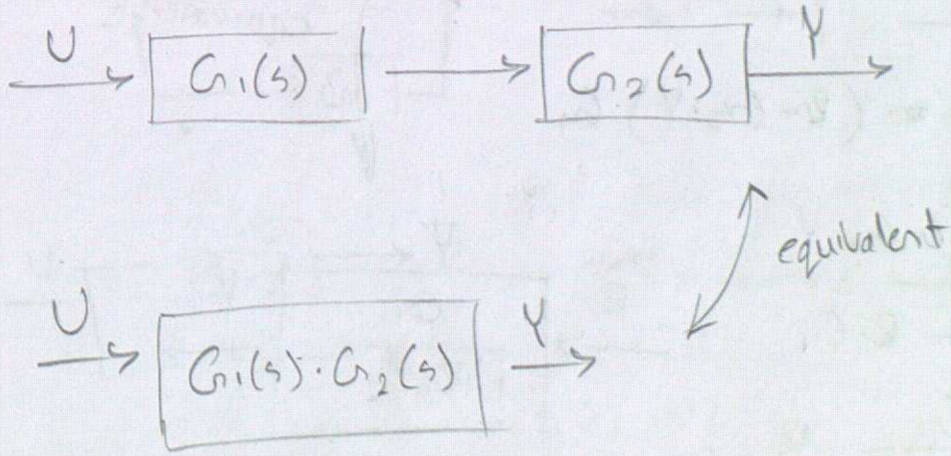
1. first make ODE into standard form
2. get a's and b's
3. put it into

$$y = [\underbrace{1/LC}_{b_0} \quad 0] \cdot x \qquad D = 0$$

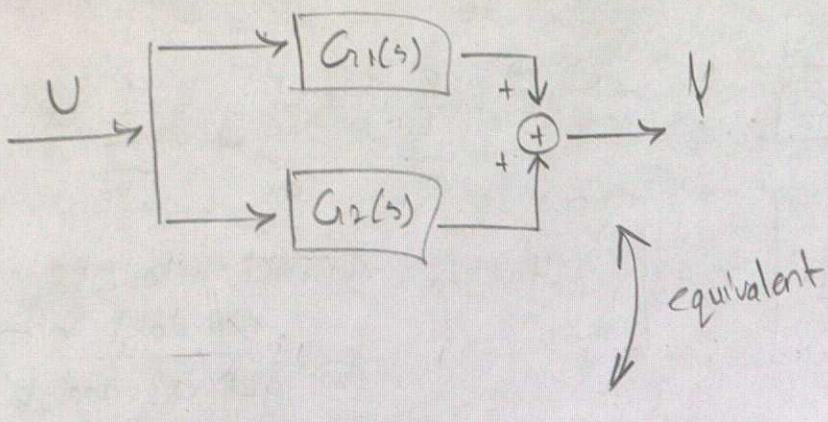
- the above SS model is different than the SS model we got in last lecture (the one from lec 2) since SS model is not unique

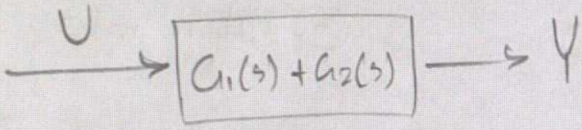
Basic Block Diagram

- cascaded connection:

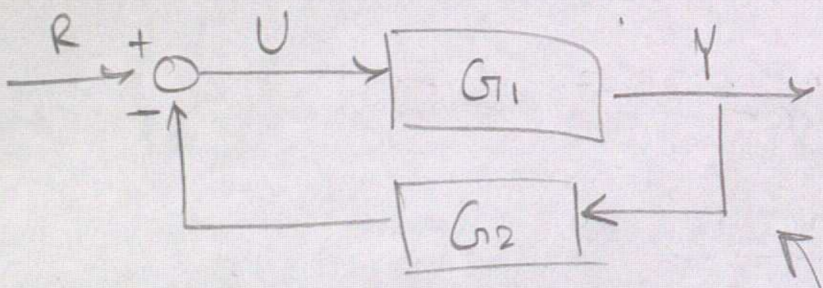


- parallel connec^{k}





feedback connec^{k}



equivalent

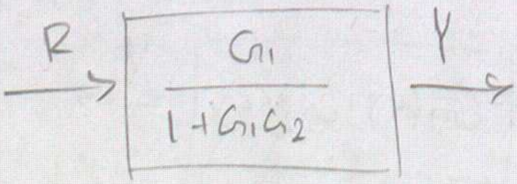
$$Y = U \cdot G_1 = (R - G_2 \cdot Y) \cdot G_1$$

$$Y = RG_1 - G_1 G_2 \cdot Y$$

$$Y(1 + G_1 G_2) = R \cdot G_1$$

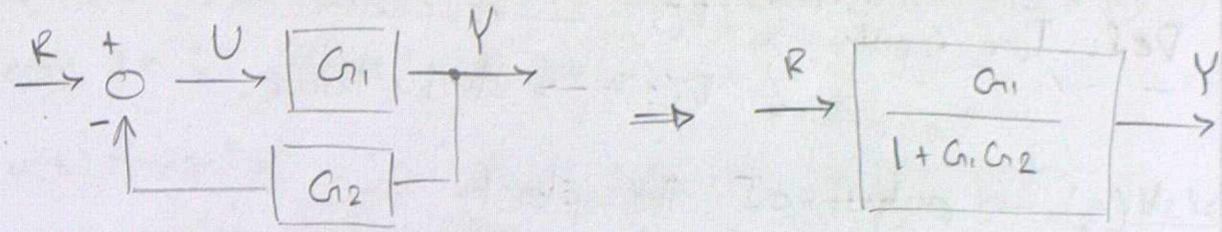
learn ⟶

$$Y = \frac{G_1}{1 + G_1 G_2} \cdot R$$



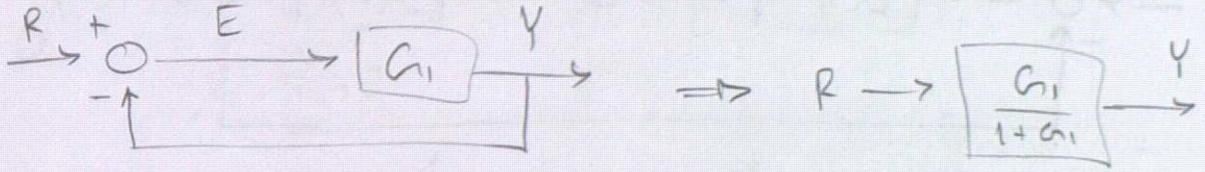
Week 4 Lecture 1 Sep 23, 2025

Block Manipulation Example

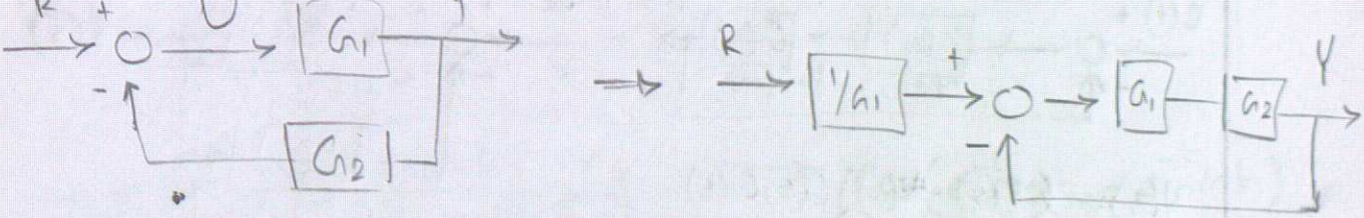


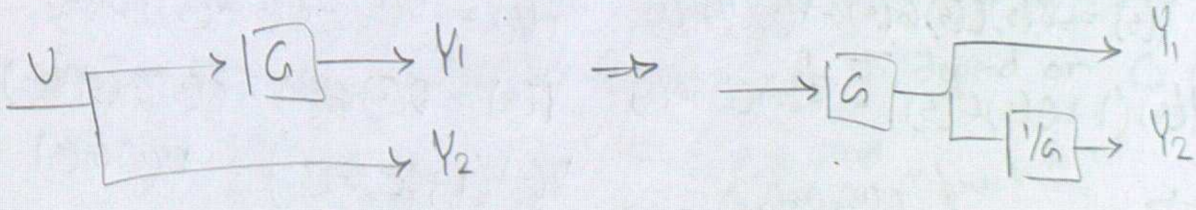
$$U = R - YG_2 = R - R\left(\frac{G_1}{1+G_1G_2}\right) = R\left(\frac{1}{1+G_1G_2}\right)$$

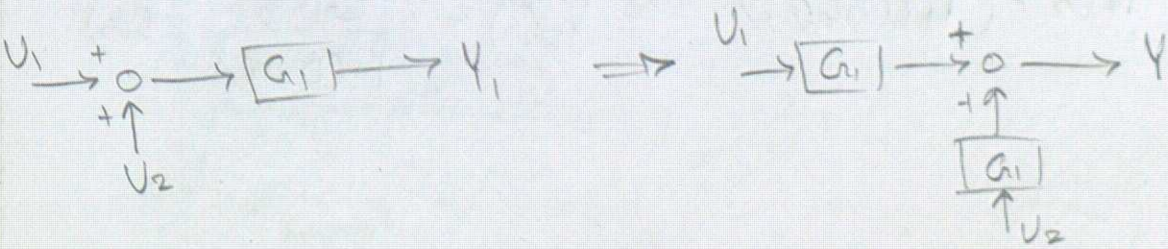
In particular, $G_2 = 1$



$$\underbrace{E}_{\text{want this to be close to zero}} = R - Y = R\left(\frac{1}{1+G_1}\right)$$





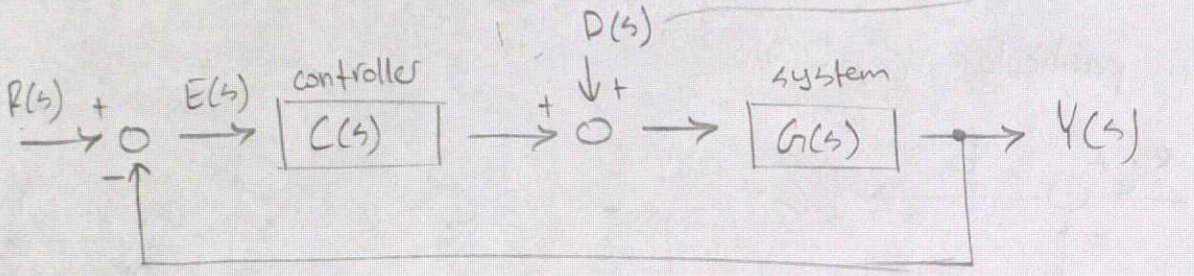


Basic Feedback Control Loop

Def: Two inputs $\begin{cases} R(s) \to \text{reference} \\ D(s) \to \text{disturbance} \end{cases}$

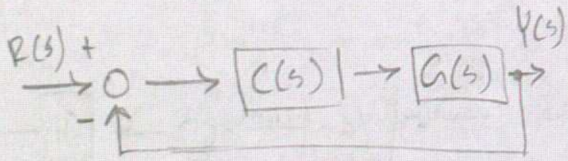
$Y(s) \to$ output of the plant

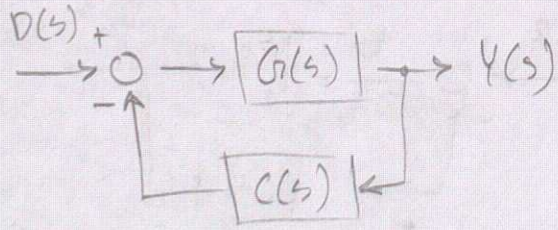
$E(s) = R(s) - Y(s)$ (tracking error)



Use superposition: $Y(s) = \underbrace{Y_R(s)}_{} + \underbrace{Y_D(s)}_{}$

$Y(s)$ when $D(s) = 0$ ↗ ↖ $Y(s)$ when $R(s) = 0$





$$Y(s) = (R(s) - Y(s))\,C(s)G(s)$$

$$Y(s) = R(s)C(s)G(s) - Y(s)C(s)G(s)$$

$$Y(s)(1 + C(s)G(s)) = R(s)C(s)G(s)$$

$$\Rightarrow Y_R(s) = \left(\frac{C(s)\,G(s)}{1 + C(s)G(s)}\right) R(s)$$

$$Y(s) = (D(s) - C(s)Y(s))\,G(s)$$

$$Y(s) = D(s)G(s) - C(s)G(s)Y(s)$$

$$Y(s)(1 + C(s)G(s)) = D(s)G(s)$$

$$Y_D(s) = \left(\frac{G(s)}{1 + C(s)G(s)}\right) D(s)$$

Time Response:

⟹ time response of complex conjugate poles (Lec #7)

Recall PMDC motor example:

u(t) : terminal

y(t) : motor shaft speed

$$G(s) = \frac{K_T / L_a I}{s^2 + \left(\frac{b}{I} + \frac{R_a}{L_a}\right)s + \frac{K_T K_e + R_a b}{L_a I}}$$

Assume: $u(t) = V_0 \cdot 1(t) \longrightarrow$ "$1(t)$" is unit step function

⟹ understand y(t) with simple math (assume numerical values)

$$G(s) = \frac{1}{s^2 + 2s + 2}$$

$$u(t) = V_0 \cdot 1(t) \xrightarrow{\mathcal{L}} U(s) = \frac{V_0}{s}$$

$$\Rightarrow Y(s) = U(s)\,G(s) = \frac{V_0}{s} \cdot \frac{1}{s^2 + 2s + 2} = \frac{V_0}{s\left[(s+1)^2 + 1\right]}$$

$p = -1 \pm j$

$p = 0$

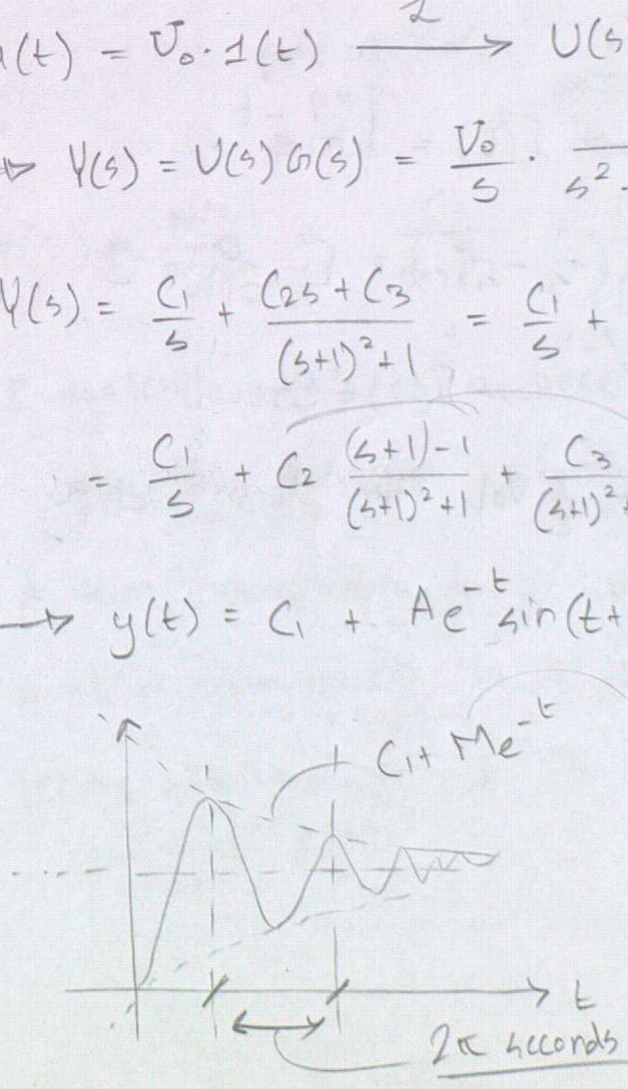
$$Y(s) = \frac{C_1}{s} + \frac{C_2 s + C_3}{(s+1)^2 + 1} = \frac{C_1}{s} + C_2 \frac{s}{(s+1)^2 + 1} + \frac{C_3}{(s+1)^2 + 1}$$

$$= \frac{C_1}{s} + C_2 \frac{(s+1) - 1}{(s+1)^2 + 1} + \frac{C_3}{(s+1)^2 + 1} \longrightarrow C_2\left(e^{-t}\cos t - e^{-t}\sin t\right)$$

→ can combine trig func's

$$\rightarrow y(t) = C_1 + Ae^{-t}\sin(t + \phi) \quad A, \phi \text{ depend on } C_2, C_3$$

not A since it doesn't "touch"



Final Value Theorem

Make a connection between $f(t)$ for large t and $F(s)$ for small s :

$$\lim_{t\to\infty} f(t) = sF(s)\Big|_{s=0} \quad \text{if the limit exists}$$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}\,dt = sF(s) - f(0)$$

$$\text{if } s=0 : \int_0^\infty f'(t)\cdot 1\,dt = \lim_{t\to\infty} f(t) - f(0) = sF(s) - f(0)$$

$$\Rightarrow \lim_{t\to\infty} f(t) = sF(s)$$

Ex: $F(s) = \frac{1}{s} - \frac{1}{s+1} \xrightarrow{\mathcal{L}^{-1}} f(t) = 1 - e^{-t}$

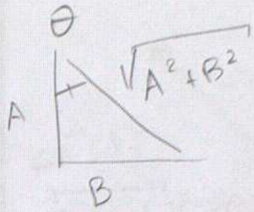
$$\lim_{t\to\infty} f(t) = sF(s)\Big|_{s=0} = s\left(\frac{1}{s} - \frac{s}{s+1}\right) = 1 - \frac{0}{0+1} = 1$$

but for $F(s) = \frac{s}{s^2+\omega^2} \xrightarrow{\mathcal{L}^{-1}} f(t) = \cos\omega t$

$\therefore \lim_{t\to\infty} f(t)$ DNE $\Rightarrow$ Final Val. Thm. doesn't appy

Week 4 Lec 2 Sep 25, 2025

$$A\sin x + B\cos x = \sqrt{A^2+B^2}\cdot\left(\underbrace{\frac{A}{\sqrt{A^2+B^2}}}_{\cos\theta}\sin x + \underbrace{\frac{B}{\sqrt{A^2+B^2}}}_{\sin\theta}\cos x\right)$$



$$\Rightarrow \sqrt{A^2+B^2}\,(\cos\theta\sin x + \sin\theta\cos x)$$

$$= \sqrt{A^2+B^2}\cdot\sin(x+\theta)$$

Generalized Function

$$\text{output } Y(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s-p_1)\left[(s+\sigma)^2+\omega_d^2\right]\cdot(s-p_1)^k\left[(s+\sigma)^2+\omega_d^2\right]^k\cdots}$$

pole at $p = p_1$ poles at $-\sigma \pm j\omega_d$

$$Y(s) = \frac{N(s)}{D(s)} = \frac{C_1}{s-p_1} + \frac{C_2 s + C_3}{(s+\sigma)^2+\omega_d^2} + \cdots$$

LTI system: $y(t)$ is a linear combination of "elementary elements" of three types

1). A term corresponding to a real pole $\frac{1}{s-p_1}$

2) A term corresponding to a pair of conjugate poles $\frac{\cdots}{(s+\sigma)^2+\omega_d^2}$

3) Terms involving repeated real or complex conjugate poles

$\Rightarrow$ our focus is 1) and 2)

Week 4 Lec 3 Sep

Representations of Complex Conjug. Poles

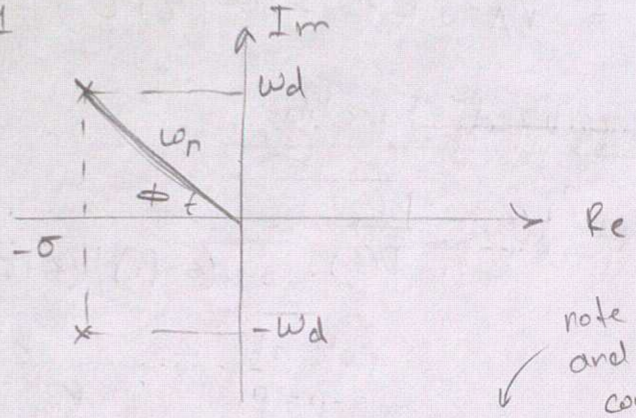
A → $Y(s) = \dfrac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$, $s = -\sigma \pm j\omega_d$

B → $Y(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

damping ratio natural frequency

$$\begin{cases} \zeta = \dfrac{\sigma}{\sqrt{\sigma^2 + \omega_d^2}} & |\zeta| \leq 1 \\ \omega_n = \sqrt{\sigma^2 + \omega_d^2} \end{cases}$$

$$\cos\phi = \frac{\sigma}{\omega_n} = \frac{\sigma}{\sqrt{\sigma^2 + \omega_d^2}} = \zeta$$



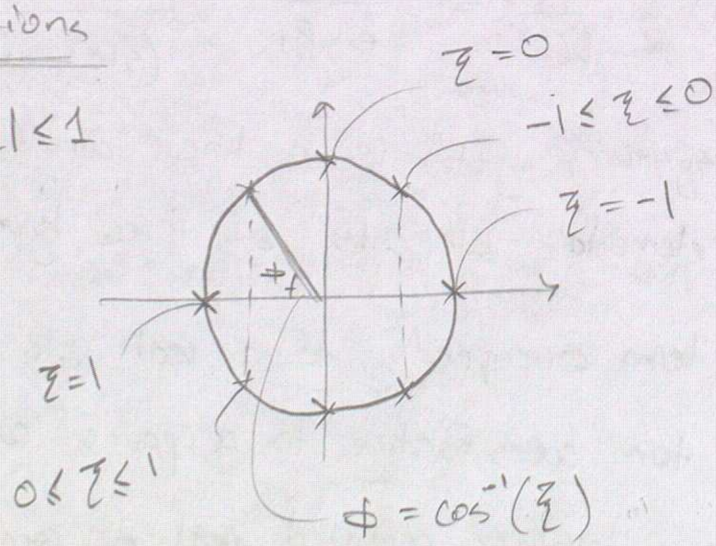
note direction and (-)ve conventions

Form A shows (x,y) of pole, Form B shows (r, ϕ)

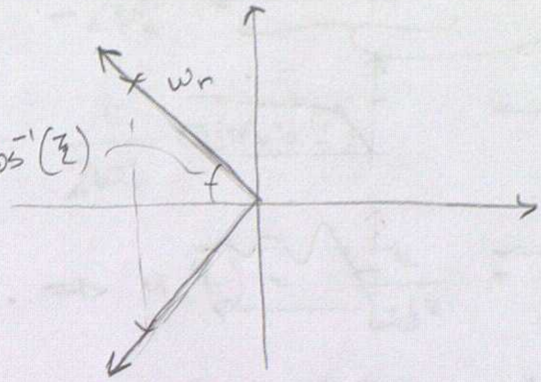
Geometric Interpretations

1) ω_n = constant, $|\zeta| \leq 1$

$$\zeta = \frac{\sigma}{\sqrt{\sigma^2 + \omega_d^2}}$$



2) ξ = constant and vary ω_n : $\omega_n \in (0, +\infty)$



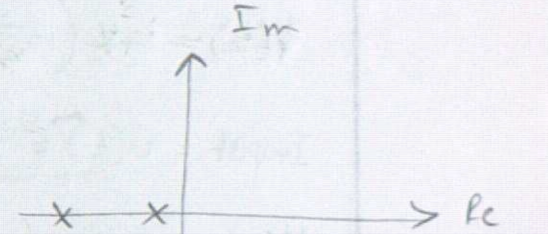
ξ fixed, angle fixed

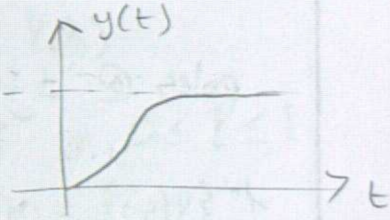
increase ω_n, poles get further from origin along same line with fixed ϕ

Poles Plot

$$R(s) \longrightarrow \underset{G(s)}{\boxed{\frac{b}{s^2+as+b}}} \longrightarrow Y(s)$$

1) $R(s) = \frac{1}{s} \longrightarrow \boxed{\frac{9}{s^2+9s+9}} \longrightarrow Y(s)$

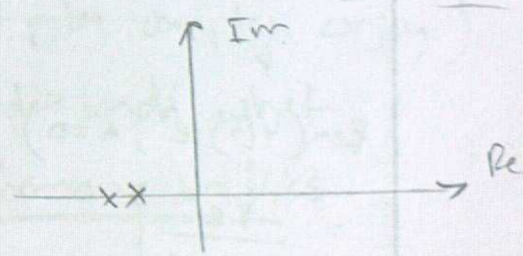


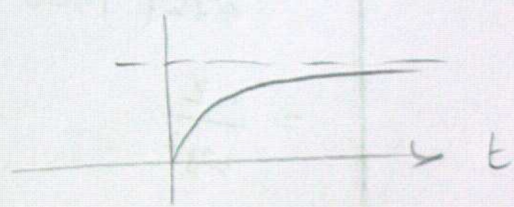


$\xi > 1$, overdamped

you reduce ξ to see effect

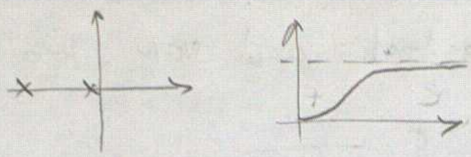
2) $R(s) = \frac{1}{s} \rightarrow \boxed{\frac{9}{s^2+6s+9}} \longrightarrow Y(s)$



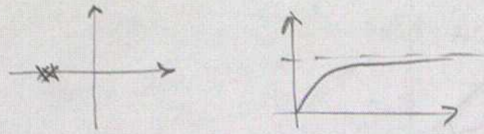


if ξ is large, then the poles are further, and if ξ is smaller, then the poles are closer together

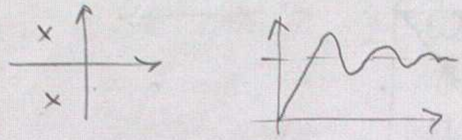
- $\xi > 1$, overdamped



- $\xi = 1$, critically damed



- $0 \leq \xi \leq 1$, underdamped



- tradeoff b/n tracking speed and controller stability

Example

$$Y(s) = -k\left(\frac{1}{s^2 + \omega_d^2}\right) \cdot U(s)$$

$$\text{Input} = u(t) = 1(t) \xrightarrow{\mathcal{L}} U(s) = 1/s$$

$$Y(s) = -k\left(\frac{1}{s(s^2 + \omega_d^2)}\right)$$ ← no final value, keep in mind

poles @ $\pm j\omega_d$

$$\mathcal{L}^{-1}\{Y(s)\} = y(t) = y_0 + A\sin(\omega t + \phi)$$

$\mathrm{Res}(Y(s) \cdot e^{st}, s = 0)$

$$= \left.\frac{-k e^{st}}{s^2 + \omega_d^2}\right|_{s=0}$$

$$= \frac{-k}{\omega_d^2}$$

$\mathrm{Res}(Y(s) e^{st}, s = j\omega_d) + \mathrm{Res}(Y(s) e^{st}, s = -j\omega_d)$

$$= \left.\frac{-k e^{st}}{s \cdot (s + j\omega_d)}\right|_{s = j\omega_d} + \left.\frac{-k e^{st}}{s \cdot (s - j\omega_d)}\right|_{s = -j\omega_d}$$

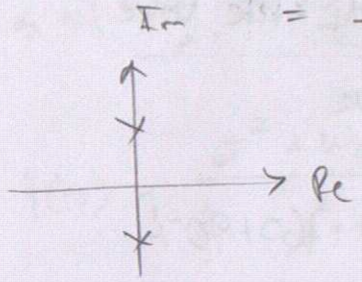
$$= \frac{-k e^{j\omega_d t}}{j\omega_d \cdot 2j\omega_d} + \frac{-k e^{-j\omega_d t}}{-j\omega_d(-2j\omega_d)}$$

$$= \frac{-k}{-2\omega_d^2}\left(\underbrace{e^{j\omega_d t} + e^{-j\omega_d t}}_{2\cos(\omega_d t)}\right)$$

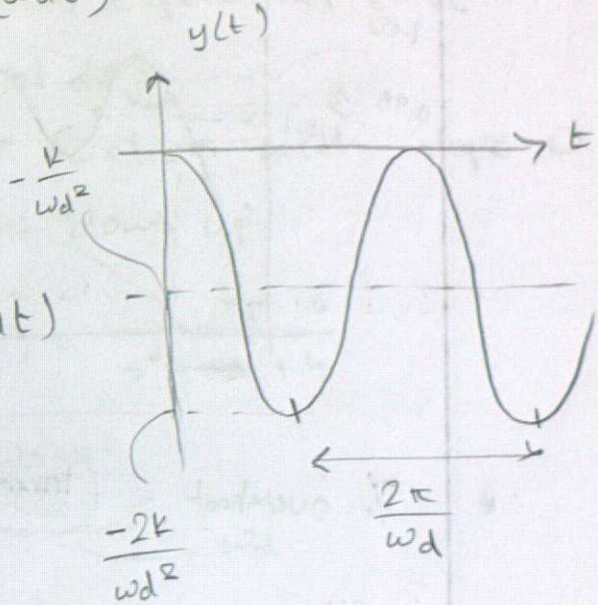
$$= \frac{k}{2\omega_d^2}\cos(\omega_d t)$$

$$\therefore \Rightarrow y(t) = \frac{-k}{\omega_d^2} + \frac{k}{2\omega_d^2}\cos(\omega_d t)$$

$$y(t) = \frac{k}{\omega_d^2}(-1 + \cos\omega_d t)$$



and $\xi = 0$

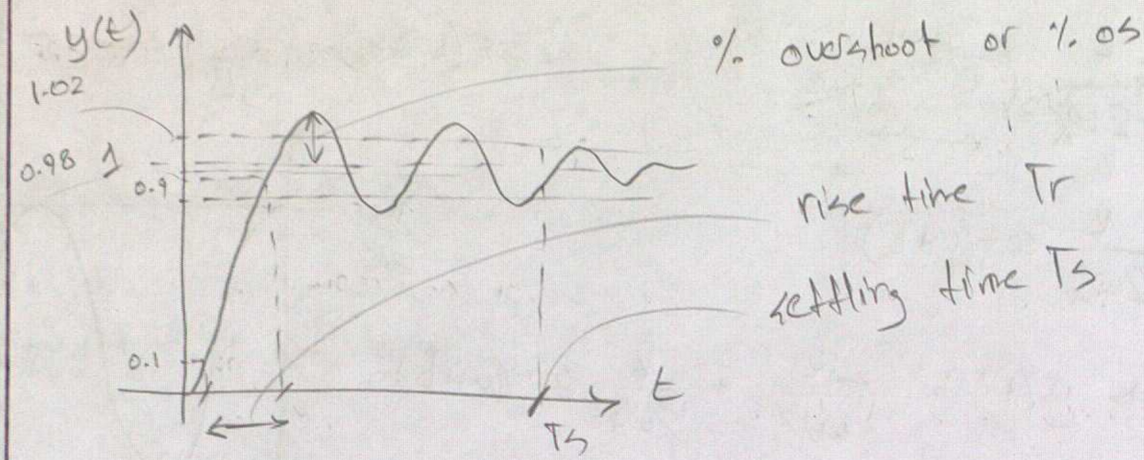


Control Specifications of 2nd Order System

objective:

$$U(s) = \frac{1}{s} \longrightarrow \boxed{\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}} \longrightarrow Y(s) \qquad 0 \le \xi \le 1$$

Using such a second order system, (2 poles, complex conjug.) in the left half plane (LHP) to understand what parameters of $y(t)$ constitutes performance metrics that could represent control specs



* % overshoot = $\dfrac{(\text{largest peak value}) - (\text{steady state value})}{\text{steady state value}}$

* Tr : time for y(t) to go from 10% to 90% of its steady state value

* Ts : settling time is time for y(t) to reach and stay within ± 2% of its steady-state value.

Check Sheet

Week 4 Lec 2 cont'd Sep 28, 25

* for real poles $\frac{1}{s+a}$, your y(t) follows exponential curve
  - ↳ if a on (-) Re axis : exponential decay
  - ↳ if a at origin : you have linear function with slope 0
  - ↳ if a on (+) Re axis : exponent blows up.

* for complex conjugate poles : $Y(s) = \frac{b}{s^2 + as + b}$

$$s = -\frac{a}{2} \pm \frac{\sqrt{a^2-4b}}{2} = -\underbrace{\frac{a}{2}}_{\sigma} \pm j \underbrace{\frac{\sqrt{4b-a^2}}{2}}_{\omega_d}$$

$$Y(s) = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2} \longrightarrow \frac{\sigma^2 + \omega_d^2}{\omega_d} \cdot \frac{\omega_d}{(s+\sigma)^2 + \omega_d^2}$$

$\mathcal{L}^{-1}\{\}$

$$y(t) = \underbrace{\frac{\sigma^2 + \omega_d^2}{\omega_d}}_{\text{some magnitude}} e^{-\sigma t} \sin(\omega_d t)$$

* σ affects exponential envelope, ω_d affects oscillation freq.
  - ↳ σ in (-)ve Re : exp. decay :
  - ↳ $\sigma = 0$: no decay or blowup :
  - ↳ σ in (+)ve Re : blow up :
  - ↳ increase ω_d → increase freq and vice versa

1) $Y(s) = \frac{b}{s^2 + as + b}$ $\quad s = -\frac{a}{2} \pm j \frac{\sqrt{4b-a^2}}{2}$

2) $Y(s) = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$ $\quad s = -\sigma \pm j\omega_d$

3) $Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ $\quad s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

$\sigma = \zeta\omega_n$ ①

$\omega_d = \omega_n\sqrt{1-\zeta^2}$ ②

ⓐ $\zeta = \frac{\sigma}{\sqrt{\sigma^2 + \omega_d^2}}$

ⓑ $\omega_n = \sqrt{\sigma^2 + \omega_d^2}$

Week 5 Tut 4 : Q1,2,6 from Hwk 2 Sep 30, 2025

Problem 1

$$\dddot{y} + 3\ddot{y} - 2\dot{y} + y = \ddot{u} - 3\dot{u} + 2u \quad , \text{ assume } IC = 0$$

i) Find the TF from u to y

ii) Find SS representation

state space includes dynamic info about x ... which is lost in the TF rep.

Laplace transform :

i) $$Y(s)(s^3 + 3s^2 - 2s + 1) = U(s)(s^2 - 3s + 2)$$

$$\frac{Y(s)}{U(s)} = \frac{s^2 - 3s + 2}{s^3 + 3s^2 - 2s + 1} \quad //$$

ii) $$\frac{Y(s)}{U(s)} = \frac{s^2 - 3s + 2}{s^3 + 3s^2 - 2s + 1} \cdot \frac{Z(s)}{Z(s)}$$

split it up and tak num. and den. seperately

start with $U(s)$

select states smartly

$$U(s) = (s^3 + 3s^2 - 2s + 1) \cdot Z(s) \quad \mathcal{L}^{-1}\{\}$$

$$u = \dddot{z} + 3\ddot{z} - 2\dot{z} + z \longrightarrow \text{Let } x = \begin{bmatrix} z \\ \dot{z} \\ \ddot{z} \end{bmatrix}$$

$$\therefore \quad \dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -3x_3 + 2x_2 - x_1 + u$$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & -3 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u \quad //$$

now do the output:

$$Y(s) = (s^2 - 3s + 2)\, Z(s) \quad \mathcal{L}^{-1}\{\}$$

$$y = \ddot{z} - 3\dot{z} + 2z$$ → already selected state when u did denom.

$$\therefore\ y = x_3 - 3x_2 + 2x_1$$

$$y = \underbrace{\begin{bmatrix} 2 & -3 & 1 \end{bmatrix}}_{C} x + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u \quad //$$

Problem 2

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

find TF from u to y

$$F(s) = C(sI - A)^{-1} B + D$$

Original way to solve : write our ODE's

$$\dot{x}_1 = x_2 + u \xrightarrow{\mathcal{L}} sX_1 = X_2 + U$$

$$\dot{x}_2 = x_3 + u \xrightarrow{\mathcal{L}} sX_2 = X_3 + U$$

$$\dot{x}_3 = -x_1 - 3x_2 - 2x_3 + u \xrightarrow{\mathcal{L}} sX_3 = -X_1 - 3X_2 - 3X_3 + U$$

$$y = x_1 \xrightarrow{\mathcal{L}} Y = X_1$$

eq1 $X_2 = sX_1 - U \longrightarrow X_2$ as $f(X_1)$

eq2 $X_3 = sX_2 - U \longrightarrow X_3$ as $f(X_2)$ where $X_2 = f(X_1)$

$$= s(sX_1 - U) - U$$

$$= s^2X_1 - sU - U$$

eq3 $X_1 = -s(s^2X_1 - sU - U) - 3(sX_1 - U) - 2(s^2X_1 - sU - U) + U$

$$\Rightarrow X_1 + 3sX_1 + 2s^2X_1 + s^3X_1 = 3U + 2sU + 2U + s^2U + sU + U$$

$Y = X_1$ $\hookrightarrow$ $Y(s)(1 + 3s + 2s^2 + s^3) = U(s)(3 + 2s + 2 + s^2 + s + 1)$

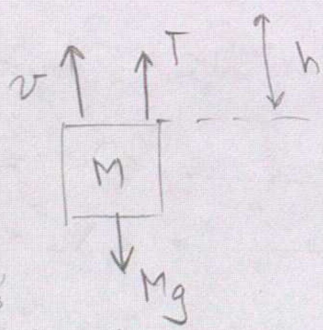
$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{s^2 + 3s + 6}{s^3 + 2s^2 + 3s + 1} \quad //$$

Problem 6

a) $h = -R\theta + \cancel{h_0}^{\,0}$

$\dot{h} = -R\dot{\theta}$

$\ddot{h} = -R\ddot{\theta}$



since viscous opposes us

torque/mount

(1) $I\ddot{\theta} = u - RT$ —— wheel

(2) $M\ddot{h} = Mg - K_v\dot{h} - T$ —— mass

(3) $\ddot{h} = -R\ddot{\theta}$

(3) into (2) : $-MR\ddot{\theta} = Mg + K_v R\dot{\theta} - T$

$T = MR\ddot{\theta} + Mg + K_v R\dot{\theta}$

sub into (1)

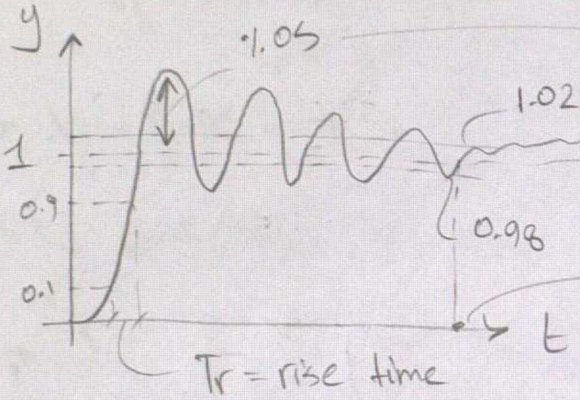
$$\begin{cases} (I + MR^2)\ddot{\theta} = u - R(Mg + K_v R\dot{\theta}) \\ h = -R\theta \end{cases}$$

Week 5 Lec 1 Sep 30, 2025

- Design Specs of 2nd Order Systems

$$G(s) = \frac{\omega_n^2}{s^2 + 2\omega_n \zeta s + \omega_n^2} \qquad U(s) = \frac{1}{s}$$

$$\Rightarrow Y(s) = \frac{\omega_n^2}{s \cdot (s^2 + 2\omega_n \zeta s + \omega_n^2)}$$



→ $\dfrac{\text{largest peak value} - \text{steady state val}}{\text{steady-state val}}$

settling time T_s is time for $y(t)$ to reach and stay within 2% of steady state value.

2nd Order System

Assumptions: Our analysis will be based on:

1) $G(s)$ has precisely 2 complex conjugate poles on LHP
2) $G(s)$ has no zeroes

→ we will discuss what happens if assumptions do not hold

$$Y(s) = \frac{\omega_n^2}{s \cdot (s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$\boxed{\begin{aligned} \sigma &= \zeta\omega_n \\ \omega_d &= \omega_n\sqrt{1-\zeta^2} \end{aligned}}$$

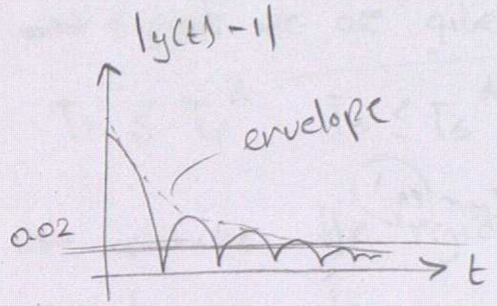
$$= \frac{\sigma^2 + \omega_d^2}{s((s+\sigma)^2 + \omega_d^2)} \quad \leftarrow \text{let's } \mathcal{L}^{-1}\{\}$$

$$y(t) = 1 - e^{-\sigma t}\left[\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right] \text{ — eq I}$$

$$= 1 - \frac{e^{-\sigma t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi) \; ; \; \phi = \cos^{-1}(\xi) \text{ — eq II}$$

Settling Time

look at $|y(t) - 1|$ and find when $|y(t)-1| \leq 0.02$



$$y(t) = 1 - \frac{e^{-\sigma t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi)$$

$$|y(t) - 1| = \underbrace{\frac{e^{-\sigma t}}{\sqrt{1-\xi^2}}}_{\text{envelope}}|\sin(\omega_d + \phi)|$$

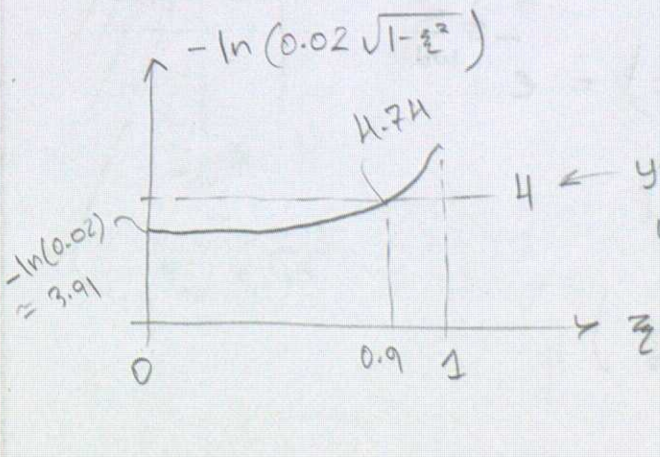
⇒ Approximation:

$T_s \simeq$ time for envelope $\frac{e^{-\sigma t}}{\sqrt{1-\xi^2}}$ enter and stay below 0.02

$$\frac{e^{-\sigma t}}{\sqrt{1-\xi^2}} = 0.02 \Rightarrow T_s = -\frac{\ln(0.02\sqrt{1-\xi^2})}{\sigma}$$

$$T_s \simeq \frac{4}{\xi \cdot \omega_n} \qquad \sigma = \xi \cdot \omega_n$$

$$\boxed{\therefore T_s \simeq \frac{4}{\xi \cdot \omega_n}}$$



← you can replace numerator with 4 since that's basically the value

Rise Time

find $t_1 > 0$ s.t. $y(t_1) = 0.1$

find $t_2 > 0$ s.t. $y(t_2) = 0.9$

very rough

$T_R = t_2 - t_1 \Rightarrow \boxed{T_R \simeq \frac{1.8}{\omega_r}}$

% Overshoot

T_P : first time $\dot{y}(t) = 0$

$\% OS = y(T_P) - 1$

From eq 1

$$\dot{y}(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\sigma t} \sin(\omega_d t)$$

$$\text{set } \dot{y}(t) = 0 = \underbrace{\frac{\omega_n}{\sqrt{1-\xi^2}}}_{\neq 0} \underbrace{e^{-\sigma t}}_{\neq 0} \underbrace{\sin(\omega_d t)}$$

$\sin(\omega_d t) = 0$

$\omega_d t = k\pi$

$\Rightarrow t = \frac{k\pi}{\omega_d}$

$\therefore T_P = \frac{\pi}{\omega_d}$

pick the first time, so $k=1$

plug into $y(T_P) - 1$

$$\Rightarrow \% OS = y\left(\frac{\pi}{\omega_d}\right) - 1$$

$$= e^{-\sigma \cdot \frac{\pi}{\omega_d}} \cos(\pi) = e^{-\sigma \frac{\pi}{\omega_d}}$$

$$= \exp\left(-\frac{\sigma \pi}{\omega_d}\right)$$

$$= \exp\left(-\frac{\xi \pi}{\sqrt{1-\xi^2}}\right) \,\|$$

Summary:

$T_s = \frac{4}{\zeta \omega_n}$ $T_r = \frac{1.8}{\omega_n}$ $\%OS = \exp\left(-\frac{\zeta \pi}{\sqrt{1-\zeta^2}}\right)$

all these are valid for complex conjugate poles and no zeroes

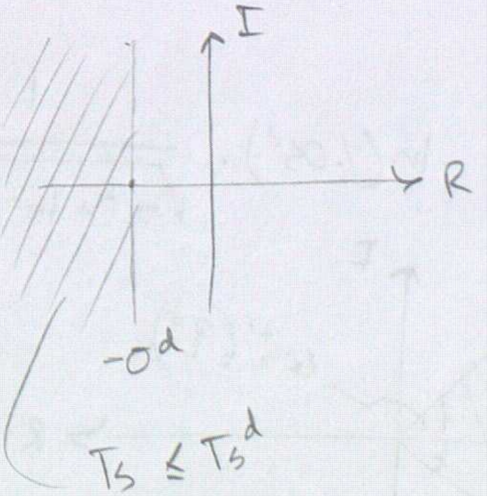
→ suppose we are given performance specs: "design"

$T_r \leq T_r^d$ $T_s \leq T_s^d$ $\%OS \leq \%OS^d$

→ Determine the region of the complex plane where the two complex conjugate poles of $Y(s)$ must be located in order to meet these specs

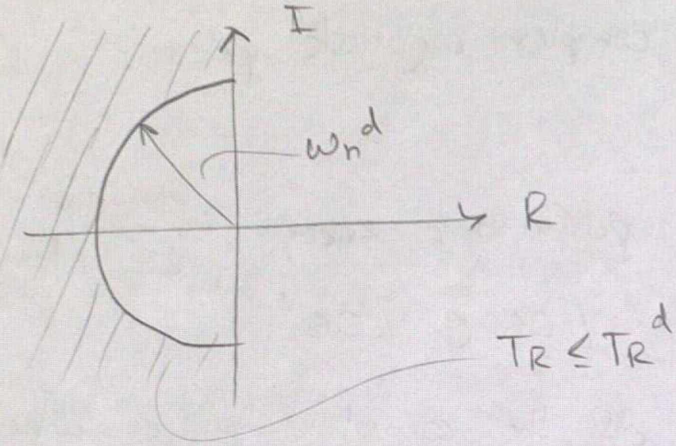
T_s in s plane

$T_s = \frac{4}{\zeta \cdot \omega_n} = \frac{4}{\sigma} \leq T_s^d$ → $\boxed{\therefore \sigma \geqslant \frac{4}{T_s^d} = \sigma^d}$



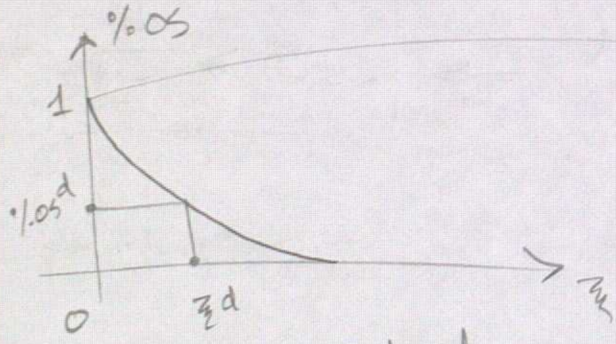
T_R in S-Plane

$T_R = \frac{1.8}{\omega_n} \leq T_R^d \Rightarrow \boxed{\therefore\ \omega_n \geq \frac{1.8}{T_R^d} = \omega_n^d}$



% OS in S-Plane

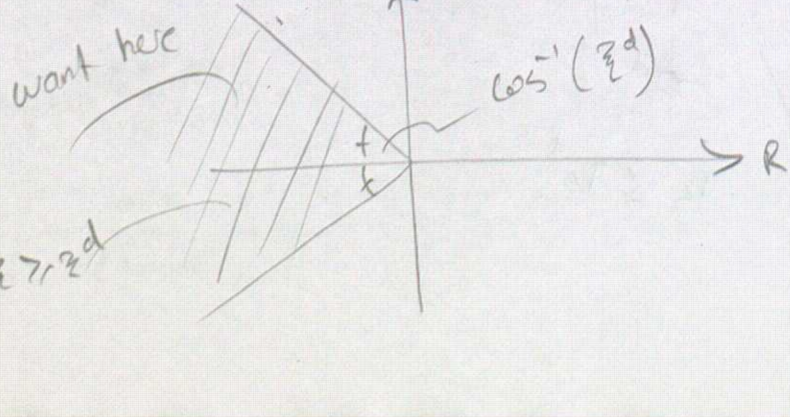
$\% OS = \exp\left(-\frac{\xi\pi}{\sqrt{1-\xi^2}}\right) \leq \% OS^d$



max oscillation since $\xi = 0$ is undamped system

Find ξ^d $\longrightarrow$ $\xi^d = -\ln(\% OS^d) \cdot \frac{1}{\sqrt{\pi^2 + \ln^2(\% OS^d)}}$

$\xi \geq \xi^d$



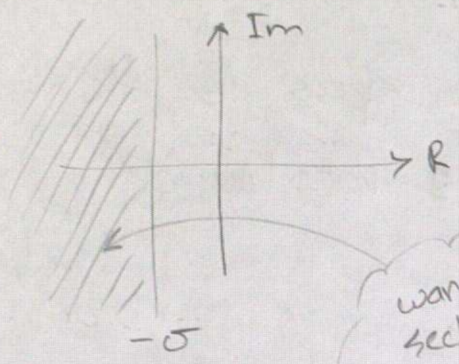
Conclusion:

the allowable region in the complex plane where the poles of $Y(s)$ should lie in order for $y(t)$ to meet 3 performance specs is the intersection of the 3 regions mentioned

Week 5 Lecture 2 Oct 7, 2025

Recap:

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma} \leq T_s^d$$

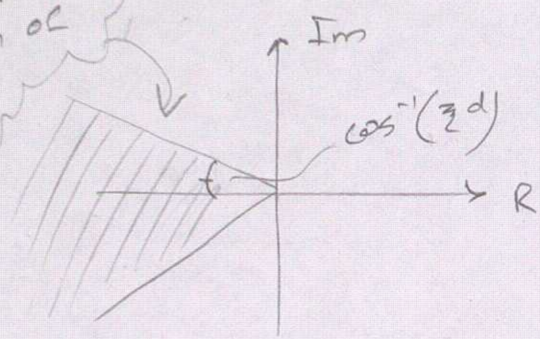


$$\%OS = \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right) \leq \%OS^d$$

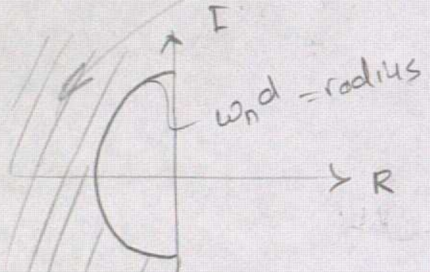
$$\Updownarrow$$

$$\zeta \geq \zeta^d$$

want intersection of these areas



$$T_r = \frac{1.8}{\omega_n} \leq T_r^d$$



PMDC Motor Control

Use the numerical parameters from Lecture 10

$$G(s) = \frac{1}{s^2+2s+2} = \frac{1}{2} \times \frac{2}{s^2+2s+2} \Leftarrow \frac{\sigma^2+\omega_d^2}{(s+\sigma)^2+\omega_d^2} \times \frac{1}{2}$$

$$\begin{cases} \sigma = 1 \\ \omega_d = 1 \end{cases} \text{ or } \begin{cases} \omega_n = \sqrt{2} \\ \zeta = \frac{1}{\sqrt{2}} \end{cases}$$

$$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$u(t) = V_0 \cdot I(t) \longrightarrow \boxed{G(s)} \longrightarrow y(t)$$

1) Transient: $T_r = \frac{1.8}{\omega_n} = 1.27$ seconds

$$T_s = \frac{4}{\xi \omega_n} = \frac{4}{\sigma} = 4 \text{ seconds}$$

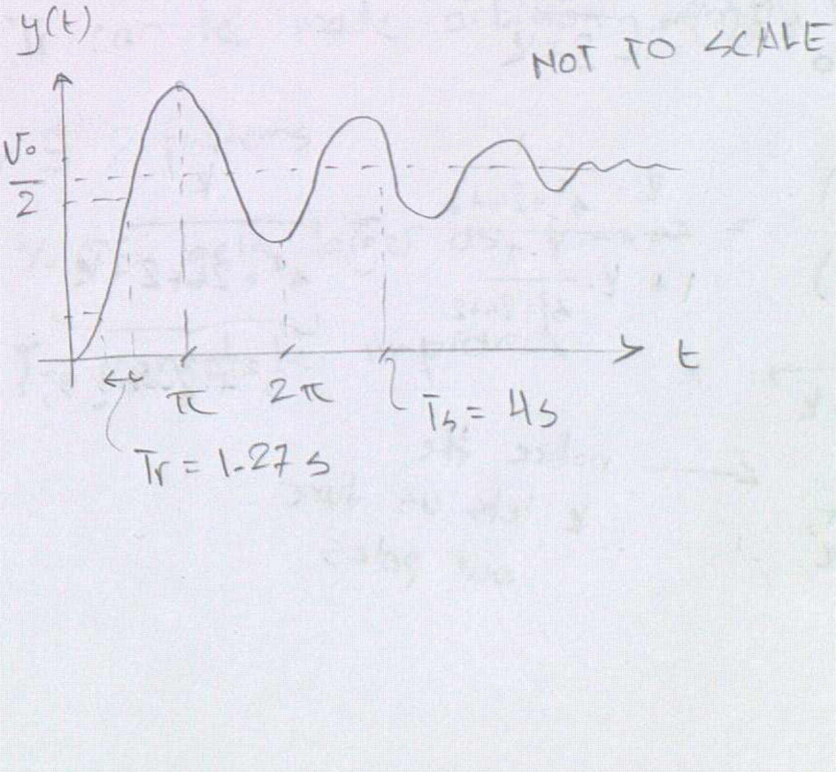
$$\% OS = \exp\left(-\frac{\xi \pi}{\sqrt{1-\xi^2}}\right) = e^{-\pi} \cong 0.04 = 4\%$$

2) Steady State:

$$\lim_{t \to \infty} y(t) \;=\!=\; \lim_{s \to 0} sY(s) = \lim_{s \to 0} s \cdot \underbrace{\frac{1}{s^2+2s+2}}_{G(s)} \cdot \underbrace{\frac{V_0}{s}}_{U(s)}$$

we know the limit exists.

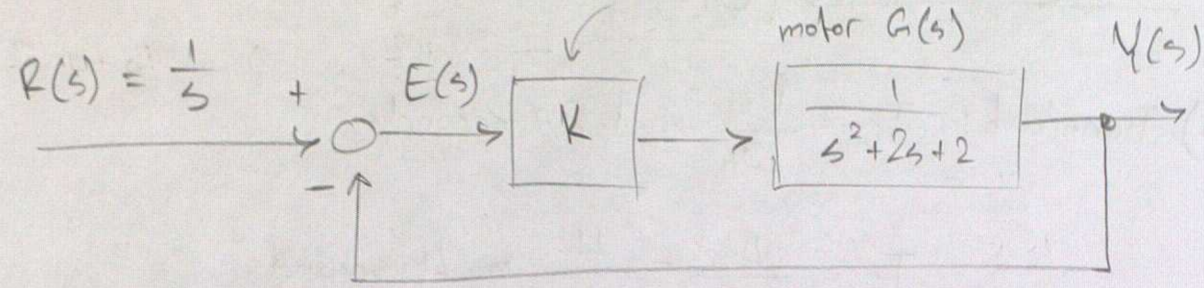
$$= \frac{V_0}{2}$$



T_s doesn't necessarily <u>have</u> to be on the $y(t)$ since we used the envelope to get the formula

P controller

Pcontroller : K is Gain



$$E(s) = R(s) - Y(s)$$

$$= \frac{1}{1 + K\,G(s)} \cdot \frac{1}{s}$$

R(s)

$$= \frac{s^2+2s+2}{s(s^2+2s+2+K)}$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{2}{2+K}$$

this means if $K \to \infty$, $e(t) \to 0$

$$\frac{Y(s)}{R(s)} = \frac{K \cdot G(s)}{1 + K\,G(s)} = \frac{K \cdot \frac{1}{s^2+2s+2}}{1 + K \cdot \frac{1}{s^2+2s+2}} = \frac{K}{\underbrace{s^2+2s+2+K}_{s^2 + 2\omega_n \xi s + \omega_n^2}}$$

$$\Rightarrow \omega_n = \sqrt{2+K}$$

$$\xi = \frac{1}{\sqrt{2+K}}$$

notice the K lets us tune our poles

$\omega_n = \sqrt{2+k} \qquad \xi = \frac{1}{\sqrt{2+k}}$

$k \to \infty$: $\omega_n \to \infty$, $\xi \to 0$, $T_r = \frac{1.8}{\omega_n} \to 0$ ... faster response

$T_s = \frac{4}{\omega_n \cdot \xi} = 4$ ... "not affected by k"

% OS :

- the controller has 2 benefits:

1) $e(\infty)$ can be made arbitrarily small by increasing k, but cant make it 0

2) T_r can be made arbitrarily small by increasing k

- but $\exists$ 2 problems

1) % OS gets larger as $k \to \infty$

2) T_s cannot be improved

Additional Poles

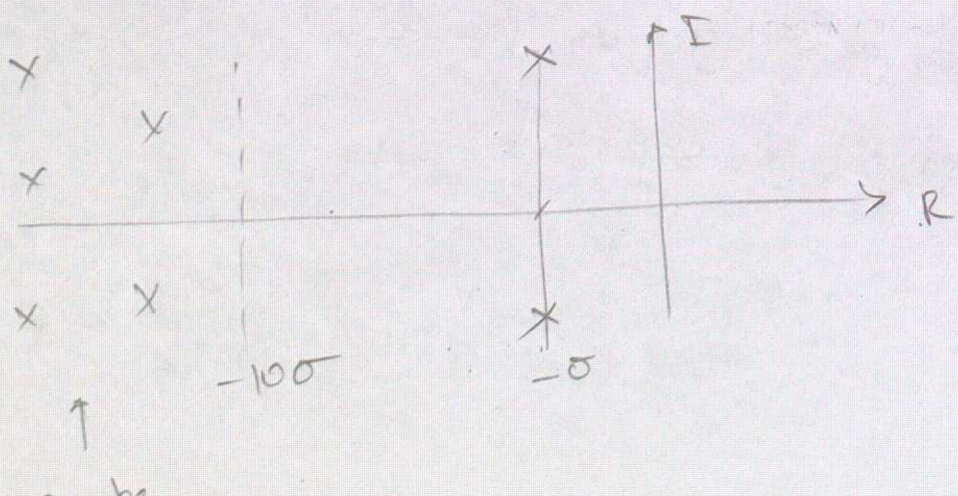
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

What if there are zeroes in $G(s)$ and/or > 2 poles?

⇒ Under certain conditions, we can still use the formulas for T_r, T_s, and %OS

⇒ Additional poles, if they are in the LHP, do not significantly affect the performance as long as their real part is "much more" negative than the real part of the conjugate poles.

↳ much more means by factor of 5 - 10 times



↑

can be ignored

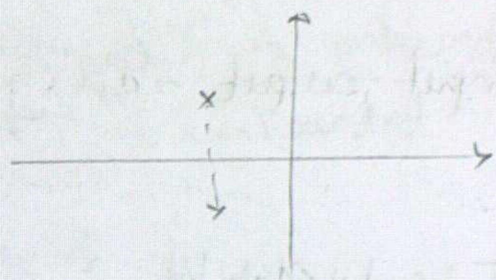
Additional Zeroes

1) zeroes have small impact as they are in LHP and their real parts are very different than

$\sigma = \xi \cdot \omega_n$

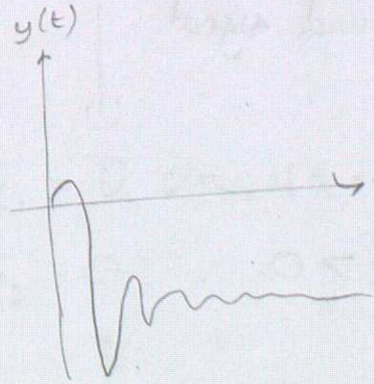
ensure

$$T_s = \frac{(\omega_n^2/a)(s+a)}{s^2 + 2\omega_n \xi s + \omega_n^2}$$



zeroes cause less issues if they are far away

2) Zeroes in the RHP are called non-minimum phase and have a major impact on the step-response b/c they can change sign of $\lim_{t \to \infty} y(t)$ or they can turn the max of the OS into a minimum



Week 5 lec 3 Oct 3, 2025

Stability

* internal stability : pertaining to state space with input $u = 0$.

* input-output stability : involve input u and output y.

Internal Stability

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \Rightarrow$$

1) set $u = 0$
2) discard y (you don't care about output)

$\Rightarrow \dot{x} = Ax$ — (system 1)

Define: system 1 is stable

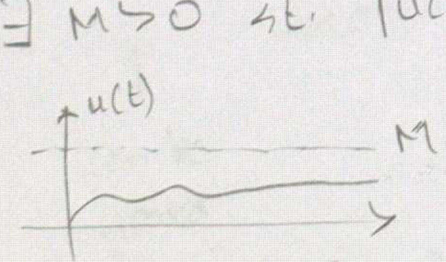
if $\forall\, x(0) \in \mathbb{R}^n$, the solution $x(t)$ is bounded

$\forall$: for all; $x(0)$: initial states; $\mathbb{R}^n$: n dim real number space

i.e: each component $x_i(t)$ is a bound signal

bounded signal $u(t)$

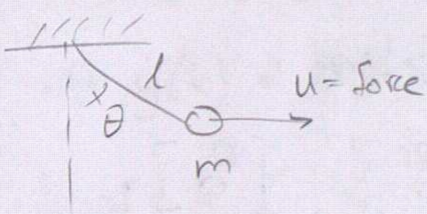
$\exists\, M > 0$ s.t. $|u(t)| < M$, $\forall\, t > 0$



* Asymptotic stable: if $\forall x(0) \in \mathbb{R}^n$, the solution $x(t)$ converges to 0: ie: each component $x_i(t) \to 0$ as $t \to \infty$

Unstable: if $\exists\ x(0) \in \mathbb{R}^n$ s.t. $x(t)$ is unbounded ie: some component $x_i(t)$ is unbounded

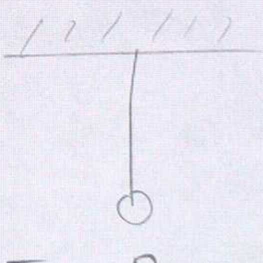
<u>Example:</u> Crane Model lec 4

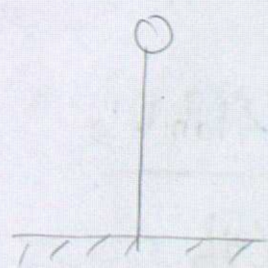


def: $x_1 = \theta$, $x_2 = \dot{\theta}$, $y = x_1$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}\sin x_1 + \frac{u}{ml}\cos x_1 \\ y = x_1 \end{cases}$$

put input $u=0$, then we know two equilibriums:





$\bar{x}_1 = 0, 2\pi, 4\pi \cdots$

$\bar{x}_2 = 0$

↑

"stable"

$\bar{x}_1 = \pi, 3\pi, 5\pi, \cdots$

$\bar{x}_2 = 0$

↑

"unstable"

if we initialize the pendulum near the equilibrum point $\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the pendulum will oscillate and stay near the equilibrium

- if we initialize the pendulum arbitrarily close to $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$, the pendulum would diverge away from this equilibrium $\longrightarrow$ instability

- physically, your pendulum might not fly off, but since it goes outside of your operating point/region, it's already unstable to your POV.

$\Rightarrow$ suppose the pendulum has friction:

if we initialize the pend. near $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

not only does the pend. stay near $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

but now $x(t) \longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

"asymptotic stability"

Input/Output Stability

$Y(s) = G(s) \cdot U(s)$ ——— system 2

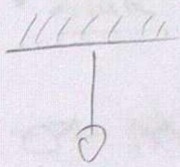
$G(s)$ assumed to be rational and proper

Define system 2 is BIBO stable.

if for any bounded input signal $u(t)$, the corresponding output signal $y(t)$ is also bounded

BIBO unstable : if $\exists$ a bounded input signal giving an unbounded output signal $y(t)$

Example:



$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

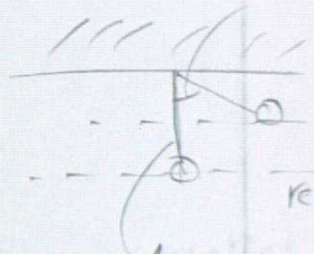
suppose $u(t)$ is a bounded signal,

$\Rightarrow y(t) = \theta(t)$ will not be bounded!

interesting when you think about it

Proof: given an arbitrarily small bound, $\varepsilon > 0$, on the signal $u(t)$, we can always make $\dot{\theta}(t) \to \infty$

$$u(t) = +\varepsilon \cdot \mathrm{sgn}(x_2 \cdot \cos x_1)$$



mechanical energy:

$$E = \underbrace{\frac{1}{2} m l^2 x_2^2}_{\text{kinetic}} + \underbrace{mgl \cdot (1 - \cos x_1)}_{\text{potential}}$$

$$\Rightarrow \frac{dE}{dt} = \frac{1}{2} m l^2 \cdot 2 x_2 \dot{x}_2 + mgl \sin x_1 \cdot \dot{x}_1$$

$$= m l^2 x_2 \cdot \left[-\frac{g}{l} \sin x_1 + \frac{u}{ml} \cos x_1 \right]$$

$$+ mgl \sin x_1 \cdot x_2$$

$$= l u x_2 \cos x_1$$

check the derivative and if we show it's always (+)ve, then we can prove that it can blow up system

$\Rightarrow l x_2 \cos x_1 \cdot \varepsilon \cdot \mathrm{sgn}(x_2 \cdot \cos x_1) \leftarrow$ plug in u

$= l \cdot \varepsilon \left| x_2 \cdot \cos x_1 \right| > 0$

$\therefore \frac{dE}{dt} > 0 \rightarrow$ energy is unbounded (energy is injected into the system as long as $x_2 \cdot \cos x_1 \neq 0$)

$\Rightarrow E(t) \rightarrow \infty \Rightarrow x_2(t) \rightarrow \infty$

Input - Output instability !!!

Week 6 Tutorial 5 Oct 7, 2025

Time Response for a State Spaces

$$\dot{x} = Ax + Bu \qquad \xrightarrow{\mathcal{L}} \qquad sX(s) - x(0) = AX(s) + BU(s)$$
$$y = Cx + Du \qquad\qquad\qquad Y(s) = CX(s) + DU(s)$$

* $\boxed{\text{state}}$ time response

$G(s) = C[sI-A]^{-1}B + D$

assumes zero state.

$$\Rightarrow sX(s) - AX(s) = x(0) + BU(s)$$
$$X(s)[sI-A] = x(0) + BU(s)$$
$$X(s) = \underbrace{(sI-A)^{-1}}_{M(s)}x(0) + \underbrace{(sI-A)^{-1}}_{M(s)}BU(s)$$

$$X(s) = M(s)x(0) + \underbrace{M(s)BU(s)}$$

$\mathcal{L}^{-1}$

$$x(t) = \underbrace{M(t)x(0)}_{\substack{IC \\ \text{response}}} + \underbrace{\int_0^t M(t-\tau)Bu(\tau)d\tau}_{\text{input response}}$$

what is $\mathcal{L}^{-1}\{(sI-A)^{-1}\}$ ← called the matrix exponential

$$\mathcal{L}^{-1}\{(sI-A)^{-1}\} = e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

scalar case:

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

HW 2 Q3 Part 2

Consider the sys... Find the TF from u to y

$$\begin{cases} \dot{x}_1 = x_2 + u \\ \dot{x}_2 = u \end{cases}$$
$$y = x_1 + x_2 + u \Rightarrow$$

$G(s) = C(sI-A)^{-1}B + D$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \end{bmatrix} u$$

$$G(s) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix}$$

$\det(Is - A) = s^2$

$$G(s) = \frac{1}{s^2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix}$$

$$= \frac{1}{s^2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+1 \\ s \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix}$$

$$= \frac{s+1+s}{s^2} + 1 = \frac{s+1+s+s^2}{s^2} = \frac{s^2+2s+1}{s^2}$$

HW 2 Q5 : $G(s) = \dfrac{Y(s)}{R(s)}$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} r$$

$$G(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 3 & 2 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

$$\det(Is-A) = s\left[s(s+5)+2\right] + 1\left[3\right] + 0$$

$$= s\left[s^2+5s+2\right] + 3 = s^3+5s^2+2s+3$$

$$\begin{bmatrix} +(s^2+5s+2) & -(3) & +(-3s) \\ -(-s-5) & +(s^2+5s) & -(2s+3) \\ +(1) & -(-s) & +(s^2) \end{bmatrix} = \begin{bmatrix} s^2+5s+2 & -3 & -3s \\ s+5 & s^2+5s & -2s-3 \\ 1 & s & s^2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} s^2+5s+2 & s+5 & 1 \\ -3 & s^2+5s & s \\ -3s & -2s-3 & s^2 \end{bmatrix}$$

$$G(s) = \frac{1}{\det} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

$$= \frac{1}{\det} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 10s \\ 10s^2 \end{bmatrix}$$

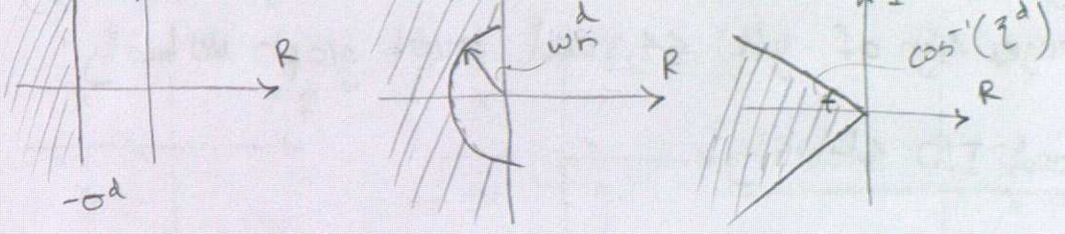
$$= \frac{10}{\det} = \frac{10}{s^3+5s^2+2s+3} \quad //$$

Week 5 and 6 Recap — Sat, Oct 12, 2025

lec13 **Control Specs and Pole loc (W5L1, lec13)**

- we looked at 3 control specifiations (settling time, rise time, and percent overshoot) and where to place poles to meet these specs.

$$T_s = \frac{4}{\zeta\omega_n} \qquad T_r = \frac{1.8}{\omega_n} \qquad \%.OS = \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$



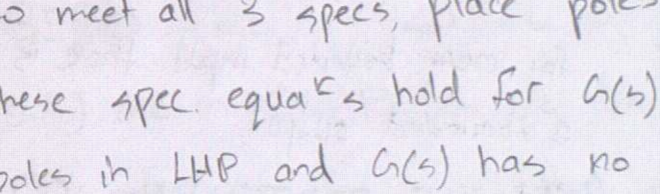
$$\zeta_d = \frac{-\ln(\%.OS^d)}{\sqrt{\pi^2 + \ln^2(\%.OS^d)}}$$

$$\sigma \geq \frac{4}{T_s^d} = \sigma^d \qquad \omega_n \geq \frac{1.8}{T_r^d} = \omega_n^d \qquad \zeta \geq \zeta^d$$

- to meet all 3 specs, place poles in intersection of above 3 areas.
- these spec. equa^{n}s hold for $G(s)$ with precisely 2 complex conjugate poles in LHP and $G(s)$ has no zeroes

lec14 **Controls Specs Example, P controller, Zeroes and Additional Poles**

- given a transfer function, you can find params σ, ωd, ωn, ζ
  - ↳ for transient, use params to find T_s, T_r, and $\%$ OS
  - ↳ for s.s. use Final Val. Thm to find ss. value of $y(t)$
- P-controller: add a gain K b4 $G(s)$ that you can control



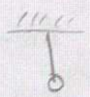
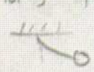
↳ if $K \to \infty$, $e(t) \to 0$ and use Transfer Func^{n} $\frac{Y(s)}{R(s)}$ to get params and see what are the effects $K \to \infty$ has on T_s T_r and $\%$ OS

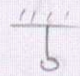
$$E(s) = R(s) - Y(s) = \left(\frac{1}{1+KG(s)}\right)R(s)$$

you want $e(t) \to 0$, so adjust K

- Additional poles: negligible effect if real part more than 5-10x more negative than your real poles of interest
- Additional zeroes:
  - ↳ negligible effect if on LHP and real part very different than $\sigma = \zeta\omega_n$
  - ↳ major effect if RHP, called non-minimum phase, and can change sign of $y(t)$ s.s. and invert graph vertically.

lec 15 Internal and I/O Stability

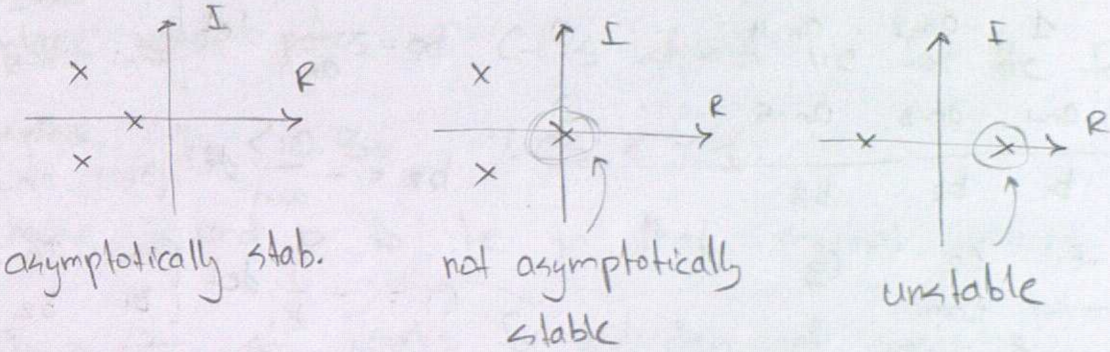
- Internal Stability deals with just system states: (matrix A)
- system is internally stable if every component (row equation) of matrix A is bounded
- system is asymptotically stable if those A matrix equations converge to 0 as $t \to \infty$
- example for crane model: 
- you are stable if you start near  since system will stay near your equilibrium point
- you are unstable if you start near  since you'll leave your linearized region.
- you are asymptotically stable if your system has friction and you start near 

- Input/Output Stability deals with $U(s)$ and $Y(s)$ → BIBO stab.
- system is BIBO stable if for every bounded input there is a bounded output.
- system is BIBO unstable if ∃ a bounded input that yields an unbounded output
- for our crane model:  the system is BIBO unstable because you can input a "signum" input that changes direction based on sign. This physically means you keep adding energy/force little by little until output $y(t) = \theta(t)$ angle blows up. You can prove this by showing derivative of energy always ≥ 0

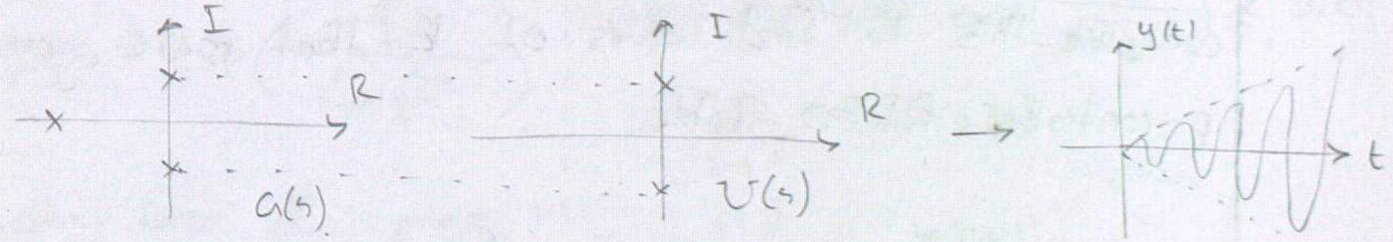
lec 16
lec 17

Asymptotic and BIBO Stability (for open loop system TF

* Asymptotic stability: system is asymptotically stable iff all eigenvalues of matrix A (aka the roots/poles of the rows $X_i(s)$, or the roots of $\det(sI-A)$) are in the open left half plane (OLHP) as per Thm 1. → OHLP ≡ real part of poles < 0



* BIBO stability: system is BIBO stable iff all poles of $G(s)$ are in OLHP as per Thm 2 → remember if your $G(s)$ has poles on Im axis, then you could choose a (bounded) $U(s)$ with those exact poles, increasing multiplicity and getting unbounded $y(t)$.



* Asymptotically stability implies BIBO stability: if a system is asymptotically stable, then it is also BIBO stable, but not the other way around →

$$G(s) = \underbrace{\left(\frac{1}{\underbrace{\det(sI-A)}_{\text{eigenvalues of A}}} \cdot C \cdot \mathrm{Adj}(sI-A) \cdot B\right) + D}_{\text{poles of } G(s).}$$

can have BIBO stable system without asymptotic stability lec18 **Routh Stability Criterion (t-dom)**

- an algorith that lets you check whether or not the roots of a polynomial are in OLHP.

$$a(s) = s^n + (a_{n-1})s^{n-1} + (a_{n-2})s^{n-2} + \cdots + a_1 s + a_0$$

s^n	1	a_{n-2}	a_{n-4}	$\cdots$	0
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$	0
s^{n-2}	b_1	b_2	b_3		
s^{n-3}	c_1	c_2	c_3		
$\vdots$					
s					
s^0					

$$b_1 = -\frac{1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-2} \\ a_{n-1} & a_{n-3} \end{bmatrix}$$

$$b_2 = -\frac{1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-4} \\ a_{n-1} & a_{n-5} \end{bmatrix}$$

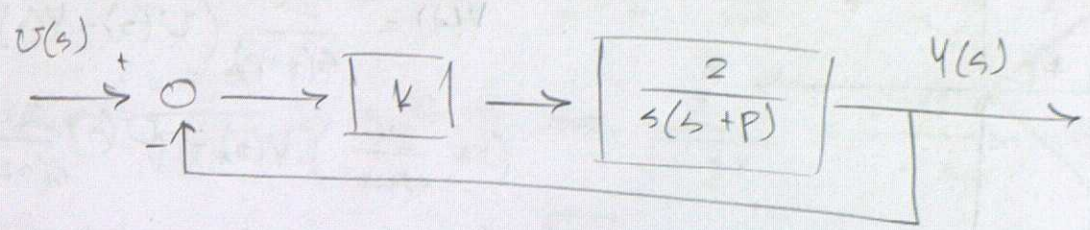
$$c_1 = -\frac{1}{b_1} \det \begin{bmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{bmatrix}$$

$$c_2 = -\frac{1}{b_1} \det \begin{bmatrix} a_{n-1} & a_{n-5} \\ b_1 & b \end{bmatrix}$$

- if this column has no sign variation $\rightarrow$ roots $a(s)$ in <u>OLHP</u>
- the number of sign variations equals # of poles in <u>ORHP</u>
- can use this to find values of K that make your p-controller BIBO stable.

Week 7 Tutorial 6 Oct 14, 2025

Problem 1: consider the C.L.S below, k and p are design params



a) Using standard formulas, sketch region in the complex plane where poles of C.L.S should lie for the following specs: $T_s \leq 0.8s$, $\%OS < \%1$

b) choose k and p to be in that region.

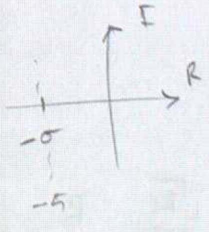
→ a) $T_s = \frac{4}{\xi \cdot \omega_n}$ ← works for underdamped systems

you know you are underdamped since ∃ some overshoot, so it can't be overdamped or critically damped

$$\%OS = \exp\left(\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right)$$

settling time: $\frac{4}{\xi \cdot \omega_n} \leq 0.8$

$$\frac{4}{0.8} \leq \underbrace{\xi \cdot \omega_n}_{\sigma}$$



$$\sigma \geq \frac{4}{0.8}$$

$$|\sigma| \geq 5$$

percent overshoot: $\exp\left(\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right) \leq 0.01$

$$-\frac{\xi\pi}{\sqrt{1-\xi^2}} \leq \ln(0.01)$$

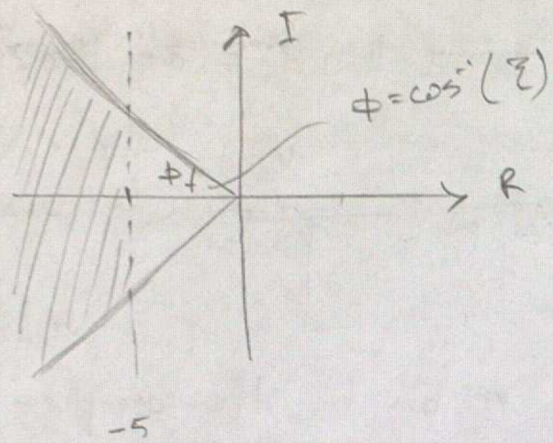
$$\frac{\xi}{\sqrt{1-\xi^2}} \geq \frac{\ln(0.01)}{-\pi}$$

$$\xi^2 \geq (1.4658)^2(1-\xi^2)$$

$$\xi \geq \sqrt{\frac{2.15}{3.02}}$$

$$\xi \geq 0.84$$

a)



b) Find TF:

$$Y(s) = \frac{2k}{s(s+p)}\left(U(s) - Y(s)\right)$$

$$\left(1 + \frac{2k}{s(s+p)}\right) Y(s) = U(s)\,\frac{2k}{s(s+p)}$$

$$\left(s^2 + ps + 2k\right) Y(s) = U(s)\, 2k$$

b) $$G(s) = \frac{Y(s)}{U(s)} = \frac{2k}{s^2 + ps + 2k} \Rightarrow \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$ } standard TF for 2nd order responce to step input

$\Rightarrow$ pick $\sigma = 6$, $\xi = 0.9$

$$\sigma = \xi\omega_n \Rightarrow \omega_n = \frac{6}{0.9} = \frac{20}{3}$$

set equal to each other and find k, p

plug this into our <u>design</u> char. poly.

$$s^2 + 2(0.9)\left(\frac{20}{3}\right)s + \left(\frac{20}{3}\right)^2 = s^2 + 12s + \frac{400}{9}$$

$$\Rightarrow p = 12 \;//\; \text{ and } \; 2k = \frac{400}{9} \Rightarrow k = \frac{200}{9} \;//$$

<u>Problem 5</u>

$$G(s) = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$ } RLC TF with cap volt. as output

design TF: $$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\omega_n = \frac{1}{\sqrt{LC}}$$

$$\xi = \frac{1}{2\omega_n}\frac{R}{L} = \frac{\sqrt{LC}}{2}\cdot\frac{R}{L} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

$$T_s = 0.002 = \frac{4}{\xi \omega_n} = \frac{4}{\frac{R}{2}\sqrt{\frac{C}{L}} \cdot \frac{1}{\sqrt{LC}}} = \frac{4}{\frac{R}{2} \cdot \frac{1}{L}} \Rightarrow T_s = \frac{8L}{R}$$

$$\exp\left(\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right) = 0.15 \Rightarrow \frac{\xi}{\sqrt{1-\xi^2}} = \frac{\ln(0.15)}{-\pi} \Rightarrow \xi = 0.517$$

$$\Rightarrow \xi = 0.517 = \frac{R}{2}\sqrt{\frac{C}{L}}$$

$$\frac{R}{\sqrt{L}} = \frac{(0.517)(2)}{\sqrt{10\times10^{-6}}} = 327.2$$

use these two
to solve L, R

$$\Rightarrow T_s = 0.002 = \frac{8L}{R}$$

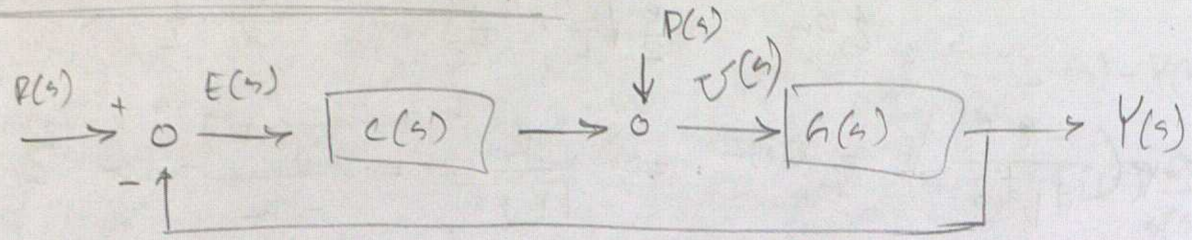
$$\frac{R}{L} = 4000$$

$$L \cdot 4000 = \sqrt{L} \cdot 327.2 \Rightarrow L = \left(\frac{327.2}{4000}\right)^2 = 6.68 \text{ mH}$$

$$R = 26.7\,\Omega$$

Week 7 Lec 1 Oct 14, 2025

Standard Feedback Loop



$R(s)$ or $r(t)$: reference signal

$D(s)$ or $d(t)$: disturbance signal

$E(s)$ or $e(t)$: error signal, or "tracking error"

$Y(s)$ or $y(t)$: output

Assumptions of Inputs:

$R(s)$ and $D(s)$ are rational, strictly proper func^{n}s with fixed set of poles, but arbitrary zeroes and gains

call $\mathcal{R}$ and $\mathcal{D}$ the classes of references and disturbances which satisfies this assumption

Ex: ① $R(s) = \dfrac{N(s)}{s^2+4}$, $N(s)$ must be a poly. order ≤ 1 with no zeroes at $\pm 2j$

$\mathcal{R}$ = { class of purely sinusoidal signals with freq of 2 rad/s with zero average }

= { $M_1 \sin(2t+\Phi)$, $M_1, \Phi \in \mathbb{R}$ are arbitrary }

② $P(s) = \dfrac{N(s)}{s(s^2+1)}$, $N(s)$ should be poly. order ≤ 2 with no zeroes at 0 and $\pm j$

$\mathcal{D}$ = { class of purely sinusoidal signals with sin. freq. of 1 rad/s and non-zero average }

= { $M_1 \sin(t+\Phi) + C$, $M, \Phi, C \in \mathbb{R}$ are arbitrary }

③: $R(s) = \dfrac{N(s)}{s^k}$, $k \ge 1$

$\mathcal{R}$ = { poly. of order $\le k-1$, poly func'n of t }

Basic Controll Problem

BCP: design a controller $C(s)$ s.t. the following 3 specs are met:

1) stability: for any Bounded ref $r(t)$ and dist. $d(t)$, both $u(t)$ and $e(t)$ are bounded.

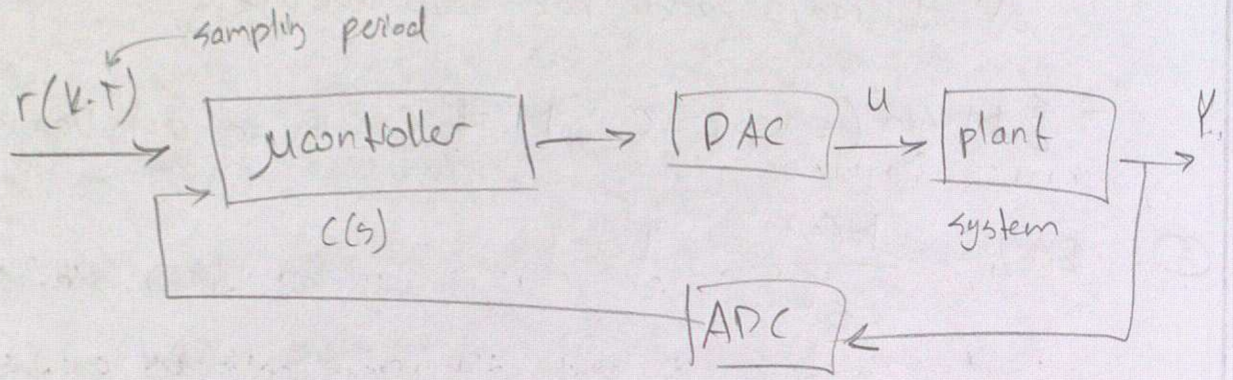
2) Asymptotic tracking: assume $d(t) = 0$, for each $r(t) \in \mathcal{R}$ the tracking error $e(t) = r(t) - y(t) \to 0$ as $t \to \infty$

3) disturbance rejection: assume $r(t) = 0$, for each $d(t) \in \mathcal{D}$ the output $y(t) \to 0$ as $t \to \infty$

How to implement $C(s)$?

1) Analog implementation : op-amp

2) Digital " : microcontroller or FPGA



$\dot{x}(t) = a \Rightarrow x[KT]$

- pole zero cancellation can introduce new poles but from numerator

- Learn Gang of 4 TF's

Week 7 Lec 2 Oct 16, 2025

Spec (b): Asymptotic Tracking

- FVT: Analytic $F(s)$ (except for finite # of poles)
  - ↳ all poles have real ≤ 0 and at most one pole on im axis at $s=0$ ← lets you know if ∃ a final val.

Asymptotic Tracking

- set disturbance to zero and put in a polynomial reference
- for different signals, you'd need different controllers and ∴ cond^{n}s
- goal: find cond^{n}s that $e(t) = r(t) - y(t) \to 0$ as $t \to \infty$ (the error goes to zero)
- assume that controller $C(s)$ has been designed s.t. BIBO stab. of sys. has been met.

  spec (a)
- get $E(s)$ from Gang of 4 TF's and plug in $R(s)$
- → you find that s^k pole must get cancelled
- you use primes (') to denote fun^{n}s with poles factored out
  - ↳ once you cancel s^k, just check if $E(s)$ has poles in OLHP as regular.
- new spec/requirement if your $C(s)G(s)$ has one pole less than k (from your reference) → you have non-zero $e(t)$ s.s.

- but if your $C(s)G(s)$ has more than 1 pole less than ref, then you would blow up → no s.s. $e(t)$, it → ∞
- idea: you need an even faster or at least nearly as fast contoller if you have a fast reference.

Terminology

- l-type: $C(s)G(s)$ has exactly l poles at $s=0$
- idea: must use $C'(s)$ and $G'(s)$ method to explicitly be able to pull out (and later cancel) s^k poles.
- → type $k-1$: your $e(t)$ has non-zero s.s. value

→ type k: your $e(t)$ goes to zero

→ type $k-2$ or smaller : your $e(t)$ blows up → cannot track.

→ diff. ref. signals have diff. requirements

Week 7 Lec 3 Oct 18, 2025

PMDC Example

- first check if spec (a) → BIBO stab. is met.

spec (a) : $\begin{cases} \text{no unstable pole/zero cancellation} \\ \text{roots of } 1 + C(s)G(s) \text{ in OLHP} \end{cases}$

spec (b) : Final Value Thm

Disturbance Rejection — spec (c).

- If spec (a) is met and $r(t) = 0$ and suppose

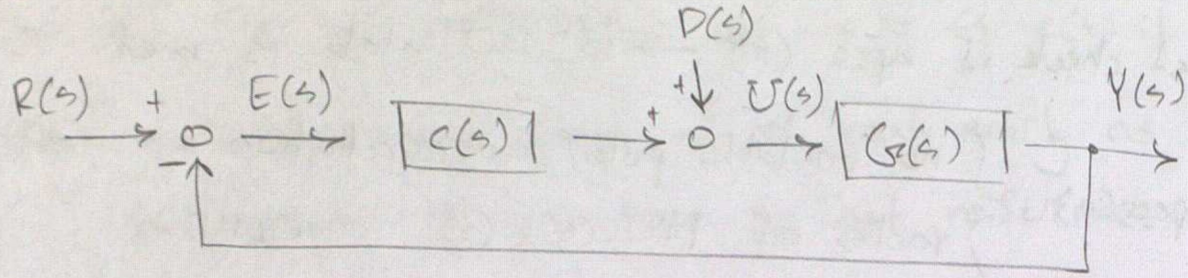
$D(s) = \dfrac{M_D(s)}{s^k}$, $k \geq 1$, then

→ disturb. rejection is established iff $C(s)$ has type k → similar to spec (b) but you can't avail poles of $G(s)$ @ $t=0$

Week 7 Recap — Oct 29, 2025

lec19

Standard Feedback Loop and Basic Control Problem



$R(s)$ $[r(t)]$: reference signal

$D(s)$ $[d(t)]$: disturbance signal

$E(s)$ $[e(t)]$: tracking error

$Y(s)$ $[y(t)]$: output

to derive gang of four transfer functions:

$$E(s) = (\quad)R(s) + (\quad)D(s)$$

$$U(s) = (\quad)R(s) + (\quad)D(s)$$

Use superposition to find $E(s)$ and $U(s)$

set $D=0$:
$$E = R - Y$$
$$E = R - E \cdot C \cdot G$$
$$E + ECG = R$$
$$E = \frac{1}{1+CG} R \quad //$$

set $R=0$:
$$E = -Y$$
$$E = -(EC + D)G$$
$$E = -ECG - DG$$
$$E + ECG = -DG$$
$$E = \frac{-G}{1+CG} D \quad //$$

set $D=0$:
$$U = C(R-Y)$$
$$U = C(R - GU)$$
$$U = CR - CGU$$
$$U + CGU = CR$$
$$U = \frac{C}{1+CG} R \quad //$$

set $R=0$:
$$U = C(-Y) + D$$
$$U = -CGU + D$$
$$U + UCG = D$$
$$U = \frac{1}{1+CG} D \quad //$$

$$\therefore \; E(s) = \frac{1}{1+C(s)G(s)} R(s) + \frac{-G(s)}{1+C(s)G(s)} D(s)$$

$$U(s) = \frac{C(s)}{1+C(s)G(s)} R(s) + \frac{1}{1+C(s)G(s)} D(s)$$

← Gang of 4 transfer functions.

- For this closed-loop-system, you care about 3 specs:
  a) BIBO stability
  b) Asymptotic tracking
  c) Disturbance Rejection

a) BIBO stability

- Your CLS is BIBO stable if gang of 4 TF's is BIBO stable
  - ↳ notice that they all have $1+C(s)G(s)$ as denominator
  - ↳ this means you'd want roots of $1+C(s)G(s)$ ($\equiv$ poles of $\frac{1}{1+CG}$) are in OLHP
  - ↳ however, notice that one TF has $G(s)$ in numerator and a second one has $C(s)$ in numerator
  - ↳ this means we must be cautious of pole-zero cancellations when we multiply $C(s)$ with $G(s)$, since that introduces an additional pole as such:

$$\frac{G(s)}{1+C(s)G(s)} = \frac{G'(s)/s-p}{1+C'(s)\cancel{(s-p)}\cdot\frac{G'(s)}{\cancel{(s-p)}}} = \frac{G'(s)}{(s-p)(1+C'(s)G'(s))}$$

additional pole appears in overall TF

  - ↳ that extra pole is okay as long as it's in OLHP

∴ Thm 1: The CLS is BIBO stable iff:

1) poles of $\frac{1}{1+C(s)G(s)}$ are in OLHP

2) the product $C(s)G(s)$ does not have a pole/zero cancellation where the cancelled pole was in RHP

b/c the cancelled pole appears again as pole in at least one of the 4 TF's lec 20 b) Asymptotic Tracking

- given a polynomial reference signal, we are trying to make error go to 0: $e(t) \to 0$ as $t \to \infty$
- we use the concept of Final Value Thm: If $F(s)$ is
  1) Analytic except for a finite # of poles
  2) All poles are in LHP (real part ≤ 0)
  3) At most 1 pole on imaginary axis at $s=0$

  $\to$ then FVT says that $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\cdot F(s)$

- if you're given an $r(t) = [C_0 + C_1 t + C_2 t^2 + \dots + C_{k-1} t^{k-1}]u(t)$

then $R(s) = \frac{N_R(s)}{s^k}$ $\longrightarrow$ eg: $r(t) = (1 + 2t + 3t^2)u(t)$, $k=3$

$$R(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{3\cdot 2!}{s^3} = \frac{s^2}{s^3} + \frac{2s}{s^3} + \frac{3\cdot 2!}{s^3} = \frac{N_R(s)}{s^3}$$

- basically, any polynomial $r(t)$ order $k-1 \xrightarrow{\mathcal{L}} R(s) = \frac{N_R(s)}{s^k}$ becomes $R(s)$ with k poles at $s=0$.

- Assume that $C(s)$ is designed s.t. system is BIBO stable → spec (a)

$$E(s) = \frac{1}{1 + C(s)G(s)} R(s) = \frac{N_R(s)}{s^k(1 + C(s)G(s))}$$

$C(s) \triangleq \frac{N_C(s)}{D_C(s)}$

$G(s) \triangleq \frac{N_G(s)}{D_G(s)}$

$$E(s) = \frac{N_R(s)}{s^k\left(1 + \frac{N_C N_G}{D_C D_G}\right)} = \frac{N_R(s)}{s^k\left(\frac{D_C D_G + N_C N_G}{D_C D_G}\right)}$$

$$E(s) = \frac{N_R(s) D_C D_G}{s^k(D_C D_G + N_C N_G)}$$

$\leftarrow$ we need to "cancel" s^k, and if $C(s)G(s)$ has k poles at $s=0$, then top becomes $N_R \cdot s^k D_C D_G$ and we can cancel s^k

- Note: we say $C(s)G(s)$ is type k if it has k poles at $s=0$
- see how having a
  - type k $C(s)G(s)$ fully cancels s^k $\rightarrow \lim_{t\to\infty} e(t) = 0$
  - type $k-1$ $C(s)G(s)$ leaves one "s" $\rightarrow \lim_{t\to\infty} e(t) =$ some finite value
  - type $k-2$ or less $C(s)G(s)$ leaves more than one "s" in $E(s)$ denominator, so FVT blows up $\rightarrow \lim_{t\to\infty} e(t)$ DNE

$\therefore$ Assuming spec (a) BIBO stability and $d(t) = 0$, and your $r(t) =$ some polynomial $\xrightarrow{\mathcal{L}} R(s) = \dfrac{N_R(s)}{s^k}$, $k \geq 1$

Asymptotic tracking is achieved iff $C(s)G(s)$ is type k

$\hookrightarrow$ if type $k-1$, $e(\infty) =$ finite but not zero

$\hookrightarrow$ if type $k-2$ or less, $e(\infty) = \infty$

lec 21 c) Disturbance Rejection

- recall that to meet spec (b) asymptotic tracking, $C(s)G(s)$ must be type k (must have k poles at $s=0$)
- now, for disturbance rejection, it's the same idea: we must have $C(s)G(s)$ cancel the $s=0$ poles introduced by $D(s)$ $\rightarrow$ but there's a catch:
- recall gang: $E(s) = \dfrac{1}{1+CG} R + \dfrac{-G}{1+CG} D$

notice this G? this means that any of G's poles at $s=0$ won't be useful

- $C(s)G(s)$ needs k poles at $s=0$, but any pole of $G(s)$ at $s=0$ won't be helpful since it would get absorbed/cancelled by $G(s)$ in denominator and not work to cancel the s^k introduced by $D(s)$ $\rightarrow$ so must come from $C(s)$ to be valid

Assume $R(s)$ type k and $D(s)$ type j:
- spec(b): $C(s)G(s)$ type k
- spec(c): just $C(s)$ type j

Week 9 Tutorial 8 Nov 4, 2025

Problem 2

$r(t) = R_0 t \cdot 1(t)$, compute steady state tracking error

$$E(s) = R(s) - Y(s)$$

$$= R(s) - \left(E(s) \cdot \frac{K}{s(\tau s+1)}\right)$$

$$E(s) = \frac{R(s)}{1 + \frac{K}{s(\tau s+1)}} = \frac{R(s)}{1 + G(s)} = \frac{1}{1 + \frac{K}{s(\tau s+1)}} \cdot \frac{R_0}{s^2}$$

$$E(s) = \underbrace{\frac{s(\tau s+1)}{\tau s^2 + s + K}}_{\substack{\tau > 0,\ K > 0 \\ \text{then OLHP} \checkmark}} \cdot \underbrace{\frac{R_0}{s^{\cancel{2}}}}_{\substack{\text{1 pole} \\ \text{at } s=0 \checkmark}}$$

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

$\Rightarrow \therefore$ can apply FVT

$$\lim_{t \to 0} e(t) = \lim_{s \to 0} \left[\frac{R_0(\tau s+1)}{\tau s^2 + s + K}\right] = \frac{R_0}{K}$$

can set tracking error by playing with K

Problem 5

a) $e(\infty) = 1 - \infty$ ← tracking error blows up.

$$E(s) = U(s) - Y(s)$$

$$e(\infty) = 1 - \infty$$

b) $E(s) = U(s) - s\,Y(s)$

$$= U(s) - \cancel{s}\left[\frac{1}{\cancel{s^2}} \cdot \frac{s(s+1)}{(s+2)} \cdot E(s)\right]$$

$$1 + \frac{s+1}{s+2}$$

$$\frac{s+2+s+1}{s+2} = \frac{2s+3}{s+2}$$

$$E(s) = U(s) - \frac{(s+1)}{(s+2)} E(s)$$

$$E(s)\left(1 + \frac{s+1}{s+2}\right) = U(s)$$

$$E(s) = \frac{(s+2)\,U(s)}{2s+3} = \frac{s+2}{2s+3} \cdot \frac{1}{s} \quad //$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0}\left[\frac{s+2}{2s+3} \cdot \frac{1}{\cancel{s}}\right] \cdot \cancel{s} = \frac{2}{3}$$

c) $Y(s) = Y_R(s) + Y_D(s)$

$$Y_D(s) = D(s) + \frac{1}{s+1} \cdot \frac{1}{s} \cdot (-Y_D(s))$$

$$= \frac{1}{1 + \frac{1}{s(s+1)}} D(s) = \frac{s(s+1)}{s^2+s+1} D(s)$$

$$Y_R(s) = \frac{1}{s+1} \cdot \frac{1}{s} \left(U(s) - Y_R(s)\right)$$

$$= \frac{\frac{1}{s(s+1)}}{1 + \frac{1}{s(s+1)}} = \frac{1}{s^2+s+1} U(s)$$

$$E(s) = U(s) - \frac{1}{s^2+s+1} U(s) - \frac{s(s+1)}{s^2+s+1} D(s)$$

$$= \frac{1}{s} - \frac{1}{s^2+s+1} \cdot \frac{1}{s} - \frac{s+1}{s^2+s+1} = 0\,?$$

Week 9 Lec 1 (Lec 23) Nov 4, 2025

Cauchy Residue Thm

$\oint_D f(s)\, d \quad = \qquad \qquad \leftarrow$ LHS ①

- difference of complex logs is basically the integer multiple of the "turns" $\longrightarrow \left[\angle G(\gamma(2\pi)) - \angle G(\gamma(0))\right]_j$

$2\pi \sum_{j=1}^{k} \text{Res}(f(s), P_i)$

consider zeroes as well since we do dG/ds

- poles of $\dfrac{dG/ds}{G} = \{\text{zeroes of } G\} \cup \{\text{poles of } G\}$

- $f(s) = \dfrac{dG/ds}{G}$ was chosen deliberately s.t. the residues of the poles (and zeros in this case) equal to $-r_2$ and r_1 respectively

The Argument Principle

$\Rightarrow$ lets you know "illustratively" how many poles/zeroes are in a certain area of the s-plane.

Loop transfer func^{k}: $L(s) = C(s)\, G(s)$

Week 9 Lec 2 (Lec 24) Nov 6, 2025

- using Nyquist plot to check stability
- break up contour $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ D_1  D_2
- adding (+1) to $C(s)G(s)$ only adds to the zeroes, not the poles

$\Rightarrow$ poles of $1 + L(s)$ is just poles of $L(s)$

Assumptions

1) plot of $\underline{\underline{1 + L(\gamma(\theta))}}$ encircles $\underline{\underline{\omega = 0}}$ k times $\equiv$ plot of $\underline{L(\gamma(\theta))}$ encircles $\underline{\underline{\omega = -1}}$ k times

$\Rightarrow$ you can stop worrying about the +1 in $1 + L(\gamma(\theta))$

Nyquist Things

since D is just Im axis

- for D_1: You have actually $L(j\omega) \rightarrow$ frequency response

$\Rightarrow$ $L(j\omega)$ in complex plane is called the Nyquist plot or polar plot $\rightarrow$ denoted by $\mathcal{L}$

eg: $\left|\frac{s}{s^2}\right|$

- for D_2: if $L(s)$ is strictly proper, $|L(Re^{-j\theta})| \rightarrow 0$

$\Rightarrow$ plot of $L(Re^{-j\theta})$ reduces to the origin (a point)

$\Rightarrow$ later, this is why you only look at $\mathcal{L}$ and you don't worry about D_2 (on new contour, just rotate)

Argument Principle

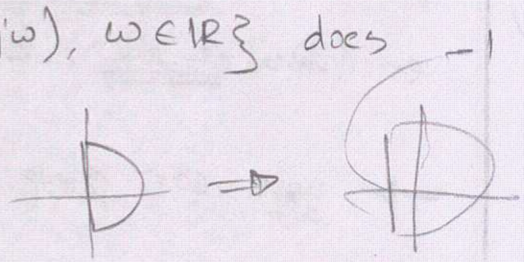
* you said ∄ pole on the contour D

  ↳ D_2 is ok since $R \to \infty$

  ↳ D_1 → poles of $1 + L(s)$ not on im axis
  → zeroes " " " " " " "

  ↳ since $1 + L(j\omega) \neq 0 \;\; \forall \omega$

  Nyquist plot $\mathcal{L} = \{L(j\omega), \omega \in \mathbb{R}\}$ does not cross -1 →



Nyquist Criterion

* for BIBO stability, # of zeroes must $= 0 \Rightarrow Z = 0$

  $\Rightarrow \mathcal{L} \leftarrow$ FDL

* if you have P controller with gain K, just adjust Nyquist Criterion accordingly (he did in blue) or your $L(s)$ can just be $L(s) = K\,C(s)\,G(s)$

Week 9 Lec 3 Nov 7, 2025

- we use Nyquist Plot to verify 1) of Thm #1
- Nyquist has more info compared to Routh
- you're using open loop TF to analyze CLS stability (without calculating Gcls of 4)

Nyquist Plot Properties

1) Low-frequency behaviour ($\omega = 0$) $\longrightarrow$ DC gain

$\longrightarrow$ this is why $\mathcal{L}$ starts on real axis at the DC gain (you plugged in $\omega = 0$ for $L(s)$)

2) $\omega \rightarrow +\infty$

$$L(j\omega)\Big|_{\omega \to \infty} = \frac{b_m}{a_n} \cdot \frac{1}{(j\omega)^{n-m}}$$

(only highest power in num and den remain ... rest is negligible)

$|L(j\omega)| \rightarrow 0$

☆ $\measuredangle L(j\omega) = \begin{cases} \\ \end{cases}$ ... lets you know how to enter origin (which angle to enter at)

3) $\mathcal{L}$ is symmetric about real axis

4) $\mathcal{L}$ intersects real axis when $\measuredangle L(j\omega)$ mod 2π is 0 or π

" " im " " " " " $\pm \frac{\pi}{2}$

Poles of $L(s)$ on Im Axis

- basically do a little indentation of contour around pole
  - ↳ you get 4 lil contours
- can do this indentation for as many poles on im axis.

Typo: Should be "Week 8 Recap"

Week 9 Recap

Nov 9, 2025

w9l1: lec22 **Internal Model Principle**

- We have been dealing with Basic Control Problem (BCP) and the 3 specs a, b, and c, but our reference $r(t)$ and disturbance $d(t)$ were just polynomials whose Laplace transform yielded only poles at $s=0$.
- Internal Model Principle lets us solve general BCP's where $r(t)$ and $d(t)$ are no longer just polynomials
  $\rightarrow$ $R(s)$ and $D(s)$ are any strictly proper rational func^{n}s of s

$\Rightarrow$ A controller $C(s)$ solves tracking problem (i.e. makes $e(\infty) \to 0$ and rejects disturbance) iff:

1) $C(s)$ makes closed loop system (CLS) BIBO stable
2) The product $C(s)G(s)$ contains the poles of $R(s)$
3) $C(s)$ contains the poles of $D(s)$

- intuitive: for robot arm to reject disturbance, it must internally be able to reproduce the disturbance $\rightarrow$ hence "internal" model principle
- note: unsolvable BCP if any zeroes of $G(s)$ are poles of $R(s)$

For designing $C(s)$ s.t. CLS is BIBO stable, can use 2 criteria of Thm 1 (w7 recap pg. 2) + Routh Table

- easier method: enter the Argument Principle and Nyquist plot and Nyquist stability criterion

W9L2: lec 23 The Argument Principle

- recall Cauchy's Residue Thm: $\oint_D f(s)ds = 2\pi j \sum_{i=1}^{k} \text{Res}(f(s), P_i)$
- that says "the contour integral of $f(s)$ along D is $2\pi j$ times the sum of all the residues" ($\exists$ rules of $f(s)$ and D that I omit)
- Now, to demonstrate The Argument Principle, we fabricate a special function $f(s) = \frac{dG(s)/ds}{G(s)} = \frac{d}{ds}\log(G(s))$ for some func^{t} $G(s)$
- you parametrize $D = \gamma(\theta)$ for $\theta \in [0, 2\pi]$ ← parametrize s.t. contour integral is done CCW

$$\underbrace{\oint_D f(s)\,ds}_{} = \underbrace{2\pi j \sum_{i=1}^{k} \text{Res}(f(s), P_i)}_{}$$

$= \int_0^{2\pi} \frac{d}{d\theta}\log[G(\gamma(\theta))]\,d\theta$

$= [\measuredangle G(\gamma(2\pi)) - \measuredangle G(\gamma(0))]j$

"some integer multiple of 2π, which tells you the number of times you circled origin: $s=0$"

the function is chosen s.t. calculating residues of its poles (which contain both the poles and zeroes of $G(s)$) gives you the number of zeroes and poles!

$\text{Res}(f(s), \text{zero of } G) = r_1$ ← order of zero

$\text{Res}(f(s), \text{pole of } G) = -r_2$ ← order of pole

$= 2\pi j[Z - P]$ ← Z = # zeroes of G, P = # poles of G

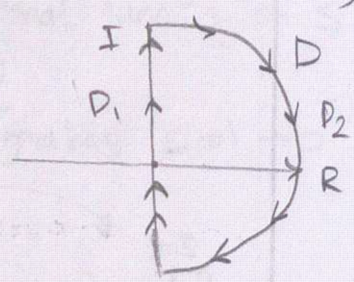
(inside contour D)

→ Result: for a CCW contour D, which you parametrize as $s = \gamma(\theta)$, you create a transformed, plotted contour $G(\gamma(\theta))$. You ∴ know that the new contour $G(\gamma(\theta))$ encircles the origin $(Z-P)$ times CCW, where Z = # zeroes of $G(s)$ and P = # poles of $G(s)$ inside D

Lec 24 or 25

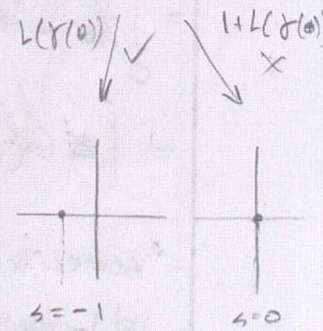
Nyquist Stability Criterion

- the story starts with our closed loop transfer function $1 + C(s)G(s)$, which we will simplify to $1 + L(s)$
- to solve the stability problem, we have to make sure that $C(s)$ keeps/makes our system BIBO stable (Thm 1)
  1) poles of $\frac{1}{1+L(s)}$, or roots/zeroes of $1+L(s)$ must be in OLHP
  2) there must not be any unstable pole-zero cancellation in $L(s)$
- We shall assume 2) and investigate 1).



$\boxed{\text{for CLS to be stable, need no poles of } \frac{1}{1+L(s)} \text{ in RHP}} \longrightarrow \boxed{\text{means no zeroes of } 1+L(s) \text{ in RHP}}$ ★

analyzing $1+L(s)$ is harder than analyzing just $L(s)$, so...



a) • notice that poles of $1+L(s)$ are the poles of $L(s)$

$$1 + \frac{N(s)}{D(s)} = \frac{D(s)+N(s)}{D(s)} \xleftrightarrow{\text{same poles as}} L(s) = \frac{N(s)}{D(s)}$$

① • so to mitigate the "+1" in $1+L(s)$, we just "shift our origin"
- plot of $1+L(\gamma(\theta))$ encircles $w=0$ the same number of times that the plot of $L(\gamma(\theta))$ encircles $w=-1$

- using a) b) and Argument principle, we can arrive at the Nyquist Stability Criterion:

* Recall the Argument Principle: Contour D (in our case, the RHP) CW

$$\begin{pmatrix} \text{\# of times } \mathcal{L} \\ \text{encircles } \omega = -1 \\ \text{CW} \end{pmatrix} = \begin{pmatrix} \text{\# zeroes of} \\ 1 + L(s) \text{ inside} \\ D = \text{RHP} \end{pmatrix} - \begin{pmatrix} \text{\# poles of} \\ 1+L(s) \text{ inside} \\ D = \text{RHP} \end{pmatrix}$$

since ①

since ☆

$$\begin{matrix} \text{\# times } \mathcal{L} \text{ encircles} \\ \omega = -1 \text{ CW} \end{matrix} = 0 - \begin{matrix} \text{\# poles of} \\ 1 + L(s) \text{ inside } D \end{matrix}$$

since a)

$$\boxed{\text{\# times } \mathcal{L} \text{ encircles } (\omega = -1) \text{ CW} = -(\text{\# poles of } L(s) \text{ inside } D)}$$

* recall that for eg: 1 turn CCW = −1 turn CW

② Along D_1, it's basically frequency response since it's only points on Im-axis: $D_1 = \{s = j\omega, \omega \in \mathbb{R}\} \Longrightarrow L(j\omega)$

③ Along D_2, it's a giant circle, so magnitude goes to 0

eg: $\left|\frac{s}{s^2}\right| \to 0$ as $s \to \infty$

$D_2 = \{s = Re^{-j\theta}, R \to \infty\}$, since L is strictly proper $|L(Re^{-j\theta})| \to 0$

④ Argument Principle assumed no poles or zeroes on D

- D_2 has no issues since $R \to \infty$
- D_1: $\begin{cases} \text{poles of } 1+L(s) = \text{poles of } L(s) \leftarrow \text{indentation of contours} \\ \text{zeroes of } 1+L(s) \leftarrow \text{must have no zeroes of } 1+L(s) \text{ on RHP to begin with since we need BIBO} \end{cases}$

$$\boxed{\begin{matrix} \text{\# times } \mathcal{L} \text{ encircles} \\ -1 \text{ counterclockwise CCW} \end{matrix} = \begin{matrix} \text{\# of poles of } L(s) \\ \text{in RHP (real part} > 0) \end{matrix}}$$

Nyquist Criterion

Assume $L(s)$ has no poles on Im-axis and is a strictly proper rational function. Assuming no unstable pole-zero cancellation

Then, roots of $1 + KL(s)$ are in OLHP iff

1) $\mathcal{L}$ does not pass thru point $-1/k$ (aka, $1+L(s)$ has no zeroes on the im-axis)

2) $\mathcal{L}$ encircles $-1/k$ P times CCW, P = # poles of $L(s)$ in RHP

(aka, no zeroes of $L(s)$ in RHP)

* if you have P controller with gain k, then make point from -1 to $-1/k$

* to actually plot $G(\gamma(\theta))$, use the phase and magnitude plots of the frequency response $L(j\omega)$

* to figure out k for BIBO stability, we are interested in the point $-1/k$ instead of -1:

1) we check the # of RHP that $L(s)$ has

2) goal is to make Nyquist plot encircle $-\frac{1}{k}$ that many times

- eg: if $L(s)$ has no RHP poles, we want values for k s.t. $-1/k$ is never inside Nyquist plot

- eg: if $L(s)$ has 1 RHP pole (eg $L(s) = \frac{1}{s-1}$), then we want values for k s.t. $-\frac{1}{k}$ is inside Nyquist plot and that it is circled one time.

ECE 311 Lab 2 Prep Nov 9, 2025

Assuming no disturbance ($d(t)=0$) and a step input $V_m(t) = V_0 \cdot \mathbb{1}(t)$

$$V_m(s) = \frac{V_0}{s} \longrightarrow \boxed{\frac{a}{s+b}} \longrightarrow V(s)$$

We use Final Value Theorem to find $\lim_{t\to\infty} V(t)$ (steady state output)

$$V(s) = \frac{a}{s+b} \cdot \frac{V_0}{s}$$

$$\lim_{t\to\infty} V(t) = \lim_{s\to 0} s V(s) = \lim_{s\to 0} \cancel{s} \cdot \frac{a}{s+b} \cdot \frac{V_0}{\cancel{s}} = \lim_{s\to 0} \frac{V_0 a}{s+b} = \frac{V_0 a}{b}$$

$\therefore$ We use FVT to show that $\boxed{\lim_{t\to\infty} V(t) = \frac{V_0 \cdot a}{b}}$

Note: We are able to use FVT because our $V(s)$ contains a finite number of poles, no poles in the RHP, and at most one pole at $s=0$

Week 10 Tutorial 9 Nov 11, 2025

Nyquist Criterion

$\boxed{N = Z - P}$

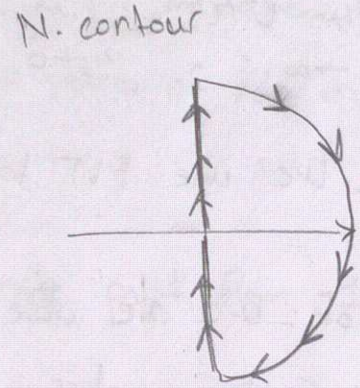
P: # of poles of OLS in RHP

Z: # of zeroes of OLS in RHP

N: # of encirclements of -1

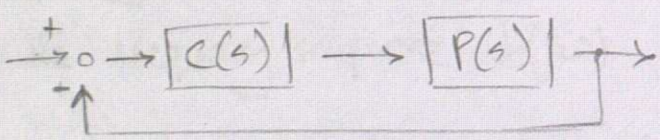
BIBO stable iff $P = -N$

OLS:



$L(s) = C(s) \cdot P(s)$

CLS:



$$G(s) = \frac{C(s)\,P(s)}{1 + C(s)\,P(s)}$$

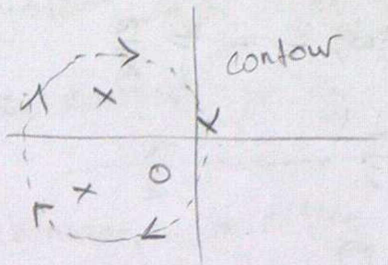
logic:

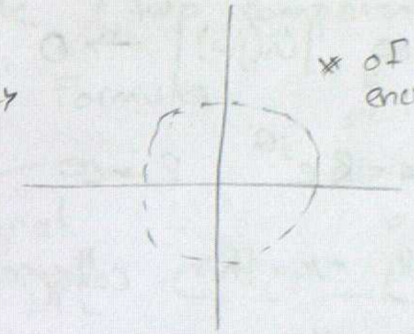
1) stable if no RHP poles in CLS
2) equivalently, no RHP zeroes of $1 + C(s)G(s)$
3) # CCW encirclements of $0 = \underset{\text{(poles)}}{n} - \underset{\text{(zeroes)}}{m}$
4) if $m = 0$ (hence no $1 + C(s)G(s)$ zeroes in RHP hence no RHP poles in CLS), then $P = -N$

OLS RHP poles

of encirclements of -1

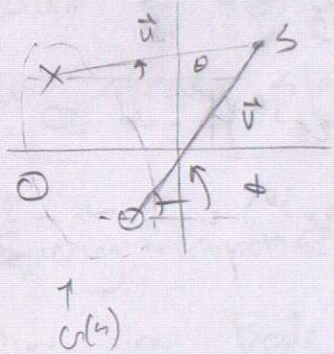
Argument Principle





\# of CCW encirclements $= \underset{(\text{poles})}{n} - \underset{(\text{zeroes})}{m}$

phase magnitude

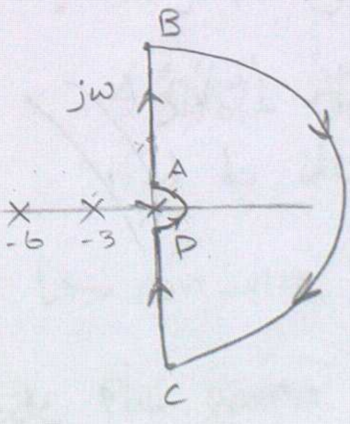


$$G(s) = \underbrace{\frac{|v|}{|u|}}_{|G(s)|} e^{\overbrace{j(\phi-\theta)}^{\measuredangle G(s)}}$$

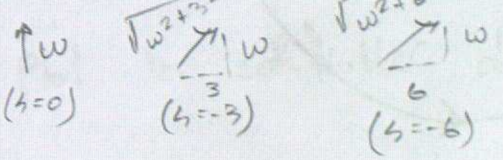
pole phase go against (opposite)

zero phase : $\underset{\text{zero}}{360} - \underset{\text{pole}}{360}$ or $\underset{\text{zero1}}{360} + \underset{\text{zero2}}{360} - \underset{\text{pole}}{360}$

Q.1a) $G(s) = \dfrac{50}{s(s+3)(s+6)}$, $C(s) \rightarrow C(s)G(s) = L(s) = \dfrac{50}{s(s+3)(s+6)}$



generic magnitude relationship:



$$|G(j\omega)| = \frac{50}{|\omega|\sqrt{\omega^2+9}\sqrt{\omega^2+36}}$$

$$\measuredangle G(j\omega) = -\frac{\pi}{2} - \operatorname{atan}\left(\frac{\omega}{3}\right) - \operatorname{atan}\left(\frac{\omega}{6}\right)$$

$\underline{AB}$: $\omega \to 0$, $|G(j\omega)| \to \infty$, $\measuredangle G(j\omega) = -\frac{\pi}{2}$

$\omega \to \infty$, $|G(j\omega)| \to 0$, $\measuredangle G(j\omega) = -3\frac{\pi}{2}$

$\underline{BC}$: set $s = Re^{j\theta}$, $R \to \infty$, $\theta = \frac{\pi}{2} \longrightarrow -\frac{\pi}{2}$

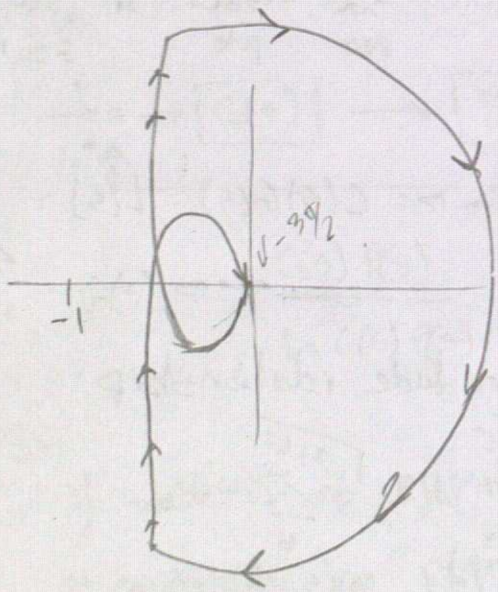
$\rightarrow$ basically everything collapses to 0

$\underline{CD}$: reflection of AB about Re axis

$\underline{DA}$: set $s = \varepsilon e^{j\theta}$, $\varepsilon \to 0$, $\theta = -\frac{\pi}{2} \to \frac{\pi}{2}$

$$L(\varepsilon e^{j\theta}) = \frac{50}{\varepsilon e^{j\theta}(\varepsilon e^{j\theta}+3)(\varepsilon e^{j\theta}+6)}, \quad |L(\varepsilon e^{j\theta})| \to \infty$$

$$\measuredangle L(\varepsilon e^{j\theta}) : \frac{\pi}{2} \to -\frac{\pi}{2}$$



L? $\measuredangle = -\pi$?

$P = -N = 0$

$\Rightarrow$ BIBO stable

Week 10 Lec 1 (Lec 26) Nov 11, 2025

- we are inputting a sinusoid to a BIBO stable system and analyzing the (two components) of the output using Residue Formula

one from $G(s)$, goes to 0 since BIBO

second one from $U(s)$

Takeaway: injecting a signal at a ω into a BIBO LTI system, your output is a signal of the same ω but scaled by $|G(j\omega)|$ and phase shifted by $\angle G(j\omega)$ in steady state (we ignore transient response due to $G(s)$)

- frequency response captured by spectrum analyzer, which gives you Bode plot
  - ▷ very powerful since it lets us get $|G(j\omega)|$ and $\angle G(j\omega)$ just by injection and measurement → don't need to derive complicated $G(s)$ from first principles.
  - ▷ can use Bode Plot for controller design.

- Bode Plot Review

Lec 27

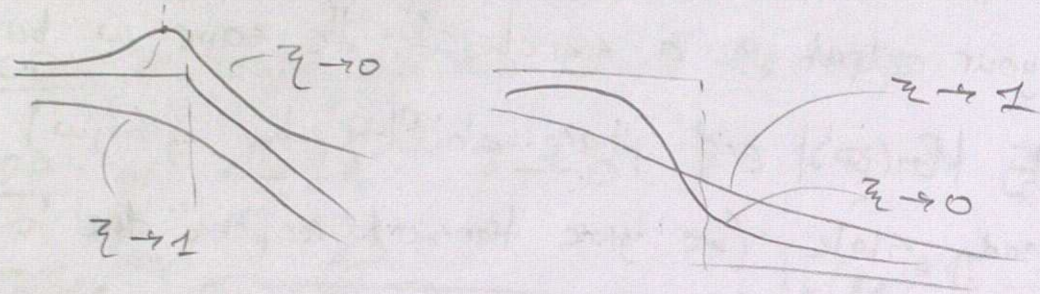
Week 10 Lec 2 (stability and phase margins) Nov 11, 2025

- Bode plots for complex second order poles:

  ↳ $\frac{\omega}{\omega_r} << 1 \longrightarrow$ easy, 0

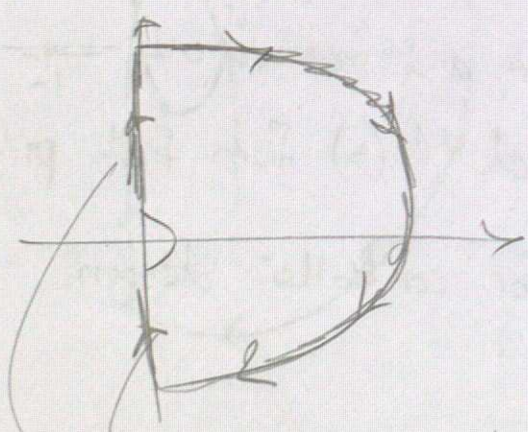
  $\frac{\omega}{\omega_r} >> 1 \longrightarrow$ regular from table

- but in middle values, take care because ξ plays role



- magnitude plot: choose a point and draw relative slopes

  phase plot: draw all individual phase plots and add /superimpose them together



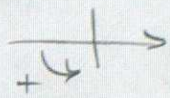
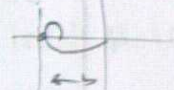
Bode plot helps you here

Stability Margin

- HF ripple in Nyquist might kick in at one point, encircle -1, and blow up your system.

lec 28

week 10 lec 8 (phase margin) Nov 14, 2025

- phase margin is where $L(j\omega)$ intersects the unit circle (the $\omega > 0$ plot half of $L(j\omega)$)
- phase margin (+)ve going 
- gain margin: $GM = \frac{1}{|L(j\bar{\omega})|}$, $\bar{\omega}$ is where $L(j\omega)$ intersects (−)ve real axis for the first time: 

  $|L(j\bar{\omega})| = \frac{1}{GM}$
- for P-controller with gain k, you don't want to encircle $-1/k$ $\Longrightarrow$ GM is the maximum value of k b4 instability.
- <u>Bode Plot Gain margin</u>: see what ω that your phase plot crosses the $-180°$ line, and go up to gain plot and see distance to 0dB $\longrightarrow$ that's your GM
- <u>Bode Plot Phase Margin</u>: see what ω that your magnitude/gain plot crosses the 0dB line, and go down to phase plot at that ω_c and see distance to $-180°$ $\longrightarrow$ that's your PM
- Bandwidth: at ω_b, $|L(j\omega)|$ is at -3dB.
  - in controls, we care about bandwidth of CLS, not just L
  - however, our bode plot is of L, not CLS TS $\frac{L}{1+L}$

  $\Longrightarrow \exists$ a correlation b/n ω_c of L and ω_b of CLS $\longrightarrow$ check lec 28

Week 11 lec 1 (lec 29) Nov 18, 2025

Transient Performance in Freq. Dom

1) 1) $PM = \operatorname{atan}\left(\dfrac{2\zeta}{-2\zeta^2 + \sqrt{1+4\zeta^4}}\right)$ a second order system performance in midterm 2

$\%OS = \exp\left(\dfrac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$ both are functions of ζ

- smaller PM → larger %OS (high ζ)
- larger PM → smaller %OS

2) 2) Settling Time : $T_s = \dfrac{4}{\zeta\omega_b}\cdot\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$

ω_b → bandwith of CLS $\dfrac{L}{1+L}$

- higher bw → settling time lower

  $\left(T_s \propto \dfrac{1}{\omega_b}\right)$

- higher ω_c → higher ω_b → smaller T_s

  ω_c → crossover freq.

  → contains more high freq. components → system responds/is faster

Summary:

larger phase margin ⟷ smaller % overshoot

larger crossover freq ⟷ smaller settling time must know how to derive the gains of a TF

Controller Design in Freq. Dom

- design $C(s)$ s.t. Thm 1 is met (BIBO stab.)

  use Nyquist Criterion to verify this

- controllers:
  - PD : proportional derivative
  - Lead
  - PI : proportional integral
  - Lag
  - PID = PI + PD : proportional, integral, derivative.

PD Controller

$$\boxed{C(s) = \underbrace{K}_{\text{gain}}(1 + \underbrace{T_D \cdot s}_{\text{eigentime}})}$$

setting $D=0$ for standard loop: $U(s) = KE(s) + KT_D sE[s]$

$$\xrightarrow{\mathcal{L}^{-1}} u(t) = \underbrace{Ke(t)}_{\text{proportional}} + \underbrace{K \cdot T_D \cdot \frac{d}{dt} e(t)}_{\text{derivative}}$$

→ corresponding Bode plot drawings

- moving /setting $\frac{1}{T_D}$ at low freq makes higher frequencies move up by $+\frac{\pi}{2}$ $\Longrightarrow$ increases phase margin
- as a result, crossover frequency also higher
- from example, $T_D = 10$ makes higher freq. go down less sharply, and at a later freq. too
- K makes the whole Bode plot move up (increasing K)

gain

- K doesn't affect phase plot of Bode
- should try to move $\frac{1}{T_D}$ as left as possible bc. our phase plot at higher freq. will overall be higher

phase

- check stab. of $C(s)G(s)$ using Nyquist

↳ can draw Nyquist plot using Bode Plot.

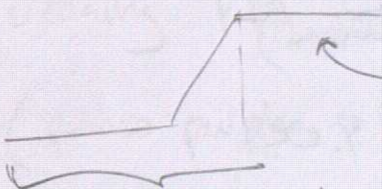
Week 11 Lec 2 Nov 20, 2025

Designing a Lead Controller ← final, study

* PD controller asks for infinite gain at higher freq, which is not possible for μcontrollers
* Lead controller helps solve this by approximating the PD

$$\boxed{C(s) = K \cdot \frac{Ts+1}{\alpha Ts+1}} \quad K, T > 0, \; 0 < \alpha < 1$$

* constant gain at freq → ∞, so no longer infinite gain



this takes care of the infinite gain issue of PD

want to keep this basic shape

* ω_{max} very important → middle of magn. plot → Δ value increase
  → peak of phase plot → ϕ_{max} value increase
  ↳ must use geometric mean since log scale
* magn. you know when ω_{max} happens and the slope of +20 dB/dec, so ~~we~~ you know value of Δ value wrt to the base of 20 log K, but if you want absolute magnitude, add Δ to the base like 20 log K + ___

- phase plot finding ϕ_{max} : you use $\measuredangle C(j\omega_c)$ to find angle b/n num vector and denom. vector with that you derive the arcsine formula for ϕ_{max}

- as $\alpha \to 0$, lead becomes PD since that  this keeps going to a very high freq. so it's basically like 

- use lead's phase plot hump to increase phase at specific frequency (which is ω_{max})
  - ↳ increased PM ⇒ smaller %OS
  - ↳ as byproduct, it also increases ω_c ⇒ decrease T_S

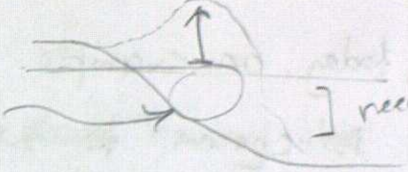
Design Example

- start at freq 1, then if s pole ($j\omega$), you come from left @ -20 dB/dec and then you think/see how other poles/zeroes affect slopes.

- bare plant has low PM → close to BIBO instability
  - ↳ use controller (lead) to increase phase margin from $0°$ to $40°$ or more.

Steps

since you have something below



] need bonus st 70° extra

1) set a k, say $k = 1$

2) you want $40° + 30°$ ← extra margin and set ϕ_{max} to this → then calculate the α needed

3) we want to put ω_{max} at the most effective frequency → crossover freq, so all that extra phase from the lead's phase hump goes straight to increasing PM → at ω_c, magnitude is 0 → LHS = 0

(know purple = 0 since $k = 1$) and

(green comes from α in step 2) and

∴ can solve for yellow

Q: what is ω_{max} to get answer (−15 dB)?

→ use Bode plot to find using ruler!

(in this case, $\omega_{max} \approx 6$ rad/sec)

now can find T by solving: $\omega_{max} = \dfrac{1}{T\sqrt{\alpha}}$

at ω_{max}, you have to crossover

(in this case, $T = 0.9623$)

4) form final lead equation $C(s)$ → 5) make new plot with your $C(s)$ in action

Week 11 Lec 3 (lec 31) Nov 21, 2025

- today, an example of a lead controller where K is chosen to satisfy tracking error requirement
- due to ζ, we have "horn peak" at ω_r ← resonant freq.
- your new mag plot must cross mag = 0 at ω_{co} that you designed to be at.
- and at that ω_{co}, you have your hump of height ϕ_{max}

PI controller

$$\boxed{C(s) = K\left(1 + \frac{1}{T_I s}\right) = \frac{K}{T_I} \cdot \frac{T_I s + 1}{s}}$$

this pole is new compared to the other controllers → lets you get ready of s.s. error.

- you don't affect phase margin if you ~~play~~ place $\frac{1}{T_I}$ low since ϕ ends up at 0 @ mid-frequencies.
- you also add some gain at low frequencies.

- green is final sys with both lead + PI controller.

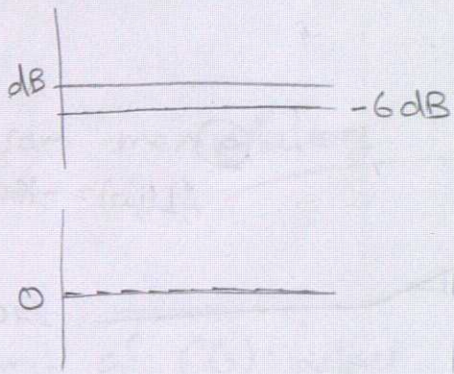
Week 12 Tutorial 11 Nov 25, 2025

Problem 1

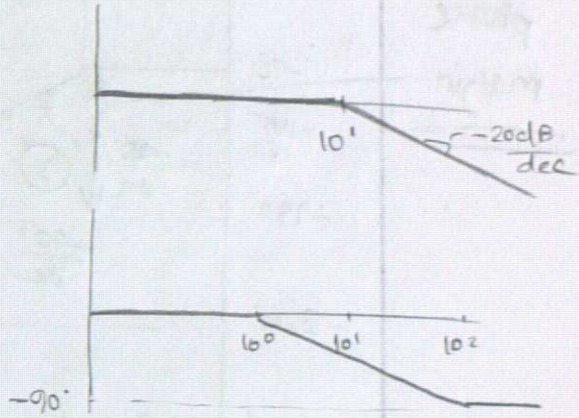
1) $L(s) = C(s)\,G(s) = \dfrac{500}{s(s+10)} \cdot \dfrac{1}{(s+100)}$ ← open-loop TF

2) $L(s) = \underbrace{\dfrac{500}{(10)(100)}}_{1/2} \cdot \dfrac{1}{s} \cdot \dfrac{1}{\left(\frac{s}{10}+1\right)} \cdot \dfrac{1}{\left(\frac{s}{100}+1\right)}$

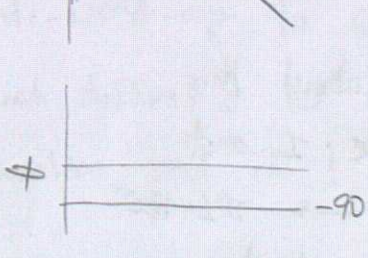
3) $K = \frac{1}{2}$



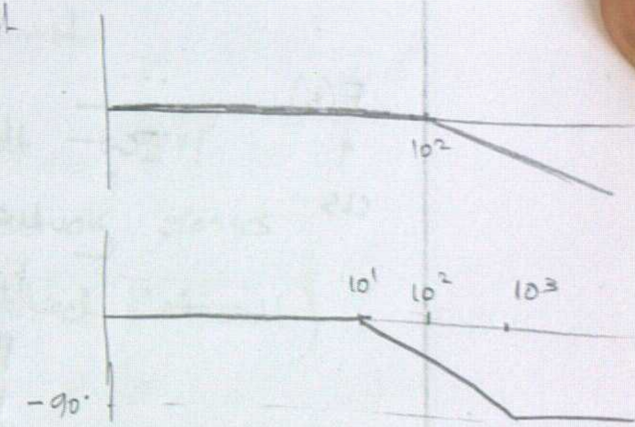
pole at $\left(\frac{s}{10}+1\right)$

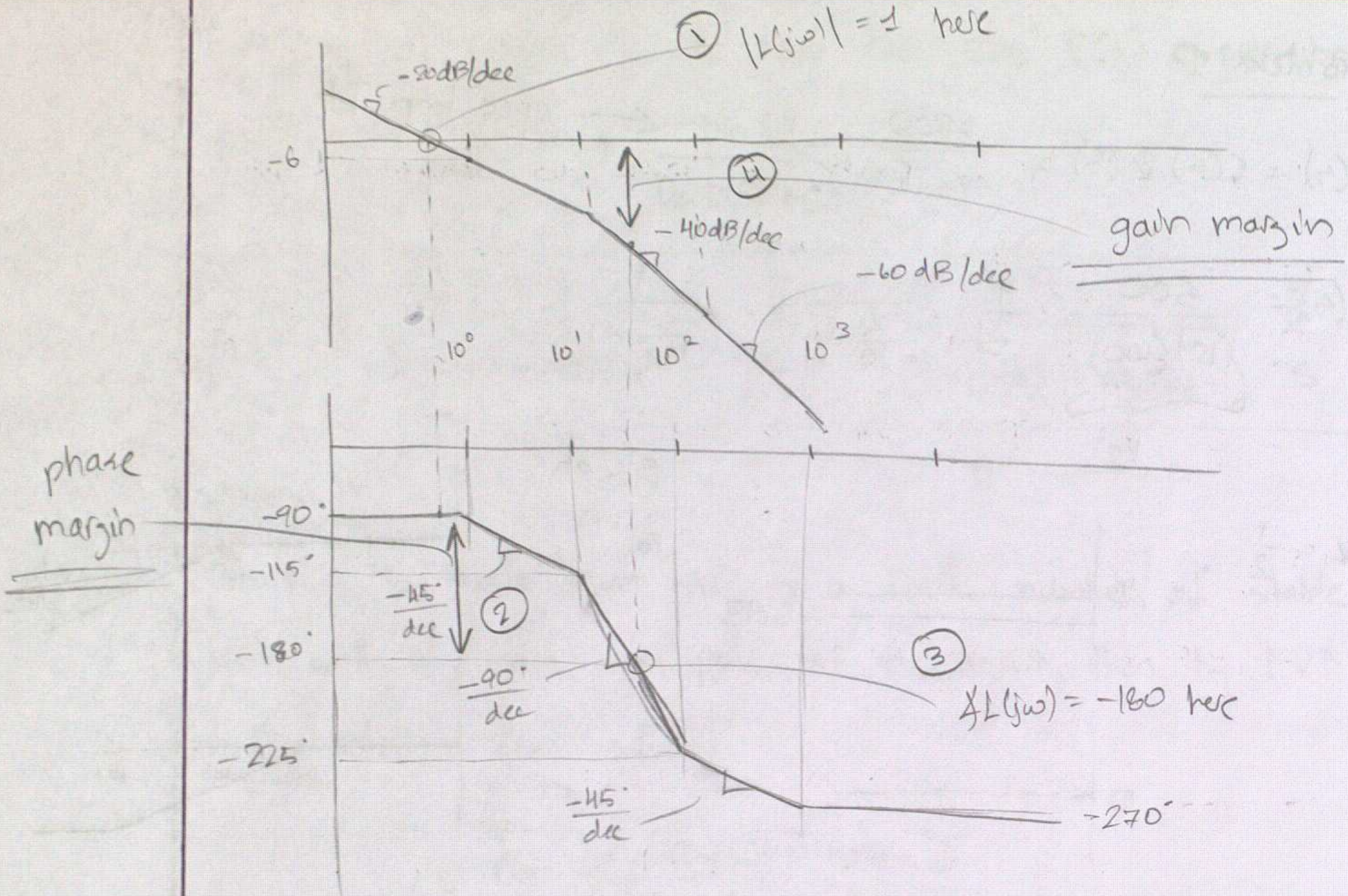


pole @ origin



pole at $\left(\frac{s}{100}+1\right)$





$$F(s) = \frac{\cdots}{1 + CG}$$

↑
CLS

we are worried about this having magnitude 1 and phase -180° ⟶ $1\angle -180°$ because then we'd have $1 - 1 = 0$ in denominator

PM and GM help us know how close $|L(j\omega)|$ is to 1 and $\measuredangle L(j\omega)$ is to -180°

- PI controller
  - at low freq, have high mag due to integral term
  - use this controller to add system type
  - don't want to affect transient performance or PM/GM work that PD or Lead controllers do

PID

- block diagram manipulation.

Limitation

- due to limit of $C(s)$ output, $e(t)$ is cut off at certain max val: $\int e(t)\,dt$ has no limit
- add an anti-wind-up to stop $\int e(t)\,dt \to \infty$
  (detect difference to see if I actually crosses saturation, and if I did, I subtract that out)

Week 9 and 10 Recap Nov 26, 2025

W9L3. Lec25

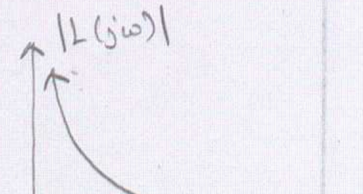
Nyquist Plot with Im axis Poles

- Recall the story: we want to know is CLS is BIBO stable, and we can do that by ensuring $1 + C(s)G(s) = 1 + L(s)$ does not have RHP zeroes (we assume no pole-zero calulation)
- We use a Nyquist Contour of the entire RHP, doing little deformation of contours for poles on Im-axis.
- By Nyquist theorem, we know our CLS is BIBO stable if the Nyquist plot circles $-\frac{1}{K}$ exactly p times ccw, where p is the number of RHP poles of $L(s)$, and we do not go through the point $-\frac{1}{K}$
- We shall use 2 methods of drawing Nyquist Plots:

Daifei Method: graphical method

$L(s) = \frac{10}{s(s+1)}$ ← assume no pole-zero cancellation

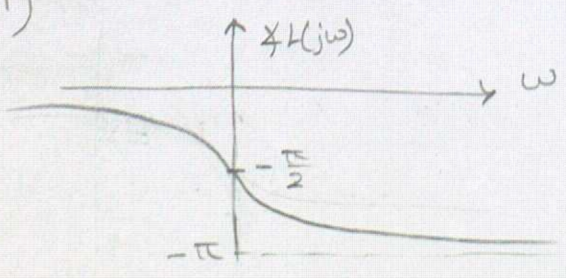
$L(j\omega) = \frac{10}{j\omega(j\omega+1)}$, $|L(j\omega)| = \frac{10}{|\omega|\sqrt{\omega^2+1}}$



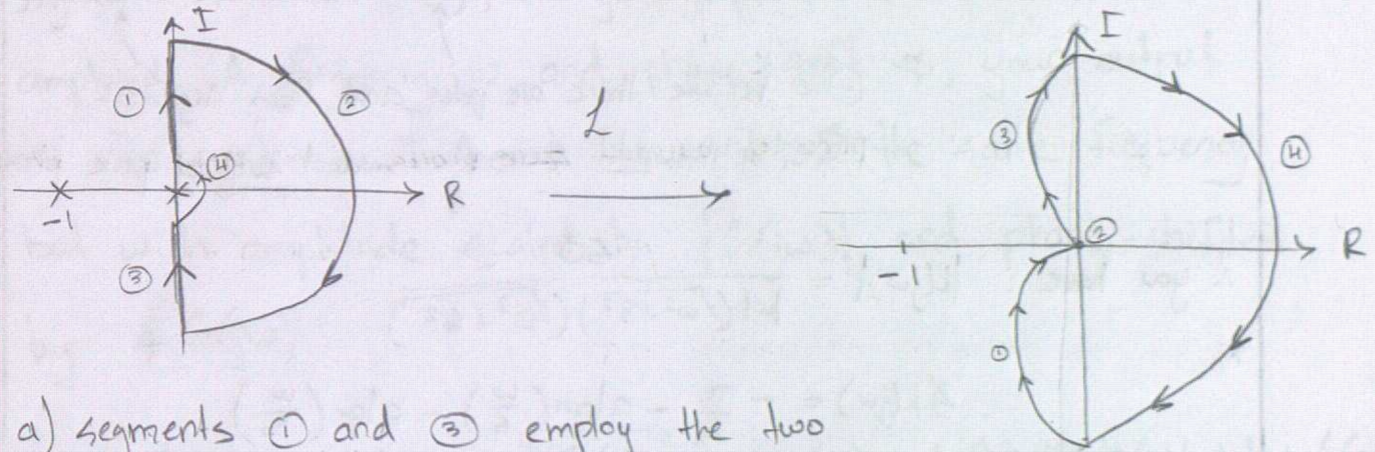
$\measuredangle L(j\omega) = (\measuredangle 10) - (\measuredangle j\omega + \measuredangle j\omega+1)$

$= 0 - (\frac{\pi}{2} + \text{atan}\,\omega)$

$= -\frac{\pi}{2} - \text{atan}\,\omega$



- you look at the components of $L(j\omega)$ to find magnitude and phase.
- To plot Nyquist plot, you start at low ω on $|L(j\omega)|$ and observe that at low ω, $|L(j\omega)| \to \infty$ and at high ω, $|L(j\omega)| \to 0$
- then, observe at $\omega = 0$, $\measuredangle L(j\omega) = -\frac{\pi}{2}$ and at high ω, $\measuredangle L(j\omega) \to -\pi$
- this tells you that from $\omega \geq 0$ to $\omega \to \infty$, your Nyquist plot starts at very far distance with angle $-\frac{\pi}{2}$ and ends at very small distance with angle $-\pi$:



a) segments ① and ③ employ the two sides ($\omega > 0$, $\omega < 0$ respectively) of the phase and magnitude plots.

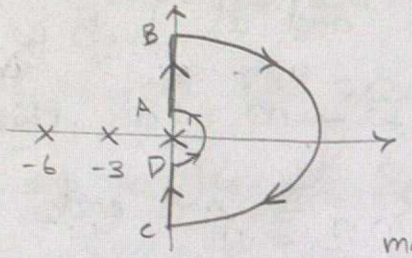
b) for segment ②: Let $\mathcal{L} = \{ s = Re^{j\theta},\ \theta \in [\frac{\pi}{2}, -\frac{\pi}{2}],\ R \to \infty \}$
since $L(s)$ strictly proper, $\mathcal{L}$ collapses to origin ($|L(Re^{j\theta})| \to 0$)

c) for segment ③: Let $\mathcal{L} = \{ s = \varepsilon e^{j\theta},\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}],\ \varepsilon \to 0 \}$
simplify $L(\varepsilon e^{j\theta})$ to $\frac{10}{r} e^{-j\theta}$, so $|L(\varepsilon e^{j\theta})| \to \infty$ and rot. CW.

$\to$ since $L(s)$ has <u>zero</u> RHP poles and Nyquist plot encircle $-\frac{1}{k} = -\frac{1}{1} = -1$ <u>zero</u> times, CLS will be BIBO stable

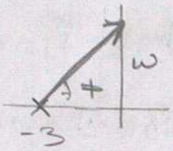
Jesse TA Method

$$L(s) = \frac{50}{s(s+3)(s+6)} \xrightarrow{s=j\omega} L(j\omega) = \frac{50}{j\omega(j\omega+3)(j\omega+6)}$$



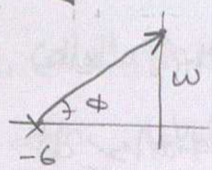
mag: $|\omega|$

phase: $-\frac{\pi}{2}$



mag: $\sqrt{\omega^2+3^2}$

phase: $-\text{atan}\left(\frac{\omega}{3}\right)$



mag: $\sqrt{\omega^2+6^2}$

phase: $-\text{atan}\left(\frac{\omega}{6}\right)$

(-)ve because these are poles, and from Argument principle you <u>add</u> zero phases and <u>subtract</u> pole phases

$\therefore$ you have: $$|L(j\omega)| = \frac{50}{|\omega|\left(\sqrt{\omega^2+3^2}\right)\left(\sqrt{\omega^2+6^2}\right)}$$

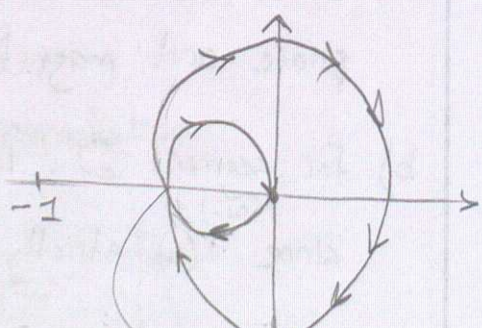
$$\measuredangle L(j\omega) = -\frac{\pi}{2} - \text{atan}\left(\frac{\omega}{6}\right) - \text{atan}\left(\frac{\omega}{3}\right)$$

$\omega: \quad 0 \to \infty$

AB: $|L(j\omega)|: \infty \to 0$

$\measuredangle L(j\omega): \left(-\frac{\pi}{2} - 0 - 0\right) \to \left(-\frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2}\right)$

$-\frac{\pi}{2} \to -\frac{3\pi}{2}$



BC: collapses to origin

CD: mirror image of AB over Re axis

DA: connect up, CW since $L(\varepsilon e^{j\theta}) = \frac{50}{\varepsilon 18} e^{-j\theta}$

Turns out, at $\omega \simeq 4$, $\measuredangle L(j\omega) = -\pi$, and $|L(j4)| < 1$. Since $\mathcal{L}$ encircles $-\frac{1}{k} = -\frac{1}{1} = -1$ <u>zero</u> times and $\exists$ <u>zero</u> RHP poles of $L(s)$

$\Rightarrow \therefore$ CLS is BIBO stable.

to find where $\mathcal{L}$ crosses Re axis, set $\measuredangle L(j\omega) = -\pi$ and solve for ω. Plug that ω into $|L(j\omega)| \to$ If magnitude < 1 then we encircle $-\frac{1}{1} = -1$ zero times

Week 10 Recap Nov 30, 2025

Lec 26
Lec 27

Frequency Response and Bode Plots

* Up until now, we've been doing everything using our actual TF, be it the OLS $G(s)$ or the CLS transfer function (s)
* Sometimes, it's hard to know the exact mathematical model of a system (aka the transfer func^n). What to do?
* Recall from ECE216 that given a BIBO stable LTI system with TF $G(s)$, if you input a sinusoid of amplitude A, frequency ω, and phase shift ϕ, your output of the system will be a sinusiod of the same frequency, but with amplitude scaled by $|G(j\omega)|$ and phase-shifted by $\measuredangle G(j\omega)$.

$G(s)$ has no poles in RHP

$$u(t) = A\sin(\omega t + \phi) \longrightarrow \boxed{G(s)} \longrightarrow y(t) = A|G(j\omega)|\sin(\omega t + \phi + \measuredangle G(j\omega))$$

* This is useful because we can input a sinusoid of known amplitude and frequency and phase, measure output, and figure out $G(s)$ based on $|G(j\omega)|$ and $\measuredangle G(j\omega)$.
* A spectrum analyzer outputs Bode Plots, which use a log-log plot for $|G(j\omega)|$ and a semi-log plot for $\measuredangle G(j\omega)$

  $\Rightarrow$ we will use Bode plots to analyze our loop transfer function $L(s) = C(s)G(s)$

Bode Plots

1) A constant gain : $\boxed{K_0}$

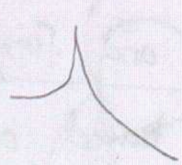
2) A pole or zero at 0 : $\boxed{\frac{1}{j\omega} \text{ or } j\omega}$

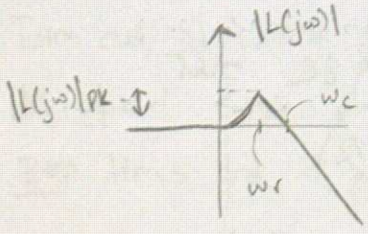
3) A pole at $s=p$ or zero at $s=z$: $\boxed{\frac{1}{1+\frac{j\omega}{p}} \text{ or } 1+\frac{j\omega}{z}}$

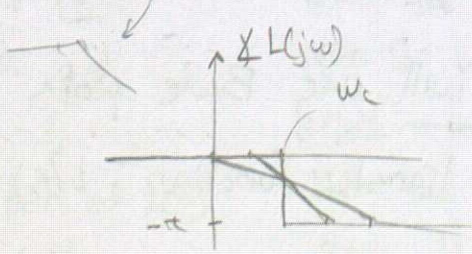
4) Complex conjugate poles or zeroes:

$$\boxed{\frac{1}{1+\frac{2j\omega\xi}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2} \quad \text{or} \quad 1+\frac{2j\omega\xi}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2}$$

- For all of the above, we use the same straight-line approximations as learned in ECE 212 (eg. $20\log(K_0)$)...

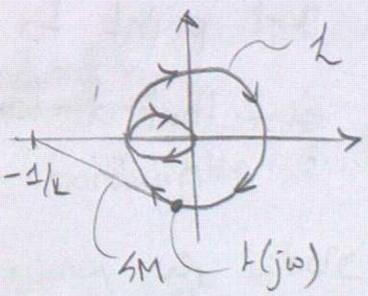
- Note: for complex conjugate poles, $\exists$ an effect due to ξ
  - if $0 < \xi < \frac{\sqrt{2}}{2}$, then $|L(j\omega)|$ has a resonant peak at $\boxed{\omega_r = \omega_n\sqrt{1-\xi^2}}$, and the height of that peak is $\boxed{|L(j\omega)|_{pk} = \frac{1}{2\xi\sqrt{1-\xi^2}}}$ 
  - as $\xi \to 0$, $|L(j\omega)|$: peak $\to \infty$, $\measuredangle L(j\omega)$: steeper 
  - as $\xi \to \frac{\sqrt{2}}{2}$, $|L(j\omega)$: peak $\to 1$, $\measuredangle L(j\omega)$: gentler 





Lec 28 ## Stability Margins

- Let us talk about stability. We know that according to Nyquist Stab. Criterion, given an $L(s)$, our CLS will be BIBO stable iff (1) the Nyquist Plot doesn't cross $-1/k$ and (2) the Nyquist plot encircles $-1/k$ exactly P times, where P is the number of RHP poles of $L(s)$ (D is CW and $\mathcal{L}$ is CCW)
- Let us saw $L(s)$ has no RHP poles. This means that $\mathcal{L}$ cannot encircle or cross $-1/k$ at all
- the stability margin is therefore the minimum distance from $-1/k$ to $\mathcal{L}$ → the higher the stability margin, the more uncertainty (or "jitter") that our system can tolerate



However, quantifying the stability margin is hard to do from the Nyquist Plot, so we use the Bode Plot to quatify SM using two things: gain margin and phase margin

- What do we not want? Remember, our gang of 4 all have $1 + L(s)$ in the denominator. Therefore, we do not want denominator to equal -1 s.t. $1-1=0$ (undefined, division by 0). Let us think about how we can get $1 + L(s) = 1-1 = 0$

- Note that if $|L(s)| = 1$ and $\measuredangle L(s) = -180^\circ = -\pi$, then the unthinkable happens! $1 + 1\angle{-180^\circ} = 1 - 1 = 0$
- We <u>do not</u> want our $|L(s)|$ to be near 1 and $\measuredangle L(s)$ to be near -180° at the same time. So what to do?
- We define the phase margin (PM) and gain margin (GM)
- <u>Phase Margin</u>: how much phase tolerance is there?

1 – find where $|L(j\omega)| = 1$ ← happens at ω_c, where the $|L(j\omega)|$ plot crosses the horizontal axis

2 – go down to the point on the $\measuredangle L(j\omega)$ where $\omega = \omega_c$ and see vertical distance from that point to where $\measuredangle L(j\omega)$ passes -180° line ← that distance is your phase margin!

- <u>Gain Margin</u>: how much gain tolerance is there?

3 – find where $\measuredangle L(j\omega)$ passes -180° ← call it $\bar{\omega}$

4 – go up to the point on $|L(j\omega)|$ that corresponds to $\bar{\omega}$ and see the vertical distance from that point to where $|L(j\omega)|$ crosses horizontal axis ← that distance is your gain margin!

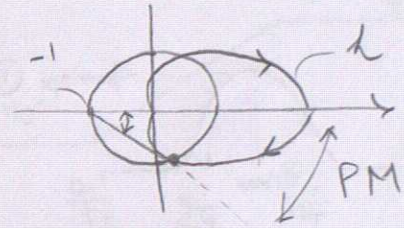
go 8 pages earlier to see 1–4 labelled on tutorial Bode Plot

PM and GM on Nyquist Plot

1) $PM = \measuredangle L(j\omega_c) - (-\pi)$

$= \measuredangle L(j\omega_c) + \pi$

ω_c is where $|L(j\omega)| = 1$, or basically unit circle intersection with L (take (+)ve ω_c)

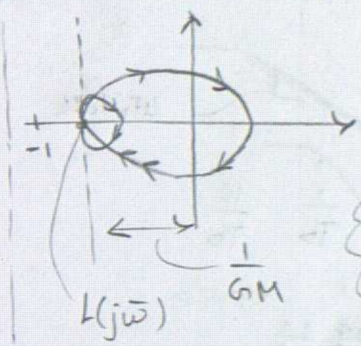


PM > 0 : "suggest" no encircl. of -1

PM < 0 : "suggest", encirclement of -1

2) $GM = \frac{1}{|L(j\bar{\omega})|}$

$\bar{\omega}$ is the frequency at which you have $\measuredangle L(j\omega) = -180^\circ = -\pi$



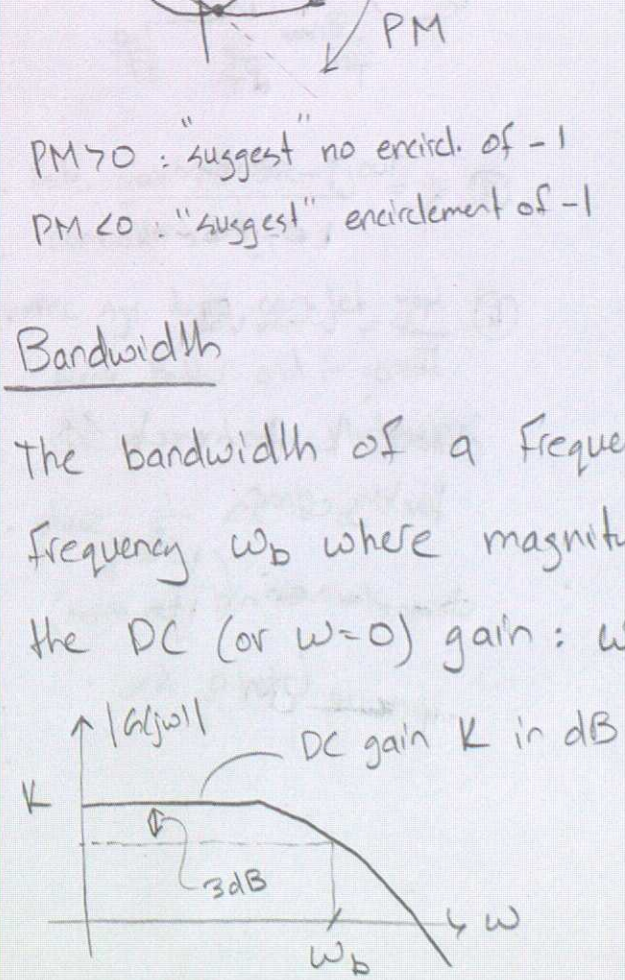
want $\frac{1}{GM} < 1$

aka $GM > 1$

$K < GM \Rightarrow$ BIBO stability, since $\frac{1}{K} > \frac{1}{GM} \Rightarrow K < GM$

Bandwidth

* The bandwidth of a frequency respons $G(j\omega)$ is the frequency ω_b where magnitude of $G(j\omega)$ is 3dB below the DC (or $\omega = 0$) gain : ω_b s.t. $|G(j\omega)| = 20\log|G(0)| - 3\text{dB}$



our Bode Plots are of $L(j\omega)$, but we care about CLS bandwidth : $\frac{L}{1+L}$

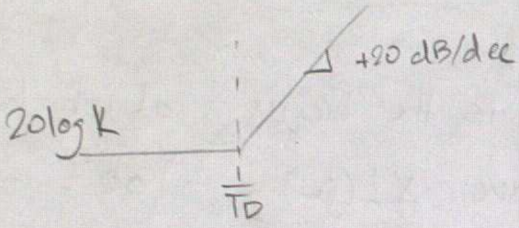
$\Rightarrow \exists$ a correlation b/n ω_c (cutoff freq) of $L(j\omega)$ and CLS bandwidth ω_b : $\boxed{\omega_c < \omega_b < 2\omega_c}$

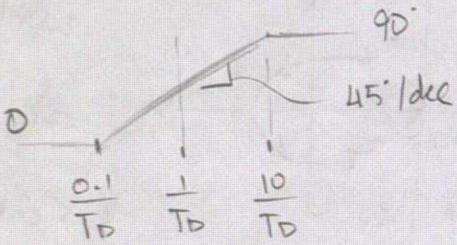
$\frac{L}{1+L}$ @ $\omega = 0$ = 1 $\qquad \frac{L}{1+L}$ @ $\omega \to \infty$, $\to 0$?Q.

Controllers:

PD Controller

$$C(s) = K(1 + T_D \cdot s)$$

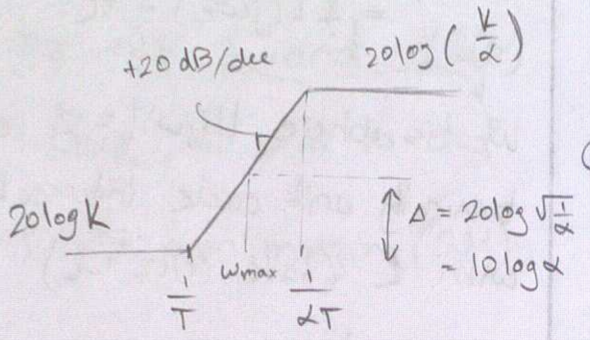


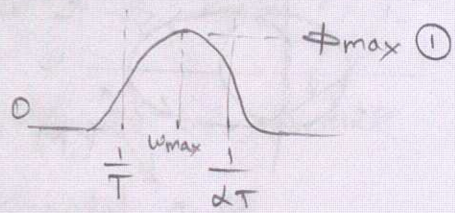


- Place $\frac{1}{T_D}$ before ω_c
- increases PM
- increases ω_c
- not implementable due to infinite gain at high freq.

Lead Controller

$$C(s) = K \frac{Ts + 1}{\alpha Ts + 1} \qquad K, T > 0 \qquad 0 < \alpha < 1$$





② $\alpha = \dfrac{1 - \sin\phi_{max}}{1 + \sin\phi_{max}}$

④ $\dfrac{1}{T} = \omega_{max}\sqrt{\alpha}$

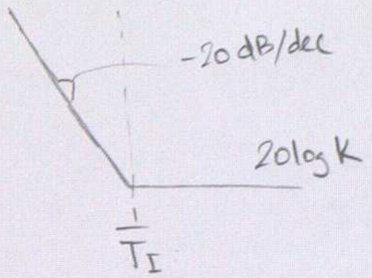
choose K to meet SS tracking error.

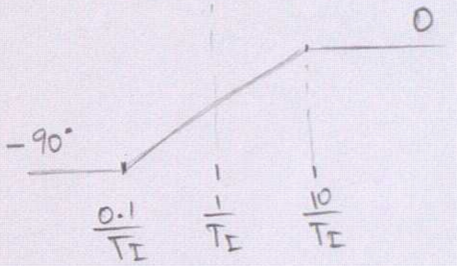
$$\phi_{max} = \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right)$$

- increase PM

PI Controller

$$C(s) = K\left(1 + \frac{1}{T_I s}\right) = \frac{K}{T_I} \cdot \frac{T_I s + 1}{s}$$





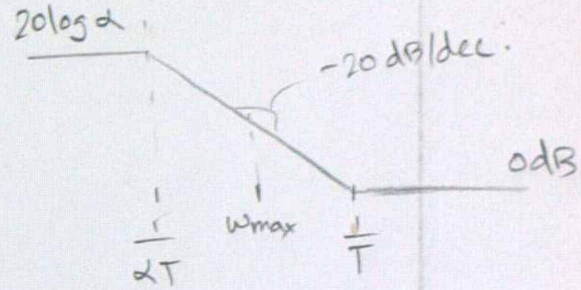
- lets you increase your controller's type
- inc sys. type can let you track better and in general affects mid-freq. behaviour
- place $\frac{1}{T_I} << \omega_c$ (want $\frac{1}{T_I}$ b/n $0.01\omega_c$ and $0.1\omega_c$)
-

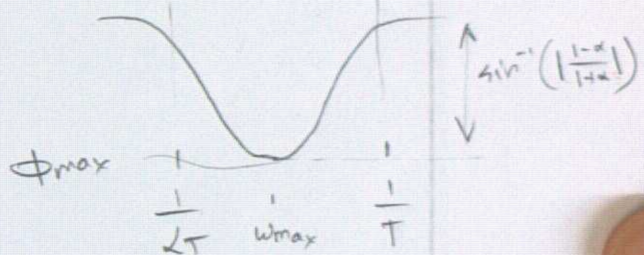
Lag Controller

$$C(s) = \alpha \frac{Ts + 1}{\alpha T s + 1}$$

$T > 0$

$\alpha > 1$





$$\phi_{max} = \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right)$$

$$\omega_{max} = \frac{1}{T\sqrt{\alpha}}$$

- amplify DC gain