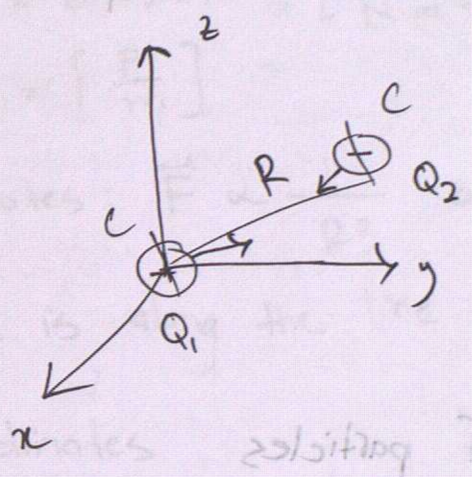
Week 1 : Lecture 1 Jan 6, 2025

## Maxwell's Equations

Faraday's Law: $\nabla \times \vec{E} = 0$

Ampere's Law: $\nabla \times \vec{H} = \vec{J}$

## Coulomb's Law




$$|\vec{F}_e| = \left| k \frac{Q_1 \cdot Q_2}{R^2} \right|$$

a) $|\vec{F}_e|$ depends on $Q_1, Q_2$

b) $|\vec{F}_e| \propto \frac{1}{R^2}$ ← R is distance b/n $Q_1, Q_2$

c) Direction of force $|\vec{F}_e|$ acted along line connecting $Q_1, Q_2$

d) like charges : repel
opposite charges: attract

$$\boxed{\vec{F}_e = k \frac{Q_1 Q_2}{R^2}}$$

$\vec{F}_e \to [N]$, $Q_1 Q_2 \to [C^2]$, $R^2 \to [m^2]$

$k = 9 \times 10^9 \left( \frac{Nm^2}{C^2} \right)$

$$\boxed{\vec{F}_e = \frac{Q_1 Q_2}{4\pi \epsilon_0 R^2}}$$

$\epsilon_0 = 8.85 \times 10^{-12}$ (F/m)

permeability for free space

$$\boxed{\therefore k = \frac{1}{4\pi\epsilon_0}}$$

Week 1: Lecture 1 Jan 7, 2025

## Maxwell's Equa$^n$s

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t} B \qquad \nabla \cdot \vec{D} = \rho$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t} D \qquad \nabla \cdot \vec{B} = 0$$

- under static conditions, $\vec{E}$ and $\vec{B}$ are decoupled
- math tools needed
  - ↳ vector algebra
  - ↳ vector calculus

## Electric Charge

- an intrinsic characteristic of particles making an object
  - → same sign ⇒ repel   call it "q" or "Q"
  - → opposite sign ⇒ attract
- charge can be transferred, but not created or destroyed
- unit charge

  election $= -1.6 \times 10^{-19}$ [C]

  $q_1$ ⊕   $q_2$ ⊕ ⟶ $\vec{F}$

## Coulomb's Law

- gives the amount of force b/n two point charges:

$$\vec{F} = k \frac{q_1 q_2}{R^2} \hat{R}$$

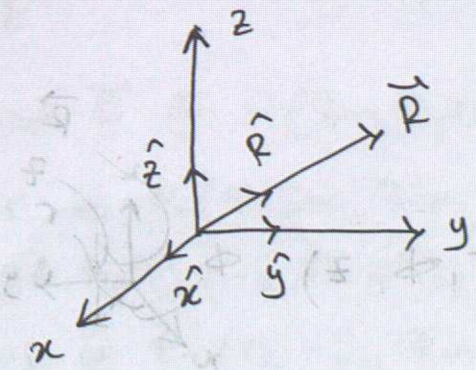
$k$ = electrostatic constant
$= 8.99 \times 10^9 \simeq 9 \times 10^9 \left[\frac{N \cdot m^2}{C^2}\right]$
$= \frac{1}{4\pi\epsilon_0}$

$\epsilon_0$ = permittivity
$= 8.85 \times 10^{-12} \left[\frac{C^2}{N \cdot m^2}\right]$
$= \left[\frac{F}{m}\right]$

- Notes: $\vec{F} \propto \frac{1}{R^2}$ and $F \propto q_1 q_2$
- $\hat{R}$ is along the line connecting $q_1$ and $q_2$

## Coordinates

→ Cartesian (rectangular) : $(x, y, z)$



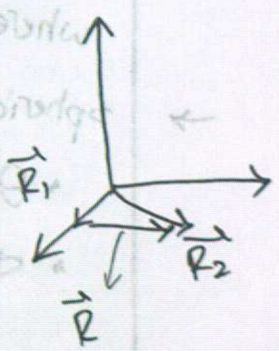

$\vec{R} = x\hat{x} + y\hat{y} + z\hat{z}$

$\hat{R} = \frac{\vec{R}}{|\vec{R}|} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2 + y^2 + z^2}}$

→ Position vector: is a directed distance from origin to a point

- need to decide on $\begin{cases} \text{a common origin} \\ \text{a coordinate system} \end{cases}$

e.g. $\vec{R}_1 = 8\hat{x}$ (m)   $\vec{R}_2 = 4\hat{x} + 8\hat{y}$ (m)



→ Distance Vector : is the separation b/n two points

e.g. $\vec{R} = \vec{R}_2 - \vec{R}_1$

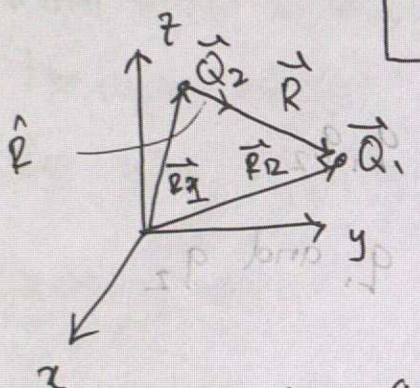
$$= 4\hat{x} + 8\hat{y} - 8\hat{x} = -4\hat{x} + 8\hat{y} \ (m)$$

## Electric Force

* Coulomb's Law : $\boxed{\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{R^2} \hat{R}}$

unit vector $\hat{R}$

distance $R$



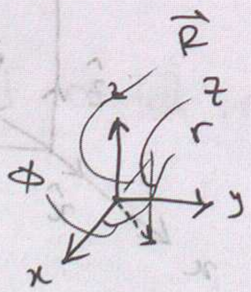

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{R}_2 - \vec{R}_1|^2} \cdot \frac{\vec{R}_2 - \vec{R}_1}{|\vec{R}_2 - \vec{R}_1|}$$

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{R}_2 - \vec{R}_1|^3} (\vec{R}_2 - \vec{R}_1)$$

units are [N]

$$\hat{R} = \frac{\vec{R}_2 - \vec{R}_1}{|\vec{R}_2 - \vec{R}_1|}$$

→ Cylindrical Coordinate : $(r, \phi, z)$




$\vec{R} = r\hat{r} + z\hat{z}$

where $\hat{r} = R\cos\phi\,\hat{x} + R\sin\phi\,\hat{y}$

→ Spherical Coordinate : $(R, \theta, \phi)$

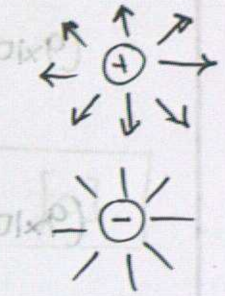
* $\theta$ is angle from (+)ve z-axis
* $\phi$ is angle from (+)ve x-axis (on xy plane)

Week 1: Lecture 2 Jan 8, 2025

## Electric Field ($\vec{E}$)

* is a vector field defined in terms of electrostatic force ($\vec{F}$) that is exerted on a positive test charge (q) at every point in space

$$\boxed{\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \hat{R}} \quad [V/m] \text{ or } [N/C]$$



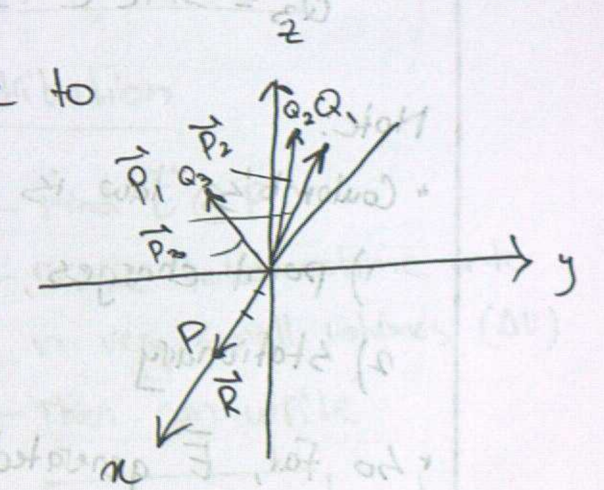
* $\vec{E}$ is one way to explain $\vec{F}$ by considering $\vec{E}$ around each charge
* Direction of $\vec{E}$ is same as $\vec{F}$ that acts on positive test charge
* Field exists regardless of $q_0$
* charge q (not test charge) is not affected by its own $\vec{E}$.

Ex: Find $\vec{E}$ @ P(3m, 0, 0) due to

$Q_1 = 1\mu C$ @ $P_1(0, 2m, 3m)$

$Q_2 = -2\mu C$ @ $P_2(-2, 0, 3m)$

$Q_3 = 3\mu C$ @ $P_3(0, -2, 3m)$




Knowledge: $\vec{E}_{total} = \sum_{i=1}^{3} \frac{Q_i}{4\pi\epsilon_0} \frac{(\vec{R} - \vec{R}_i)}{|\vec{R} - \vec{R}_i|^3}$

$k = 9\times10^9$

$\vec{R} = 3\hat{x}$ , $\vec{R_1} = 2\hat{y} + 3\hat{z}$, $\vec{R_2} = -2\hat{x} + 3\hat{z}$ , $\vec{R_3} = -2\hat{y} + 3\hat{z}$

$$\vec{E} = (9\times10^{9})(10^{-6})\,\frac{3\hat{x} - 2\hat{y} - 3\hat{z}}{(\sqrt{3^2+2^2+3^2})^3} +$$

$$(9\times10^{9})(-2\times10^{-6})\,\frac{3\hat{x} + 2\hat{x} - 3\hat{z}}{(\sqrt{\cancel{3^2+2^2}+3^2})^3} + \quad 5^2$$

$$(9\times10^{9})(3\times10^{-6})\,\frac{3\hat{x} + 2\hat{y} - 3\hat{z}}{(\sqrt{3^2+2^2+3^2})^3}$$

$$\vec{E} = 591.8\hat{x} + 348.4\hat{y} - 773.1\hat{z} \left[\frac{V}{m}\right] \text{ or } \left[\frac{N}{C}\right]$$

H.W. Ex: $\vec{E}$ - ? @ $P(3m, 0^\circ, 0m)$ (cylindrical coordinates)

$Q_1 = 1\mu C$ @ $P_1(2m, 90^\circ, 3m)$

$Q_2 = -2\mu C$ @ $P_2(2m, 180^\circ, 3m)$

$Q_3 = 3\mu C$ @ $P_3(2m, 270^\circ, 3m)$

Note:

* Coulomb's law is valid if charges are:
  1) point charges, and
  2) stationary
* So far, $\vec{E}$ generated by a single point charge
* How about → line charge, → surface charge, → volume charge } → uniform, → nonuniform

Charge distribution in 3D { rect., cylind., sph.

## Charge Densities

* at atomic scale, charge distribution is discrete

$$\rho_V = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V} = \frac{dq}{dV} \Longrightarrow dq = \rho_V\, dV$$

↳ volume charge density

then total charge $\boxed{q = \int_V \rho_V\, dV = \iiint \rho_V\, dV \ [C]}$

if $\rho_V$ is uniform, $\boxed{q = \rho_V V\ [C]}$

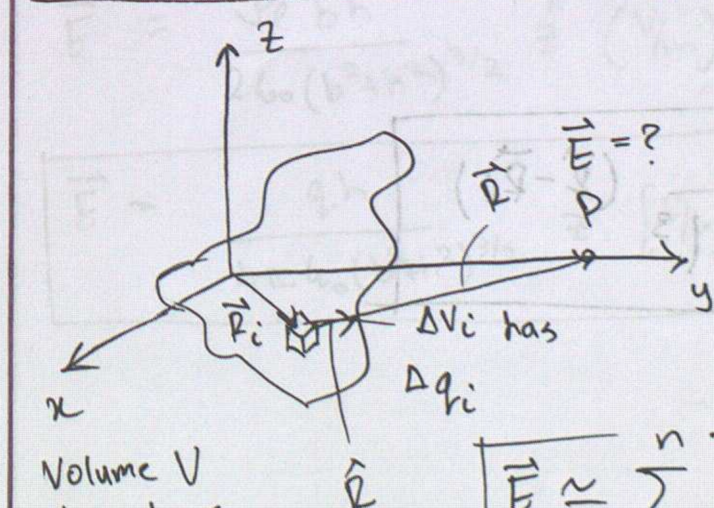
* surface charge density ($\rho_S$) ($C/m^2$)

$$\rho_S = \lim_{\Delta S \to 0} \frac{\Delta q}{\Delta S} = \frac{dq}{dS} \Longrightarrow \boxed{q = \int_S \rho_S\, dS\ [C]}$$

* line charge density ($\rho_\ell$) ($C/m$)

$$\rho_\ell = \lim_{\Delta\ell \to 0} \frac{\Delta q}{\Delta \ell} = \frac{dq}{d\ell} \Longrightarrow \boxed{q = \int_\ell \rho_\ell\, d\ell\ [C]}$$

4.3.2 Electric Field due to a Charge Distribution




Volume V w/h volume charge density $\rho_V$

- Find $\vec{E}$ @ P
- Divide total volume into n very small volumes ($\Delta V$)
- Then can write

$\boxed{\Delta q = \rho_V \Delta V}$ and

$$\boxed{\vec{E} \simeq \sum_{i=1}^{n} \frac{\Delta q_i}{4\pi\varepsilon_0 |\vec{R} - \vec{R_i}|^3}(\vec{R} - \vec{R_i})}$$

• at the limit when

$n \to \infty \Rightarrow \begin{matrix} \Delta V = dV \\ \Delta q = dq \end{matrix}$ and E becomes exact

$$\Rightarrow \vec{E} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\Delta q_i}{4\pi\epsilon_0 |\vec{R}-\vec{R_i}|^3}(\vec{R}-\vec{R_i})$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int_V \frac{dq}{|\vec{R}-\vec{R_i}|^3} (\vec{R}-\vec{R_i})$$ where $dq = \rho_V dV$

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \iiint_V \frac{\rho_V dV}{|\vec{R}-\vec{R_i}|^3} (\vec{R}-\vec{R_i}) \left[\frac{V}{m}\right]}$$

• similarly, $\vec{E}$ due to surface charge distribution

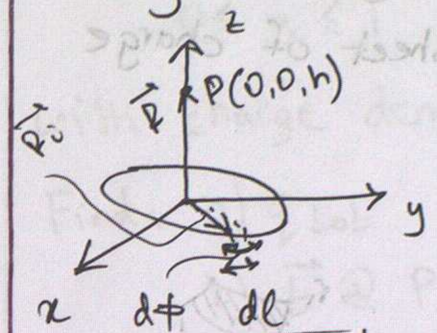
$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \iint_S \frac{\rho_s dS}{|\vec{R}-\vec{R_i}|^3} (\vec{R}-\vec{R_i})}$$

• for line

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \int_l \frac{\rho_l dl}{|\vec{R}-\vec{R_i}|^3} (\vec{R}-\vec{R_i})}$$



Ex: A ring of radius b and uniform line charge density of positive $\rho_\ell$, find $\vec{E}$ @ $P(x,y,z) = (0,0,h)$



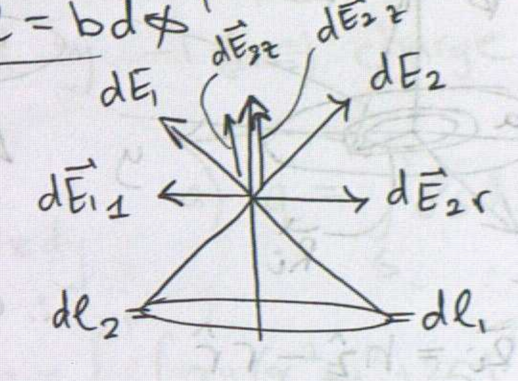

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\rho_\ell d\ell}{|\vec{R}-\vec{R_i}|^3} (\vec{R}-\vec{R_i})$$

need $d\ell$, $\vec{R}$, and $\vec{R_i}$

$\boxed{dq = \rho_\ell d\ell}$

~~dφ =~~

$\boxed{d\ell = b\, d\phi}$




∴ "r" comp. cancel out, only $\hat{z}$ comp. remains

$$\vec{R}-\vec{R_i} = h\hat{z} - b\hat{r}$$
$$= -b\hat{r} + h\hat{z}$$

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\rho_\ell b\, d\phi}{(b^2+h^2)^{3/2}} (-b\hat{r} + h\hat{z})$$

0, cancels out

$$\vec{E} = \frac{\rho_\ell b h}{4\pi\epsilon_0 (b^2+h^2)^{3/2}} \int_0^{2\pi} d\phi\, \hat{z}$$

recall $q = 2\pi b \rho_\ell$

$\Rightarrow \rho_\ell = \frac{q}{2\pi}$

$$\vec{E} = \frac{\rho_\ell b h}{2\epsilon_0 (b^2+h^2)^{3/2}} \hat{z} \ (V/m)$$

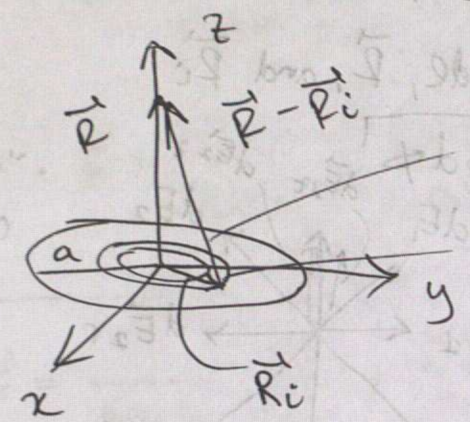
$$\boxed{\vec{E} = \frac{qh}{4\pi\epsilon_0 (b^2+h^2)^{3/2}} \hat{z} \ [V/m]}$$

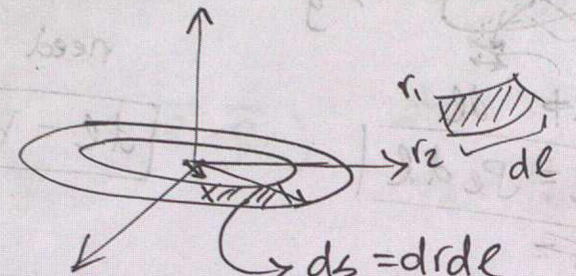
electric field in ~~that~~ terms of "q" instead of charge density

**Ex:** Find $\vec{E}$ @ $P=(0,0,h)$ due to a disk of

a) radius $a$ and uniform charge ~~density~~ distribution $\rho_s$

then evaluate $\vec{E}$ due to infinite sheet of charge density $\rho_s$ by letting $a \to \infty$




$\to ds = dr\,dl$

$\Rightarrow dq = \rho_s\, ds$

$\therefore ds = r\,dr\,d\phi$

$\vec{R}-\vec{R}_i = h\hat{z} - r\hat{r}$

$= -r\hat{r} + h\hat{z}$, we know $d\vec{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{\rho_s ds}{|\vec{R}-\vec{R}_i|^{3/2}}(\vec{R}-\vec{R}_i)$

$$d\vec{E} = \frac{\rho_s}{4\pi\epsilon_0}\frac{\cancel{-r\hat{r}} + h\hat{z}}{(r^2+h^2)^{3/2}}\, r\,dr\,d\phi$$

0 due to symmetry

$$d\vec{E} = \frac{\rho_s}{4\pi\epsilon_0}\frac{h}{(r^2+h^2)^{3/2}}\, r\,dr\,d\phi$$

$$\vec{E} = \frac{\rho_s h}{4\pi\epsilon_0}\int_0^a\int_0^{2\pi}\frac{r}{(r^2+h^2)^{3/2}}\,dr\,d\phi$$

$$= \frac{\rho_s h}{2\epsilon_0}\int_0^a \frac{r\,dr}{(r^2+h^2)^{3/2}}$$

$$\boxed{\Rightarrow \vec{E} = \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{h}{\sqrt{a^2+h^2}}\right]\hat{z} \quad [V/m]}$$

b) if $a \to \infty$ ?

$$\boxed{\vec{E} = \frac{\rho_s}{2\epsilon_0}\hat{z}\left[\frac{V}{m}\right]}$$

**Ex:** Given that sheet of

$0 \le x \le 1$
$0 \le y \le 1$ (m) on $z=0$ plane

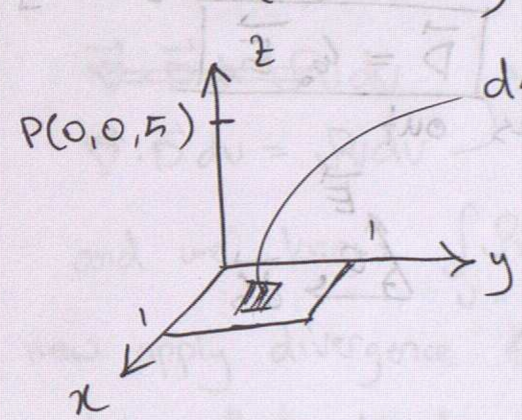
with charge density of $\rho_s = xy(x^2+y^2+25)^{3/2}$ $\left[\frac{nC}{m^2}\right]$

non-uniform

Find: a) $q_{tot}$

b) $\vec{E}$ @ $P(0,0,5)$ m

c) force experienced by $-1$ (mC) charge at $(0,0,5m)$




$ds = dx\,dy$

a) $q = \int_s \rho_s\, ds$

$$q = \int_0^1\int_0^1 xy(x^2+y^2+25)^{3/2}\,dx\,dy$$

$$\boxed{q = 33.15\ [nC]}$$

b) $d\vec{E} = \dfrac{\rho_s ds}{4\pi\epsilon_0|\vec{R}-\vec{R}_i|^{3}}(\vec{R}-\vec{R}_i)$

$\vec{R}-\vec{R}_i = 5\hat{z} - (x\hat{x}+y\hat{y}) = -x\hat{x} - y\hat{y} + 5\hat{z}$

$$d\vec{E} = \frac{\rho_s\,dx\,dy}{4\pi\epsilon_0(x^2+y^2+5^2)^{3/2}}(-x\hat{x} - y\hat{y} + 5\hat{z})$$

$$d\vec{E} = \frac{xy\cancel{(x^2+y^2+25)^{3/2}}}{4\pi\epsilon_0\cancel{(x^2+y^2+25)^{3/2}}}\; -x\hat{x} - y\hat{y} + 5\hat{z}\; dx\,dy$$

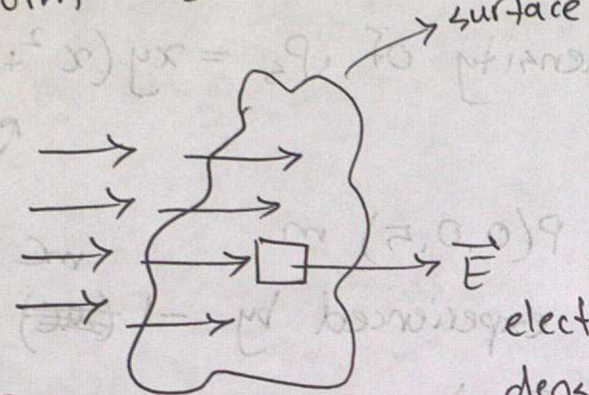
$$\boxed{\vec{E} = -1.5\hat{x} - 1.5\hat{y} + 11.25\hat{z} \quad [V/m]}$$

Week 2: Lecture 1 Jan 14, 2025

Today: Gauss' Law

→ integral form } examples
→ point form }

Consider:




electric field intensity $\vec{E}$ $\left[\frac{N}{C}\right]$ or $\left[\frac{V}{m}\right]$

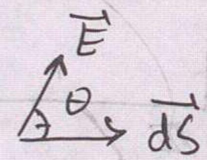
electric field density $\vec{D}$ $[C/m^2]$

$$\boxed{\vec{D} = \varepsilon_0 \vec{E}}$$

- Find out how much $\vec{E}$ comes out

if $\theta = 0 \longrightarrow$ max flux

$\theta = 90^\circ \longrightarrow 0$ flux




$$\Rightarrow \boxed{E\,ds\cos\theta = \vec{E}\cdot\vec{ds}}$$

Gauss's Law (1777 – 1855)

- constitutes one of the fundamental laws of electromagnetism
- states that total electric flux ($\Phi$, or $\Psi$) through any closed surface is equal to the total charge enclosed by that surface (Gaussian Surface)

$$\boxed{\Phi = Q_{enc}} \quad \boxed{\Phi = \oint_S \vec{D}\cdot\vec{ds}} \rightarrow$$

$$\boxed{\oint_S \vec{D}\cdot\vec{ds} = Q_{enc}} \Longrightarrow \boxed{\varepsilon_0 \oint_S \vec{E}\cdot\vec{ds} = Q_{enc}} \text{ integral form of Gauss Law}$$

$$\vec{D} = \varepsilon_0 \vec{E}$$

Differential Form Gauss Law

- First Maxwell equation

$$\boxed{\vec{\nabla}\cdot\vec{D} = \rho_V}$$ "divergence" of $\vec{D}$ is $\rho_V$

now let's derive integral form from here

$\cancel{\vec{\nabla}\cdot\vec{D}\,dv}\ \rho_v\,dv$

$\vec{\nabla}\cdot\vec{D}\,dv = \rho_v\,dv \rightarrow \int_V \vec{\nabla}\cdot\vec{D}\,dV = \int_V \rho_v\,dV$ ①

and we know $\int_V \rho_v\,dV = Q_{enc,tot}$ ②

now apply divergence theorem (aka Gauss's theorem)

- states that closed surface integral of a vector equals the volume integral of the divergence of the vector

$$\int_V \vec{\nabla}\cdot\vec{A}\,dv = \oint_S \vec{A}\cdot\vec{ds} \Longrightarrow$$ can write $$\int_V \vec{\nabla}\cdot\vec{D}\,dV = \oint_S \vec{D}\cdot\vec{ds}$$ ③

- combining ①, ②, and ③

$$\Rightarrow \boxed{\oint_S \vec{D}\cdot\vec{ds} = Q_{enc}}$$ voila! integral form

Notes: depending on convenience, I can go back and forth b/n forms

- Gauss's law is an alternative form of Coulomb's law
- is more convenient to use when symmetry exists

## Gaussian Surface

+ hypothetical surface
+ must enclose all the space I care about
+ must make use of symmetry

**Ex:** uniformly charged sphere


- radius a [m]
- uniform charge density $\rho_V$ $[C/m^3]$ ← ∴ solid sphere
- Find $\vec{E}$ everywhere

Knowing $\varepsilon_0 \oint_S \vec{E}\cdot\vec{ds} = Q_{enc}$

$$Q_{enc} = \int_V \rho_V dv = \rho_V \int_V dv = \rho_V \frac{4}{3}\pi r^3 \quad [C]$$

$$\varepsilon_0 \oint \vec{E}\cdot\vec{ds} = Q_{enc} \Rightarrow \varepsilon_0 |E| \oint_S ds = Q_{enc}$$

$r<a$

$$\varepsilon_0 E \cdot 4\pi r^2 = \rho_V \frac{4}{3}\pi r^3 \Rightarrow \underline{\underline{\vec{E} = \frac{\rho_V}{3\varepsilon_0} r \hat{R} \quad [V/m]}}$$

since $\vec{D} = \varepsilon_0 \vec{E} \Rightarrow \underline{\underline{\vec{D} = \frac{\rho_V}{3} r \hat{R}}}$

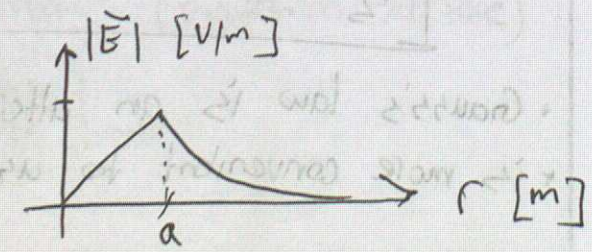
$r>a$

$$Q_{enc} = \rho_V \frac{4}{3}\pi a^3$$

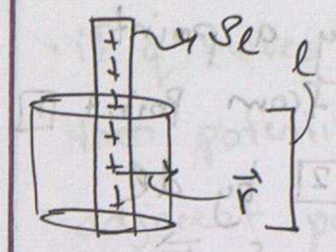
$$\varepsilon_0 E 4\pi r^2 = \rho_V \frac{4}{3}\pi a^3 \Rightarrow \underline{\underline{\vec{E} = \frac{\rho_V a^3}{3\varepsilon_0}\cdot\frac{1}{r^2}\hat{R} \quad [V/m]}}$$

$r=a$

$$\therefore \vec{E} = \begin{cases} \frac{\rho_V}{3\varepsilon_0} r \hat{R} & 0<r<a \\ \frac{\rho}{3\varepsilon_0} a \hat{R} & r=a \\ \frac{\rho_V a^3}{3\varepsilon_0}\cdot\frac{1}{r^2}\hat{R} & r>a \end{cases}$$




**Ex:** Infinite line charge with density charge $\rho_\ell$ [C/m]




→ Find $\vec{E}$ @ distance r

$$\varepsilon_0 \oint_S \vec{E}\cdot\vec{ds} = q_{enc}$$

$$\varepsilon_0 E (2\pi r \cancel{\ell}) = \rho_\ell \cancel{\ell} \quad \rightarrow \quad \underline{\underline{\vec{E} = \frac{\rho_\ell}{2\pi\varepsilon_0}\cdot\frac{1}{r}\hat{r} \quad [V/m]}}$$

Note: contributions from top + bottom = 0 since $\vec{E}$ is tangential to surface

**Ex:** Thin spherical shell with $Q_{tot} = +q$

Find $\vec{E}$ everywhere

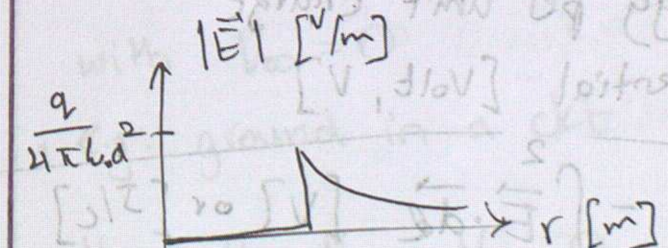
$$\varepsilon_0 \oint_S \vec{E}\cdot\vec{ds} = Q_{enc}$$

inside: $r<a$

$Q_{enc} = 0 \Rightarrow \vec{E} = 0$

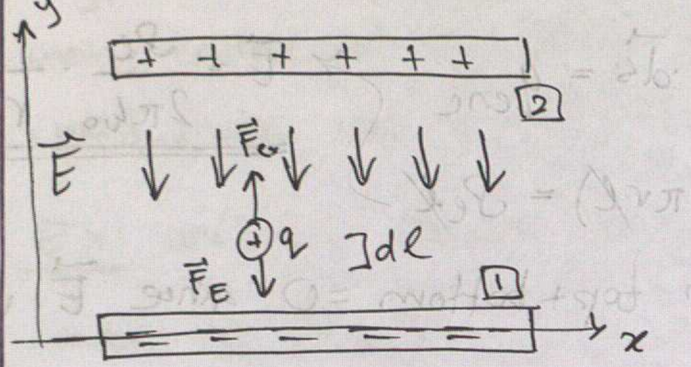
outside: $r>a \Rightarrow \varepsilon_0 E 4\pi r^2 = q \Rightarrow \underline{\underline{\vec{E} = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}\hat{R} \quad [V/m]}}$

on: $r=a \Rightarrow \underline{\underline{\vec{E} = \frac{q}{4\pi\varepsilon_0 a^2}\hat{R} \quad [V/m]}}$



Week 2: Lecture 2 Jan 15, 2024

## Electric Potential




- move slowly a point charge (q) from Point [1] to Point [2] by $d\ell$ in presence of $\vec{E}$

$$\vec{F}_{ext} = -\vec{F}_E = -q\vec{E}$$

$$dW = -\vec{F}_E \cdot \vec{d\ell}$$

- Note: negative sign indicates that work is done by an external agent
- Knowing $\vec{F}_E = q\vec{E} \Rightarrow dW = -q\vec{E}\cdot\vec{d\ell}$

$$\boxed{W = -q\int_{1}^{2}\vec{E}\cdot\vec{d\ell}}$$ ← electric potential energy [J]

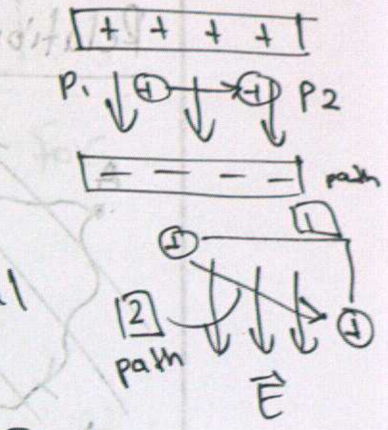
(2 → final, 1 → initial)

## Electric Potential (aka voltage, potential)

- Electric potential energy per unit charge is called electric potential [Volt, V]

$$\boxed{V_{21} = V_2 - V_1 = \frac{W}{q} = -\int_{1}^{2}\vec{E}\cdot\vec{d\ell} \quad [V] \text{ or } [J/C]}$$

## Equipotential Surfaces




- if q moves along surface ⊥ to $\vec{E}$, then potential doesn't change
- adjacent points with the same electric potential make equipotential surface
- hence work done by electrostatic force = 0 in going from $p_1$ to $p_2$
- Note: work done by electrostatic force is independent of path ... taking path 1 or 2 are same
- do example on 4-8 page 184

## Relative Potential

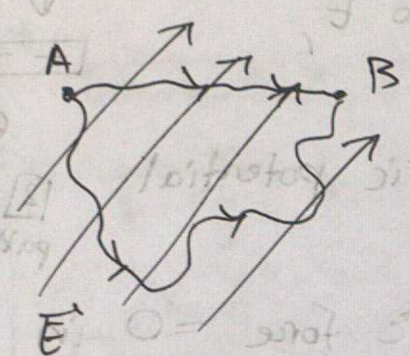
$$V_{21} = V_2 - V_1 = -\int_{1}^{2}\vec{E}\cdot\vec{d\ell}$$ is the voltage b/n two points

## Absolute Potential

- is potential w.r.t. a reference considered at infinity with $V_\infty = 0$
- e.g: ground in a ckt

$$\boxed{V_{21} = V_2 - V_1 = V_2 - 0 \Rightarrow V = -\int_{\infty}^{2}\vec{E}\cdot\vec{d\ell}}$$

## Relation B/n E and V (Maxwell's Equ.)




* Consider, knowing that potential gain is independent of path

$$V_{AB} + V_{BA} = 0$$

* recognize that physically that no net ~~net~~ work is done in moving q along any closed path

$$\boxed{\oint_{\ell} \vec{E} \cdot d\vec{\ell} = 0}$$

$\vec{E}$ is conservative field

"aka KVL"

## Stoke's Theorem

* surface integral of curl of a vector field over a surface bounded by a closed path is equal to the line integral of the vector around that path

$$\boxed{\oint_{\ell} \vec{E} \cdot d\vec{\ell} = \int_{S} (\nabla \times \vec{E}) \cdot d\vec{S}}$$

* knowing $\oint_{\ell} \vec{E} \cdot d\vec{\ell} = 0 \implies \boxed{\nabla \times \vec{E} = 0}$

* point form of 2nd Maxwell equation for static electric field
  or differential

## Derive $\vec{E}$ from V

* dV b/n two points separated by $d\vec{\ell}$ in presence of $\vec{E}$ is $dV = -\vec{E} \cdot d\vec{\ell}$

* $\vec{E}$ in cartesian coordinate

$$\vec{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$$

plug into here

$$\Rightarrow dV = -(E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) \cdot (dx\, \hat{x} + dy\, \hat{y} + dz\, \hat{z})$$

$$dV = -(E_x dx + E_y dy + E_z dz)$$

implies $E_x = -\frac{\partial V}{\partial x}$ $\quad E_y = -\frac{\partial V}{\partial y}$ $\quad E_z = -\frac{\partial V}{\partial z}$

$$\vec{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z} = -\left(\frac{\partial}{\partial x} V \hat{x} + \frac{\partial}{\partial y} V \hat{y} + \frac{\partial}{\partial z} V \hat{z}\right)$$

$$\boxed{\vec{E} = -\nabla V}$$

gradient of V

* physically, (-)ve sign implies that if V increases as +q moves along a path, then $\vec{E}$ has an opposite direction

## Poisson's Equation

- recall $\nabla \cdot \vec{D} = \rho_v$ (point form of Gauss Law)

$$\vec{D} = \epsilon_0 \vec{E} \implies \nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon_0}$$

- knowing $\vec{E} = -\nabla V$

$$\nabla \cdot (-\nabla V) = \frac{\rho_v}{\epsilon_0}$$

$$\nabla^2 V = \frac{\partial^2}{\partial x^2} V + \frac{\partial^2}{\partial y^2} V + \frac{\partial^2}{\partial z^2} V$$

$$\boxed{\nabla^2 V = \frac{-\rho_v}{\epsilon_0}}$$ where $V = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho_v}{R'} dv$

- if charge is zero in a region, then

if $q = 0 \implies \rho_v = 0$

$$\implies \boxed{\nabla^2 V = 0}$$ ← Laplace's Equation

Week 2: Lecture 3 Jan 17, 2025

Ex: Given $V = \frac{10}{R^2} \sin\theta \cos\phi$

a) Find $\vec{D}$ @ $(2, \frac{\pi}{2}, 0)$

b) Find work done in moving $10\,\mu C$ point charge from point $A(1, 30^\circ, 120^\circ)$ to $B(4, 90^\circ, 60^\circ)$

a) $\vec{D} = \epsilon_0 \vec{E}$, $\vec{E} = -\nabla V$

$$\nabla = \left( \frac{\partial}{\partial R}\hat{R} + \frac{1}{R}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi} \right)$$ ← aid sheet

$$\vec{E} = -\nabla V = -\left( \frac{\partial}{\partial R} V\hat{R} + \frac{1}{R}\frac{\partial}{\partial\theta} V\hat{\theta} + \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi} V\hat{\phi} \right)$$

$$\vec{E} = -\left( 10\sin\theta\cos\phi \left(\frac{-2}{R^3}\right)\hat{R} + \frac{1}{R}\frac{10\cos\phi}{R^2}(\cos\theta)\hat{\theta} + \frac{10}{R^2}\cdot\frac{\sin\theta}{R\sin\theta}(-\sin\phi)\hat{\phi} \right]$$

$\vec{E} = 2.5$ [V/m] $\implies \vec{D} = \epsilon_0\vec{E} = 2.5\epsilon_0$ [C/m²]

b) $V = \frac{W}{q}$, $V_{BA} = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{l}$

$W = 28.125\,\mu J$ $V_B > V_A \implies V_B - V_A > 0$

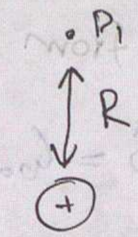
so work is done by external agent

Find V due to P.C. @ Point $P_1$

+ take test charge $q_0$ from $P_1$ to infinity
+ since path independent, then choose simplest path

$\Longrightarrow$ Radial $d\vec{l} = dR\,\hat{R}$

$\vec{E} = E_R\,\hat{R}$




$\vec{E}\cdot d\vec{l} = E\,dR$

$$V_2 - V_1 = \underbrace{V_{final}}_{=0} - V_{initial} = -\int_{in}^{fin} \vec{E}\cdot d\vec{l}$$

$$-V_{in} = -\int_R^{\infty} E\,dR' \Longrightarrow V_{in} = \int_R^{\infty} \frac{q}{4\pi\epsilon_0}\frac{dR'}{R'^2}$$

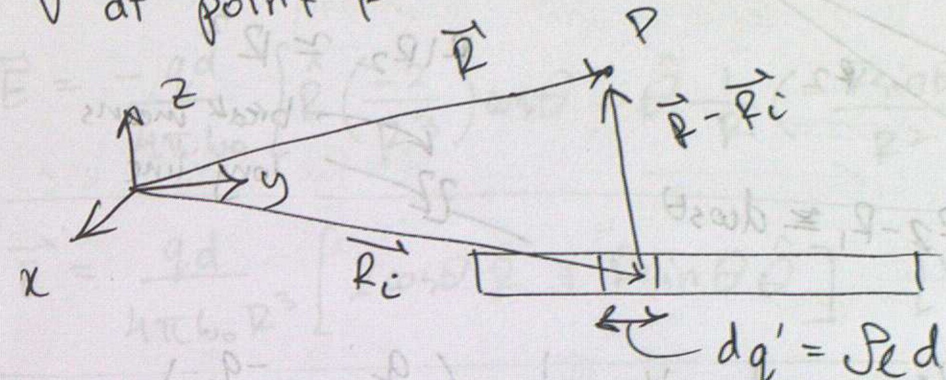
$$V = \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{R'}\right)_R^{\infty} = \frac{q}{4\pi\epsilon_0}\left(-\underbrace{\cancel{\frac{1}{\infty}}}_{\to 0} + \frac{1}{R}\right)$$

$$\boxed{V = \frac{q}{4\pi\epsilon_0}\cdot\frac{1}{R}\ [V]}$$

• potential due to many point charges

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} \frac{q_i}{|\vec{R} - \vec{R}_i|}}$$

+ If charge distribution is continuous, then V at point P

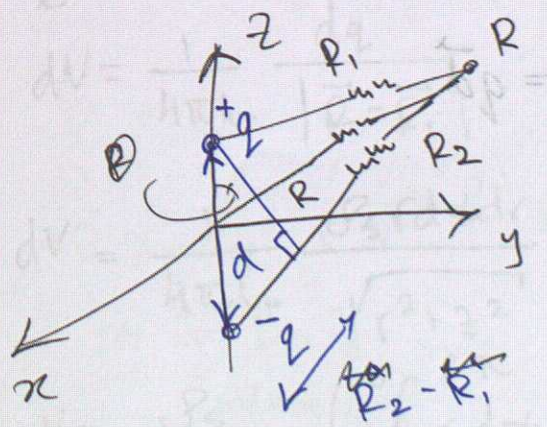



$$\boxed{V = \frac{1}{4\pi\epsilon_0}\int_l \frac{\rho_l\,dl'}{|\vec{R} - \vec{R}_i|}}$$

... line, but replace $l$ and $\rho_l\,dl'$ with $s$, $\rho_s\,ds'$ and $V$, $\rho_v\,dV'$

Electric Dipole

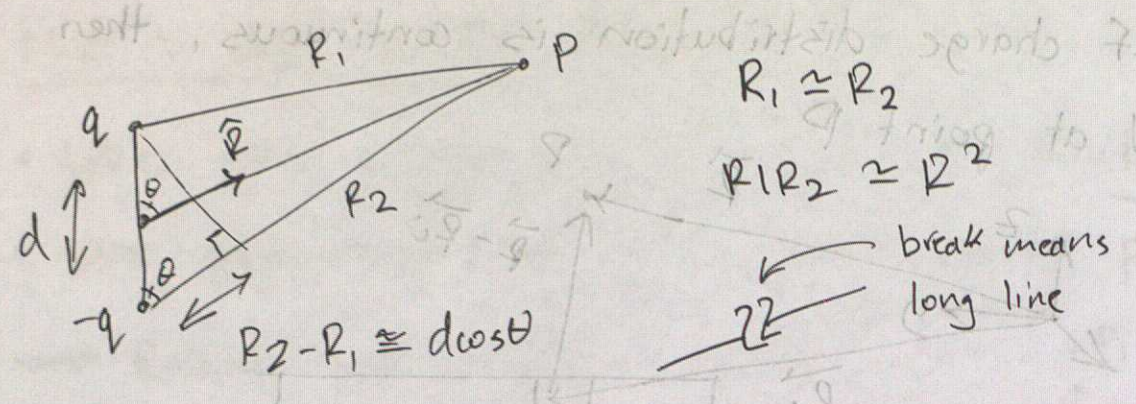
+ is two equal point charges of opposite signs separated by small distance "d"




consider drawing where $R \gg d$ ... Find V and E at $\vec{P}$

$R_2 - R_1 \simeq d\cos\theta$

Week 3 : Lecture 1 Jan 21, 2025




$R_1 \simeq R_2$

$R_1 R_2 \simeq R^2$

$V = \sum_{i=1}^{2} V_i$ and $\Longrightarrow V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{R_1} + \frac{-q}{R_2}\right)$

$V = \frac{q}{4\pi\epsilon_0}\left(\frac{R_2 - R_1}{R_1 R_2}\right) \leftarrow$ $R_2 - R_1 \simeq d\cos\theta$, $R_2 R_1 \simeq R^2$

$\Longrightarrow V = \frac{q\,d\cos\theta}{4\pi\epsilon_0 R^2}$

Note : $q\,d\cos\theta = q\,\vec{d}\cdot\hat{R}$

Define: Dipole moment as $\vec{P} = q\vec{d}$

$\Longrightarrow \boxed{V = \frac{\vec{P}\cdot\hat{R}}{4\pi\epsilon_0 R^2} \quad [V]}$

Find $\vec{E}$ from V

$\vec{E} = -\nabla V$ and spherical coordinates

Knowing from Table

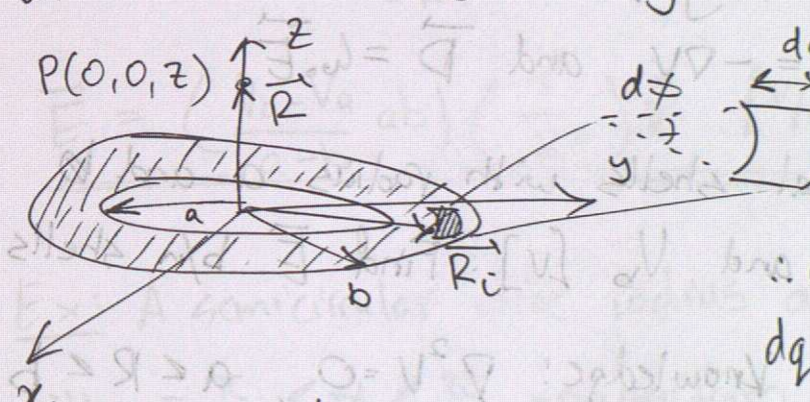
$$\vec{E} = -\left(\hat{R}\frac{\partial V}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\right)$$

$$\vec{E} = -\nabla\left(\frac{q\,d}{4\pi\epsilon_0}\frac{\cos\theta}{R^2}\right)$$

$$\vec{E} = \frac{-q\,d}{4\pi\epsilon_0}\left[\hat{R}\left(\frac{-2}{R^3}\right)\cos\theta + \hat{\theta}\frac{1}{R}\left(-\frac{R\sin\theta}{R^2}\right)\right]$$

$$\boxed{\vec{E} = \frac{q\,d}{4\pi\epsilon_0 R^3}\left[2\cos\theta\,\hat{R} + R\sin\theta\,\hat{\theta}\right] \quad [V/m]}$$

Ex: A disk with a hole in the middle (Annulus) with uniform charge density $\rho_s$ [$C/m^2$], Find V @ P




$dl = r\,d\phi$

$\therefore ds = r\,d\phi\,dr$

$dq = \rho_s\, r\,d\phi\,dr$

$\vec{R} - \vec{R_i} = -r\hat{r} + z\hat{z}$

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{|\vec{R} - \vec{R_i}|}$$

$$dV = \frac{1}{4\pi\epsilon_0}\frac{\rho_s\, r\,d\phi\,dr}{\sqrt{r^2 + z^2}}$$

$$V = \frac{\rho_s}{4\pi\epsilon_0}\int_a^b\int_0^{2\pi}\frac{r\,d\phi\,dr}{\sqrt{r^2 + z^2}}$$

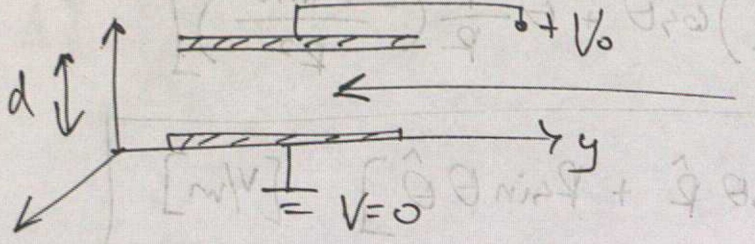
$u = r^2 + z^2 \Longrightarrow dr = \frac{1}{2r}du$

$$V = \frac{\rho_s}{2\epsilon_0}\int \frac{1}{2}u^{-1/2}du$$

$$\vdots$$

$$\frac{\rho_s}{2\epsilon_0}\left(\sqrt{b^2 + z^2} - \sqrt{a^2 + z^2}\right)$$

Note: Poisson's and Laplace's equations are used to calculate V in regions with boundary conditions.



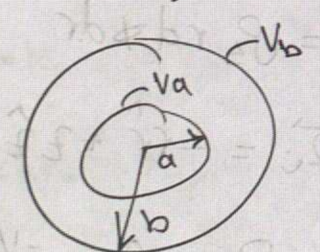

No free charge, use

$$\nabla^2 V = 0$$

$$\begin{cases} V_{z=0} = 0 \\ V_{z=d} = V_0 \end{cases} \leftarrow \text{boundary conditions}$$

This provides V at any point inside capacitor then can find $\vec{E} = -\nabla V$ and $\vec{D} = \epsilon_0 \vec{E}$

<u>Ex:</u> Two spherical shells with radius a and b and voltages $V_a$ and $V_b$ [V]. Find $\vec{E}$ b/n shells




Knowledge: $\nabla^2 V = 0$, $a < R < b$

From Table:

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta} \ldots$$

$$\Rightarrow \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) = 0 \quad \text{since } \hat{\theta} \text{ and } \hat{\phi} \text{ are equipontial}$$

$$R \neq 0, \infty \quad \frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) = 0$$

$$R^2 \frac{\partial V}{\partial R} = A \leftarrow \text{constant}$$

$$\Rightarrow \frac{\partial V}{\partial R} = \frac{A}{R^2} \Rightarrow V = -\frac{A}{R} + B$$

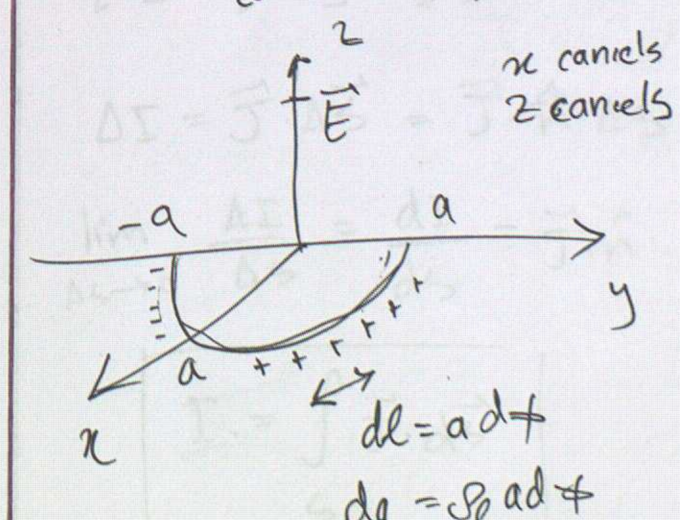
$$\begin{cases} V_a = -\frac{A}{R} + B \quad R \to a \\ V_b = -\frac{A}{R} + B \quad R \to b \end{cases} \Rightarrow \quad A = \frac{V_a - V_b}{a-b} ab \qquad B = \frac{aV_a - bV_b}{a-b}$$

$$V = \left(\frac{V_b - V_a}{a-b} ab\right)\frac{1}{R} + \frac{aV_a - bV_b}{a-b} \quad [V]$$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial R}\hat{R} \longrightarrow \frac{\partial}{\partial R}\left(\frac{1}{R}\right) = -\frac{1}{R^2}$$

$$\vec{E} = \left(\frac{V_b - V_a}{a-b} ab\right)\left(\frac{1}{R^2}\right)\hat{R} \quad [V/m]$$

<u>Ex:</u> A semicircular wire radius a [m] in xy-plane with $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$ nonuniform charge density $\rho_\ell = \rho_{\ell 0} \sin\phi$ [C/m] ... Find $\vec{E}$ @ (0,0,z)




$$\rho_\ell = \rho_{\ell 0}\sin\left(-\frac{\pi}{2}\right) = -\rho_{\ell 0}$$

$$\rho_\ell = \rho_{\ell 0}\sin\left(\frac{\pi}{2}\right) = \rho_{\ell 0}$$

$$\rho_\ell = \begin{cases} +ve & \text{for } 0 < \phi \le \frac{\pi}{2} \\ -ve & \text{for } -\frac{\pi}{2} \le \phi < 0 \end{cases}$$

$$\vec{R} = z\hat{z} \implies \vec{R}-\vec{R_i} = -a\cos\phi\,\hat{x} - a\sin\phi\,\hat{y} + z\hat{z}$$

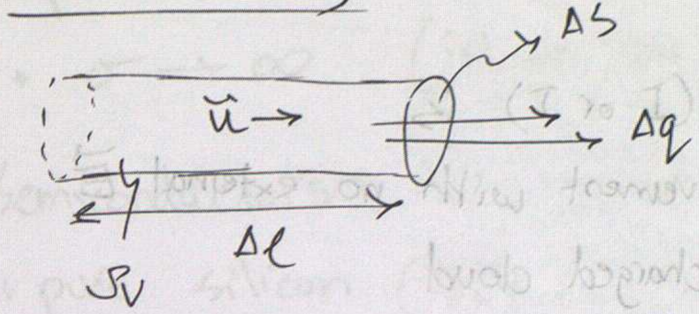
$$\vec{R_i} = a\hat{r} = a\cos\phi\,\hat{x} + a\sin\phi\,\hat{y}$$

$$d\vec{E} = \frac{\rho_{l0}\sin\phi\, a\, d\phi}{4\pi\varepsilon_0}\;\frac{-a\cos\phi\,\hat{x} - a\sin\phi\,\hat{y} + z\hat{z}}{(a^2\cos^2\phi + a^2\sin^2\phi + z^2)^{3/2}}$$

$$\vec{E} = \frac{-\rho_{l0}\,a^2}{8\varepsilon_0(a^2+z^2)^{3/2}}\,\hat{y}\;[V/m]$$

Week 3: Lecture 2 Jan 22, 2025

## Current Density




consider charges that move through $\Delta s$ with velocity $\vec{v}$

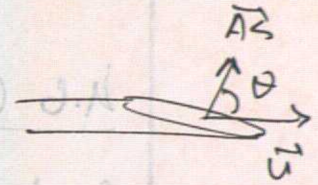
$\Delta q = \rho_v dV$ and

$$\Delta l = u\Delta t$$
$$\Delta V = \Delta l \Delta s$$
$$\Delta V = u\Delta t\Delta s$$

$\Delta q = \rho_v u\Delta t\Delta s$ ~~$dV = \Delta l\Delta s$~~ ~~$\Delta l =$~~

$\underline{\Delta q = \rho_v u \Delta t \Delta s}$

- in general, to account for effective $\Delta s$




$$\implies \Delta q = \rho_v u\Delta t\Delta s\cos\theta$$
$$\Delta q = \rho_v \Delta t\,\vec{u}\cdot\vec{\Delta s}$$

$$\Delta I = \frac{\Delta q}{\Delta t} = \frac{\rho_v\Delta t\,\vec{u}\cdot\vec{\Delta s}}{\Delta t} = \rho_v\vec{u}\cdot\vec{\Delta s}$$

Define $\vec{J} = \rho_v\vec{u}$ current density $\rightarrow \Delta I = \vec{J}\cdot\vec{\Delta s}$

$$[A] = [C/s] = [I] \implies [\vec{J}] = [A/m^2]$$

$$\Delta I = \vec{J}\cdot\vec{\Delta s} = \vec{J}\cdot\hat{n}\,\Delta s \implies \frac{\Delta I}{\Delta s} = \vec{J}\cdot\hat{n}$$

$$\lim_{\Delta s\to 0}\frac{\Delta I}{\Delta s} = \frac{dI}{ds} = \vec{J}\cdot\hat{n} \implies \int dI = \int \vec{J}\cdot d\vec{s}$$

$$\boxed{I = \int_s \vec{J}\cdot d\vec{s}}$$

## Types of Currents

* convection current (I or J) is
→ due to charge movement with no external $\vec{E}$
→ e.g: wind driven charged cloud

* conduction current (I or J)
→ charged particles moving due to electric field

## 4.6 Conductors

Conductivity: $\left(\sigma = \left[\frac{\mho}{m}\right] = \left[\frac{S}{m}\right]\right)$

* is how easily electrons can travel in a material under $\vec{E}$ applied

Conductors:

* have abundance of free electrons
* eg: metals, mercury ($10^6$ S/m)
* note: as ↑ temperature, ↓ $\sigma$

Insulator:

* does not have free electrons
* $\sigma$: $10^{-10} - 10^{-17}$ [S/m]
* perfect dielectric, $\sigma = 0$

Superconductor:

* $\sigma \to \infty$ (irl, we cool down to 0°K)

Semiconductors:

* pure silicon (Si)
* can add impurities to increase free electrons or holes (by doping)

Drift Velocity:

* in a conductor:
  * when $i = 0$
    Then free electrons move randomly at a speed of $\simeq 10^6$ m/s
  * when $i \neq 0$
    then free electrons still move randomly but at the same time, "drift" in opposite to $\vec{E}$ direction at a drift velocity $\simeq \begin{matrix}10^{-5}\\10^{-4}\end{matrix}$ [m/s]

* Drift velocity in presence of $\vec{E}$ is given
→ for electrons by: $\boxed{\vec{u}_e = -\mu_e \vec{E}}$ → electron mobility $\mu_e$ (material property)

$\mu_e = \left[\frac{m^2}{V \cdot s}\right]$, a measure of how quickly an electron moves through a solid material

→ for holes: $\boxed{\cancel{\vec{u}_h = \mu_h \vec{E}}}$ $\boxed{\vec{u}_h = \mu_h \vec{E}}$

$\mu_h$ : "hole mobility"

+ Recall: $\vec{J} = \rho_v \vec{u}$

↳ total current density for both $\vec{J}_e$ and $\vec{J}_h$

$\Rightarrow \vec{J} = \vec{J}_e + \vec{J}_h = \rho_{ve}\vec{u}_e + \rho_{vh}\vec{u}_h$ [A/m²]

where $\rho_{ve} = -N_e e$ ← charge of electron is $-e$

number of electrons

$\rho_{vh} = N_h e$ ← charge of holes is $e$

number of holes

∴ can write

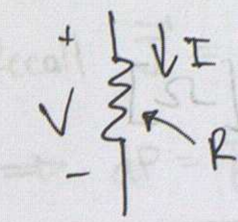
$$\vec{J} = \underbrace{\left(-\rho_{ve}\mu_e + \rho_{vh}\mu_h\right)}_{\text{conductivity}}\vec{E} = \sigma\vec{E}$$

$\boxed{\vec{J} = \sigma\vec{E}}$ • good ~~ele~~ conductor

$N_h\mu_h << N_e\mu_e$

$\Rightarrow \sigma = -\rho_{ve}\mu_e = N_e\mu_e e$

+ perfect conductor: $\sigma \to \infty$, $\vec{E} = \dfrac{\vec{J}}{\sigma} = 0$




$V = RI$

$V = (0)I$

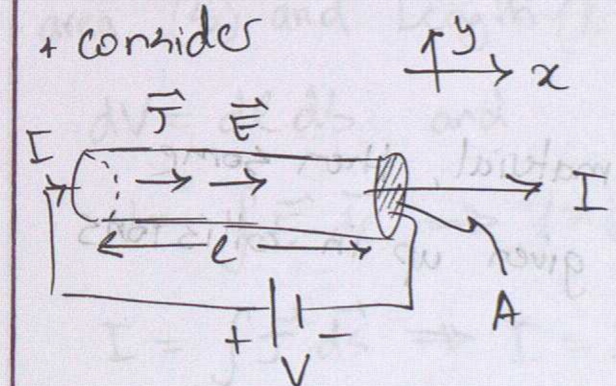
+ perfect insulator $\sigma = 0$, $\vec{J} = 0$

+ please do example 4-10

## 4-6.2 Resistance

+ a measure of opposition to flow of current

+ consider




$V = -\int_l \vec{E}\cdot d\vec{l} = El$ [V]

$I = \int_s \vec{J}\cdot d\vec{s}$ and $\vec{J} = \sigma\vec{E}$ [A]

↳ $I = \int_s \sigma\vec{E}\cdot d\vec{s} = \sigma E A$ ↑ area

Ohm's Law

$R = \dfrac{V}{I} = \dfrac{\cancel{E}l}{\sigma \cancel{E} A} \Rightarrow \boxed{R = \dfrac{l}{\sigma A}}$ ← definition of R

To Ask:

* constants like $\varepsilon_0$, e, k not given
* electric current/electron mobility formula not given $\rho_v = N_e e$ ...
* W3L1: last example, why not also spherical

Week 3: Lecture 3 Jan 24, 2025

## Ohm's Law

$$R = \frac{l}{\sigma A} \quad \text{and} \quad R = \frac{V}{I} = \frac{-\int_l \vec{E}\cdot d\vec{l}}{\int_S \sigma \vec{E}\cdot d\vec{s}} \quad [\Omega]$$

$$\boxed{R = \frac{-\int_l \vec{E}\cdot d\vec{l}}{\int_S \sigma \vec{E}\, d\vec{s}}}$$

$$\boxed{G = \frac{1}{R} = \frac{\int_S \sigma \vec{E}\cdot d\vec{s}}{-\int_l \vec{E}\cdot d\vec{l}}} \quad [\mho] \text{ or [Siemens]}$$

~~conductivity~~ conductance

## Joule's Law

- when charge moves thru a material, then some of the charge energy is given up in collisions with atoms of material
- called dissipated power (P)
- then moving $dq = \rho_v dv$ requires a force $d\vec{F} = dq\,\vec{E}$

$\therefore\ d\vec{F} = \rho_v dV \vec{E}$

- work is $dW = d\vec{F}\cdot d\vec{l} = \rho_v dV\, \vec{E}\cdot d\vec{l}$
- and the increment in power dissipated is

$$dP = \frac{dW}{dt} = \rho_v dV\, \vec{E}\cdot \underbrace{\frac{d\vec{l}}{dt}}_{\text{velocity of electron}} = \rho_v dV\, \vec{E}\cdot \vec{u}_e$$

- Rearrange $\Rightarrow dP = \vec{E}\cdot \underbrace{\rho_v \vec{u}}\, dV$
- Recall $\vec{J} = \rho_v \vec{u}$

$\Rightarrow dP = \vec{E}\cdot\vec{J}\,dV$ and integrate, also $\vec{J} = \sigma\vec{E}$

$$\boxed{P = \int_V \vec{E}\cdot\vec{J}\,dV} \Rightarrow \boxed{P = \int_V \sigma E^2\, dV}$$

- this is known as Joule's Law
- Suppose a conductor has uniform cross-section area (S) and Length (L), then

$dV = dl\,ds$ and

$V = -\int \vec{E}\cdot d\vec{l} \Rightarrow V = EL \Rightarrow E = \frac{V}{L}$ [V/m]

$I = \int \vec{J}\cdot d\vec{s} \Rightarrow I = JS \Rightarrow J = \frac{I}{S}$ [A/m²]

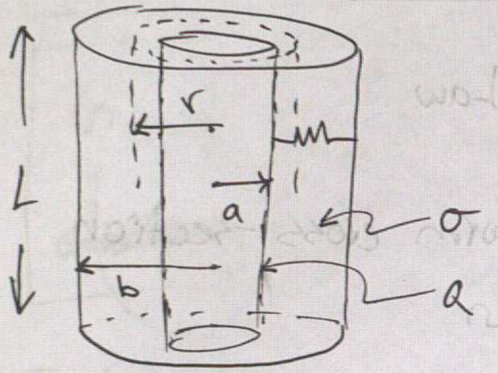
- ~~$P = \int \vec{E}\cdot\vec{J}\,dV$~~ $P = \int E\,dl \int J\,ds$

$= \frac{V}{L}\int dl\ \frac{I}{S}\int ds$

$\boxed{P = VI}$ ← more common form of Joule's Law in ckt theory

**Ex 1:** Coaxial resistor. Find Resistance (R) b/n inner ($r=a$) and outer ($r=b$) of two perfectly conducting coaxial cylinders of length $L$ filled with a material with conductivity $\sigma$ [S/m] where inner shell has a charge of $Q$ [C] on it.

Knowledge




$$R = \frac{-\int_c \vec{E}\cdot d\vec{l}}{\int_s \sigma \vec{E}\cdot d\vec{s}}$$

to find $\vec{E}$, use Gauss law

$$\epsilon_0 \oint_s \vec{E}\cdot d\vec{s} = Q_{enc} = Q$$

$$\epsilon_0 E\, 2\pi r L = Q$$

$$\vec{E} = \frac{Q}{2\pi\epsilon_0 L}\cdot\frac{1}{r}\,\hat{R} \text{ or } \hat{r}$$

$$V_{ab} = -\int_c \vec{E}\cdot d\vec{l} = V_a - V_b$$

$$= -\int_b^a \vec{E}\cdot d\vec{l}$$

$$= -\int_b^a \frac{Q}{2\pi\epsilon_0 L}\cdot\frac{1}{r}\,\hat{r}\cdot\hat{r}\,dr$$

$$V_{ab} = \frac{-Q}{2\pi\epsilon_0 L}\int_b^a \frac{dr}{r} = \frac{-Q}{2\pi\epsilon_0 L}\ln\left(\frac{a}{b}\right)$$

$$V_{ab} = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)$$

$$I = \int_s \sigma\frac{Q}{2\pi\epsilon_0 L}\cdot\frac{1}{r}\,\hat{r}\cdot r\,d\phi\,dz\,\hat{r} = \frac{\sigma Q}{2\pi\epsilon_0 L}\int_0^{2\pi} d\phi\int_0^{L} dz$$

$$I = \frac{\sigma Q}{\epsilon_0}\ [A]$$

$$\Rightarrow R = \frac{-\int \vec{E}\cdot d\vec{l}}{\int \sigma\vec{E}\cdot d\vec{s}} = \frac{\frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)}{\sigma Q/\epsilon_0} = \frac{1}{2\pi\sigma L}\ln\left(\frac{b}{a}\right)\ [\Omega]$$

**Ex 2:** Consider a 10 cm diameter cylinder with $\sigma = 10^{-3}$ [S/m] and $\vec{E} = 12 r\,\hat{z}$ [V/cm]. Find P in 1 [m] length of this cylinder

$$P = \int \vec{E}\cdot\vec{J}\,dV = \int \sigma E^2 dV \leftarrow \text{choose second}$$

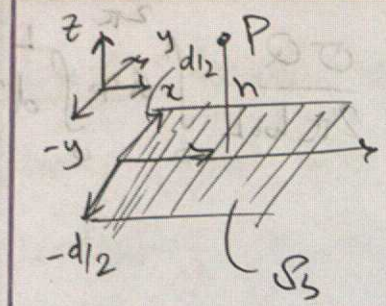
$$P = \sigma\int_0^{1m}\int_0^{2\pi}\int_0^{0.05\,m}(12\times10^2 r)^2\, r\,dr\,d\phi\,dz$$

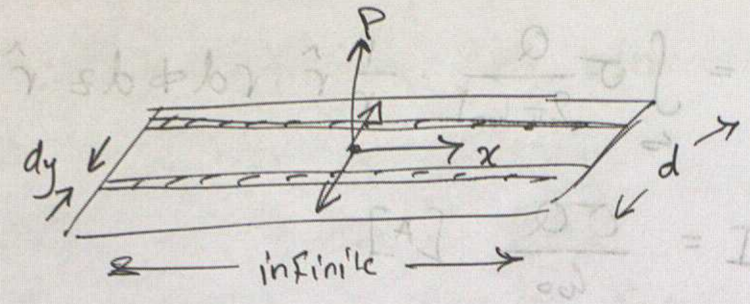
$$= (10^{-3})(144\times10^{4})\left(\frac{1}{4}(5\times10^{-2})^4\right)(2\pi)(1m)$$

$$P = 14.1\ mW$$

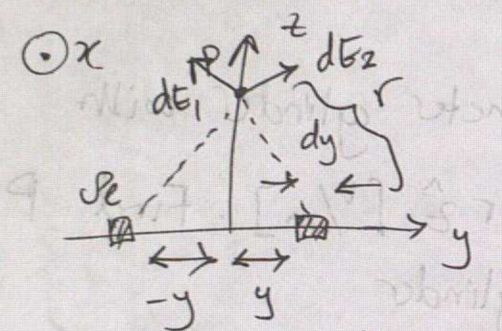
Tutorial 3

4.21




$\vec{E} = \hat{z}\,\dfrac{\rho_s}{2\epsilon_0}$ $\quad \rho_\ell = \rho_s\, dy$




$dE_1 = \hat{r}\,\dfrac{\rho_\ell}{2\pi\epsilon_0 r} = dE_2$

$dE = dE_1 + dE_2$

$= \hat{r}\,\dfrac{2\rho_\ell}{2\pi\epsilon_0 r}\cos\theta$

$E = \displaystyle\int_0^{d/2} dE = \int_0^{d/2} \dfrac{\rho_s dy}{\pi\epsilon_0 r}\cos\theta$ $\qquad = \hat{r}\,\dfrac{\rho_s dy}{\cancel{2}\pi\epsilon_0 r}\cos\theta$

$= \dfrac{\rho_s}{\pi\epsilon_0}\displaystyle\int_0^{d/2} \dfrac{\cos\theta}{r}\,dy$ $\qquad \Rightarrow y = h\tan\theta$

$\qquad dy = \dfrac{h}{\cos^2\theta}\,d\theta$

$= \dfrac{\rho_s}{\pi\epsilon_0}\displaystyle\int_0^{d/2} \dfrac{\cancel{\cos\theta}}{h/\cancel{\cos\theta}}\cdot\dfrac{h}{\cancel{\cos^2\theta}}\,d\theta$ $\qquad \Rightarrow r = \dfrac{h}{\cos\theta}$

$= \cancel{\dfrac{d\rho_s}{2\pi\epsilon_0}}\ \dfrac{\rho_s}{\pi\epsilon_0}\displaystyle\int_0^{\theta_0} \dfrac{h}{h}\,d\theta$

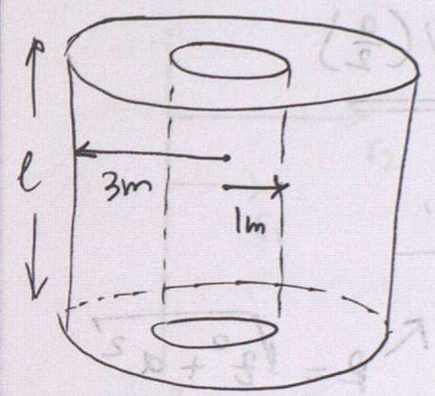
$= \dfrac{\rho_s}{\pi\epsilon_0}\left[\theta_0 - 0\right] \Rightarrow \underline{\vec{E} = \dfrac{\rho_s}{\pi\epsilon_0}\tan^{-1}\left(\dfrac{d}{2h}\right)\hat{z}}$

when considering $d \to \infty$, $\theta_0 = \tan^{-1}\left(\dfrac{d}{2h}\right) \to \dfrac{\pi}{2}$

$\Rightarrow \vec{E} = \dfrac{\rho_s}{\pi\epsilon_0}\cdot\dfrac{\pi}{2} = \underline{\dfrac{\rho_s}{2\epsilon_0}\,\hat{z}}$

4.27 Gauss's Law: $\displaystyle\oint_s \vec{E}\cdot d\vec{s} = \dfrac{Q_{enc}}{\epsilon_0}$ and $\vec{D} = \epsilon_0\vec{E}$

$\Rightarrow \displaystyle\oint_s \vec{D}\cdot d\vec{s} = Q_{enc}$




ⓐ $r < 1\,m$, $Q_{enc} = 0$

$\therefore \vec{D} = 0$

ⓑ $1 < r < 3\,m$

$\oint \vec{D}\cdot d\vec{s} = D(2\pi r\ell) = Q_{enc}$

$D(2\pi r\ell) = \rho_{v0}\pi\ell(r^2 - 1)$

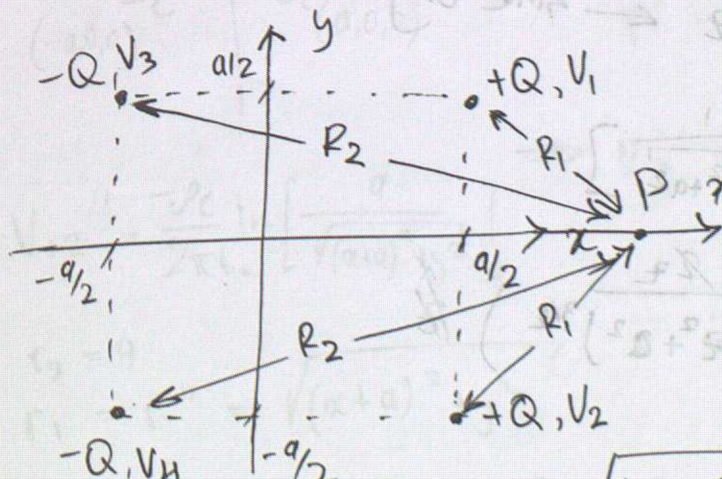
$\vec{D} = \dfrac{\rho_{v0}(r^2-1)}{2r}\,\hat{r}$

ⓒ $r > 3\,m$

$\oint D\cdot ds = D(2\pi r\ell)$

$D(2\cancel{\pi}r\cancel{\ell}) = (\cancel{\pi}3^2 - \cancel{\pi}1^2)\cancel{\ell}\rho_{v0}$ $\Rightarrow \vec{D} = \dfrac{8\rho_{v0}}{2r} = \dfrac{4\rho_{v0}}{r}\,\hat{r}$

4.30




$V = \dfrac{q}{4\pi\epsilon_0 r}$

$\Rightarrow V_1 = V_2 = \dfrac{Q}{4\pi\epsilon_0 R_1}$

$\Rightarrow V_3 = V_4 = \dfrac{-Q}{4\pi\epsilon_0 R_2}$

$R_1 = \sqrt{\left(x - \dfrac{a}{2}\right)^2 + \left(\dfrac{a}{2}\right)^2}$

$R_2 = \sqrt{\left(x + \dfrac{a}{2}\right)^2 + \left(\dfrac{a}{2}\right)^2}$

$$V = V_1 + V_2 + V_3 + V_4 = 2V_1 + 2V_3$$

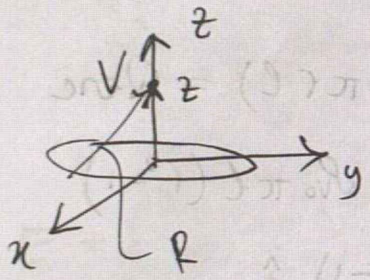
$$= \frac{Q}{2\pi\varepsilon_0 R_1} - \frac{Q}{2\pi\varepsilon_0 R_2} = \frac{Q}{2\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-a/2)^2 + a^2/4}} - \frac{1}{\sqrt{(x+a/2)^2 + a^2/4}}\right]$$

what $V$ @ $x = a/2$ ?

$$V = \frac{Q}{2\pi\varepsilon_0}\left[\frac{1}{\sqrt{\frac{a^2}{4}}} - \frac{1}{\sqrt{a^2 + \frac{a^2}{4}}}\right]$$

$$= \frac{Q}{2\pi\varepsilon_0 a}\left[2 - \frac{2}{\sqrt{5}}\right] = \frac{0.55Q}{2\pi\varepsilon_0 a} = V\left(\frac{a}{2}\right)$$

4.32:




$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{dQ'}{R} \qquad R = \sqrt{z^2 + a^2}$$

$$\partial Q = \rho_l \, dl$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi a} \frac{\rho_l \, dl}{\sqrt{z^2+a^2}} \Rightarrow V = \frac{\rho_l a}{2\varepsilon_0 \sqrt{z^2+a^2}}$$

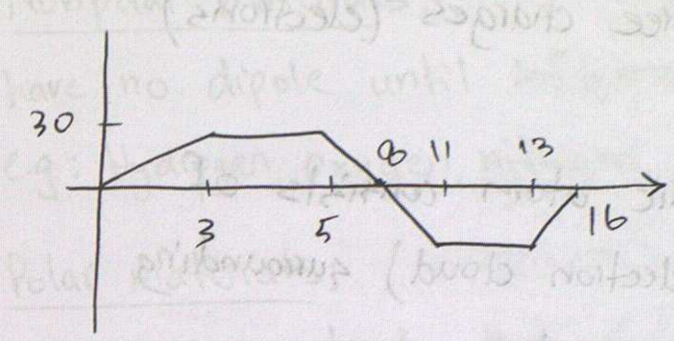
$$\vec{E} = -\nabla V = \frac{\partial}{\partial z} V \hat{z}$$ ← since only z-comp exists

$$= -\hat{z}\,\frac{\rho_l a}{2\varepsilon_0}\,\frac{\partial}{\partial z}\left[\frac{1}{\sqrt{z^2+a^2}}\right]$$

$$= -\hat{z}\,\frac{\rho_l a}{2\varepsilon_0}\left(-\frac{1}{\cancel{2}}\right)\left(\frac{\cancel{2}z}{(z^2+a^2)^{3/2}}\right)$$

$$\vec{E} = \frac{\rho a z}{2\varepsilon_0 (z^2+a^2)^{3/2}}\hat{z}$$

4.36: $\vec{E} = -\nabla V = -\hat{x}\frac{\partial V}{\partial x}$ ← since graphs show $V$ as a function of $x$ only

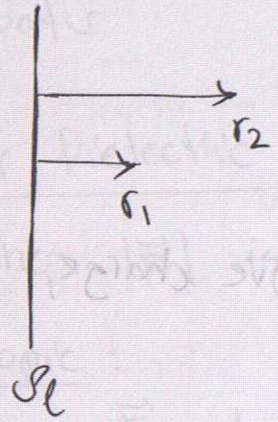



b) $V(x) = 4\sin\left(\frac{\pi}{3}x\right)$

c) $V(x) = -4\sin\left(\frac{\pi}{6}x\right)$

$$\vec{E} = -x\frac{\partial}{\partial x}V$$

4.33:



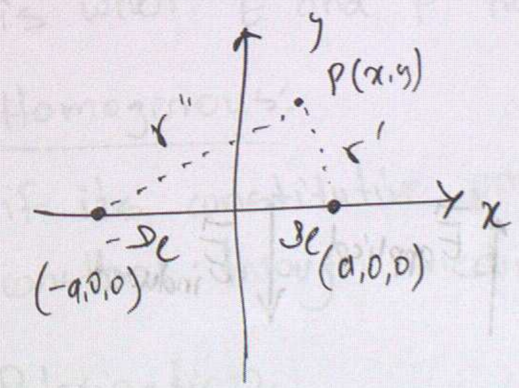

$V_{12} = -\int_l \vec{E}\cdot\vec{dl}$ and $\vec{E} = -\nabla V$

$$|\vec{E}| = \frac{\rho_l}{2\pi\varepsilon_0 r}$$

$$V_{12} = -\int_{r_2}^{r_1} \frac{\rho_l}{2\pi\varepsilon_0 r}\, dl$$

$$= \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{r_2}{r_1}\right)$$

4.37:




$$V_{12}' = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{r_2}{r_1}\right), \quad r_2 = a, \; r_1 = r'$$

$$r' = \sqrt{(x-a)^2 + y^2}$$

$$\Rightarrow V_{12}' = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left[\frac{a}{\sqrt{(x-a)^2+y^2}}\right]$$

$$V_{12}'' = \frac{-\rho_l}{2\pi\varepsilon_0}\ln\left[\frac{a}{\sqrt{(x+a)^2+y^2}}\right]$$

$r_2 = a$

$r_1 = r'' = \sqrt{(x+a)^2 + y^2}$

$$\therefore V_{12} = V_{12}' + V_{12}''$$

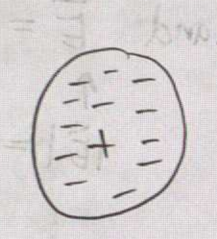
$$= \frac{\rho_l}{4\pi\varepsilon_0}\ln\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$$

$$V_{12} = \frac{\rho_l}{2\pi\varepsilon_0}\left\{\ln\left[\frac{a}{\sqrt{(x-a)^2+y^2}}\right] - \ln\left[\frac{a}{\sqrt{(x+a)^2+y^2}}\right]\right\}$$

Week 4: Lecture 1 Jan 28, 2025

## 4.7 Dielectrics

- have very few, if any, free charges (electrons) available for conduction
- at atomic level, a dielectric atom consists of a negative charge $-q$ (electron cloud) surrounding a positive charge $+q$ (nucleus)



$\vec{E}_{ext} = 0$

- since equal amount of (+)ve and (-)ve charge, then atom is electrically neutral.

→ Applying $\vec{E}$

$\uparrow \vec{E}_{applied} \equiv$ $\oplus^{+q}$ $\updownarrow d$ $\ominus -q$ $\uparrow d$ $\uparrow \vec{E}_{applied}$ $\downarrow \vec{E}_{induced}$

- causes the electron cloud to shift such that dipoles are formed that are alligned in the direction of $\vec{E}_{applied}$
- in fact, the presense of $\vec{E}_{applied}$ induces a local electric dipole moment ($\vec{P}$) and the dielectric is said to be polarized. $\vec{P} = q\vec{d}$

Two types: nonpolar materials, polar materials

1) Nonpolar materials:
- have no dipole until influenced by $\vec{E}$, then become polarized
- e.g: Hydrogen, oxygen, nitrogen, rare gases

2) Polar materials:
- have built-in dipole that are randomly oriented
- e.g: water

→ Linear Dielectric Medium
- is when $|\vec{P}|$ is proportional to $|\vec{E}|$

→ Isotropic:
- is when $\vec{E}$ and $\vec{P}$ have the same direction

→ Homogenous:
- if its constitutive parameters ($\epsilon$, $\mu$, $\sigma$) are constant through medium

Polarization
- shows the density of electric dipole moment per unit volume and is given by

$\boxed{\vec{P} = \chi_e \epsilon_0 \vec{E}}$ where $\vec{P}$ is proportional to $\vec{E}$

"chi"

susceptibility, a scalar if isotropic (for linear, isotropic, homogenous)

- $\chi_e$ shows how receptive a material is to polarization.
- note that presence of dipoles change electric flux density ($\vec{D}$) and extends D to

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \qquad \epsilon_0 \to \text{vacuum}$$

$$\vec{D} = \epsilon_0 \vec{E} + \chi_e \epsilon_0 \vec{E} \qquad \vec{D} = \epsilon_r \epsilon_0 \vec{E} \to \text{other stuff}$$

$$\underbrace{\vec{D} = (1+\chi_e)}_{\epsilon_r \text{ epsilon r (relative permitivity) (dielectric constant)}} \epsilon_0 \vec{E} \qquad \text{glass: } \epsilon_r \ 4-10$$

Q: $\vec{P}$ is electric dipole moment and also polariza?

a convenient to show $\epsilon$ relative to free space $\epsilon_0$

- $\epsilon = \epsilon_0 \epsilon_r \Rightarrow \epsilon_r = \frac{\epsilon}{\epsilon_0}$ get from tables

Dielectric Breakdown

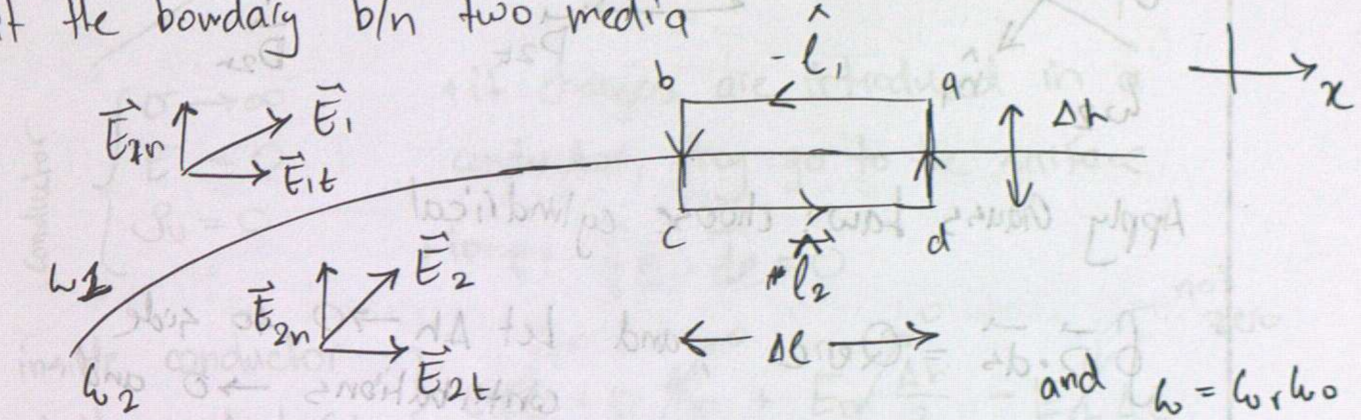
- is when dielectric becomes conducting
- the minimum $\vec{E}$ at which dielectric breakdown occurs is called dielectric strength ($E_{ds}$)
- occurs when electrons detatch from molecules and accelerate in the material as conduction current, causing sparks and damaging the dielectric.

Ex. 4-12

| material | air | glass | Quartz |
|---|---|---|---|
| $\epsilon_r$ | 1.0006 ≃ 1 | 4.5 - 10 | 3.5 - 8 |
| $E_{ds}$ [MV/m] | ≃ 3 | ≃ 25 - 40 | ≃ 30 |

☆ 4.8 Electric Boundary Conditions

- $\vec{E}$ may be continuous inside media, but can be discontinuous at the boundary
- provide information on how $\vec{E}$ ($E_t$ and $E_n$) behave at the boundary b/n two media




and $\epsilon = \epsilon_r \epsilon_0$

so $\epsilon_1 = \epsilon_{r1}\epsilon_0$

$\epsilon_2 = \epsilon_{r2}\epsilon_0$

- Find $\vec{E}$ at boundary $\to$ path a b c d

knowledge: $\oint \vec{E} \cdot d\vec{l} = 0$ and let $\Delta h \Rightarrow 0$

$$\Rightarrow \int_a^b \vec{E}_1 \cdot \hat{l}_1 dl + \int_c^d \vec{E}_2 \cdot \hat{l}_2 dl = 0 \quad \text{and} \quad \hat{l}_1 = -\hat{l}_2$$

$$\vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1n}$$

$$\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n}$$

$$\Rightarrow \left[(\vec{E}_1 \cdot \hat{l}_1) + (\vec{E}_2 \cdot (-\hat{l}_1))\right] \Delta l = 0$$

$$(\vec{E}_1 - \vec{E}_2) \cdot \hat{l}_1 = 0 \leftarrow \text{for ab}$$

$$\Rightarrow (E_{1t} - E_{2t}) \underbrace{\hat{x} \cdot \hat{x}}_{=1} + (E_{1n} - E_{2n}) \underbrace{\hat{z} \cdot \hat{x}}_{=0} = 0$$

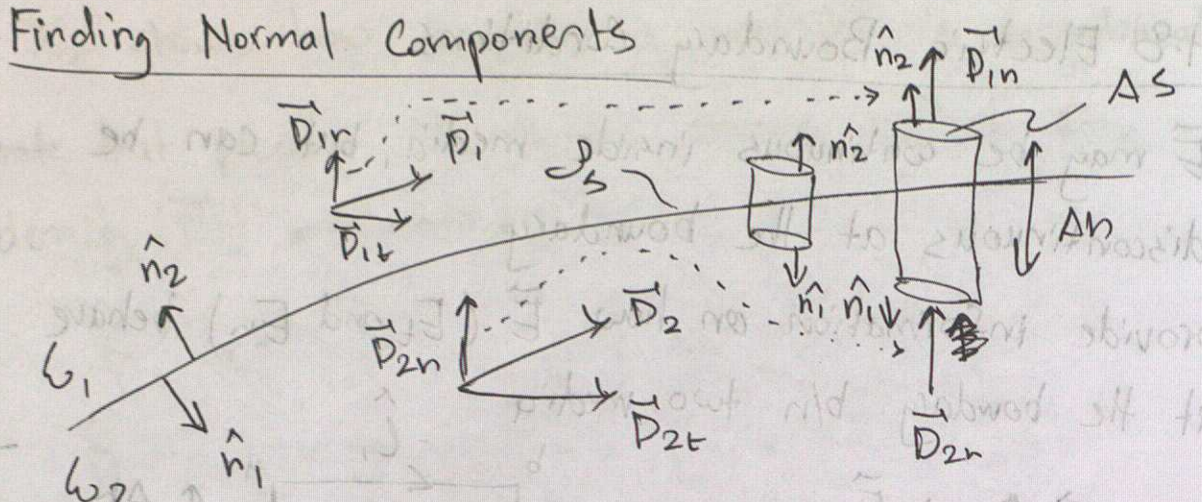
$$\boxed{E_{1t} = E_{2t}} \qquad \text{Find } \vec{D} \begin{cases} \vec{D}_{1t} = \epsilon_1 \vec{E}_{1t} \\ \vec{D}_{2t} = \epsilon_2 \vec{E}_{2t} \end{cases}$$

divide the 2

$$\frac{\vec{D}_{1t}}{\vec{D}_{2t}} = \frac{\epsilon_1 \cancel{\vec{E}_{1t}}}{\epsilon_2 \cancel{\vec{E}_{2t}}}$$

$$\boxed{\frac{\vec{D}_{1t}}{\epsilon_1} = \frac{\vec{D}_{2t}}{\epsilon_2}}$$

## Finding Normal Components




Apply Gauss Law: choose cylindrical

$\oint \vec{D}\cdot d\vec{s} = Q_{enc}$, and Let $\Delta h \to 0$ so side contributions $\to 0$ and only top/bott remains

and $Q_{enc} = \rho_s \Delta s$

$$\Rightarrow \oint \vec{D}\cdot d\vec{s} = \int_{top} \vec{D}_1\cdot\hat{n}_1\, ds + \int_{Bott} \vec{D}_2\cdot\hat{n}_2\, ds = Q_{enc}$$

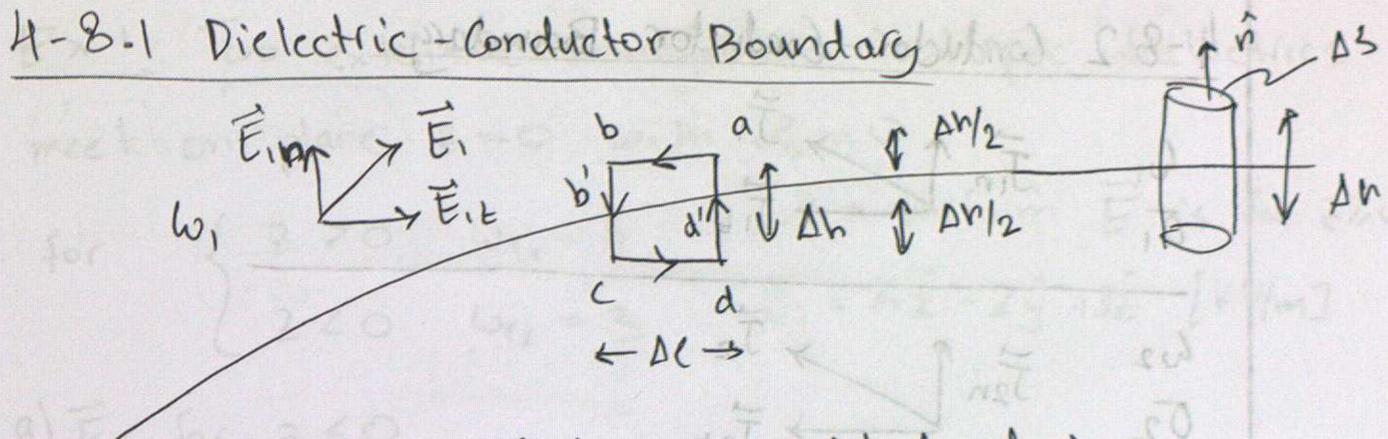
and $\hat{n}_1 = -\hat{n}_2$

$$(D_{1n} \underbrace{\hat{z}\cdot\hat{z}}_{=1} + D_{2n} \underbrace{\hat{z}\cdot(-\hat{z})}_{=-1})\cancel{\Delta s} = \rho_s \cancel{\Delta s}$$

$$\boxed{D_{1n} - D_{2n} = \rho_s} \quad \boxed{(\vec{D}_1 - \vec{D}_2)\cdot\hat{n} = \rho_s}$$

$$\boxed{\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s}$$

## 4-8.1 Dielectric - Conductor Boundary




Conductor $\begin{cases} \sigma \to \infty \\ E = 0 \\ \rho_v = 0 \end{cases}$

• if charges are introduced in a conductor, they go to the surface

• loop: $\oint \vec{E}\cdot d\vec{l} = 0$

inside conductor: / outside conductor:

$$\underbrace{0\,\Delta l}_{c\to d} + \underbrace{0\,\tfrac{\Delta h}{2}}_{d\to d'} + \underbrace{E_{1n}\tfrac{\Delta h}{2}}_{d'\to a} - \underbrace{E_{1t}\Delta l}_{a\to b}$$

(not zero)

$$\Rightarrow -E_{1t}\Delta l = 0 \qquad \underbrace{-E_{1n}\tfrac{\Delta h}{2}}_{b\to b'} + \underbrace{0\,\tfrac{\Delta h}{2}}_{b'\to c}$$

$$\boxed{\therefore E_{1t} = 0}$$

Apply Gauss Law: $\oint \vec{D}\cdot d\vec{s} = Q_{enc}$ and let $\Delta h \to 0$

inside conductor: $\vec{D} = \epsilon\vec{E} = 0$

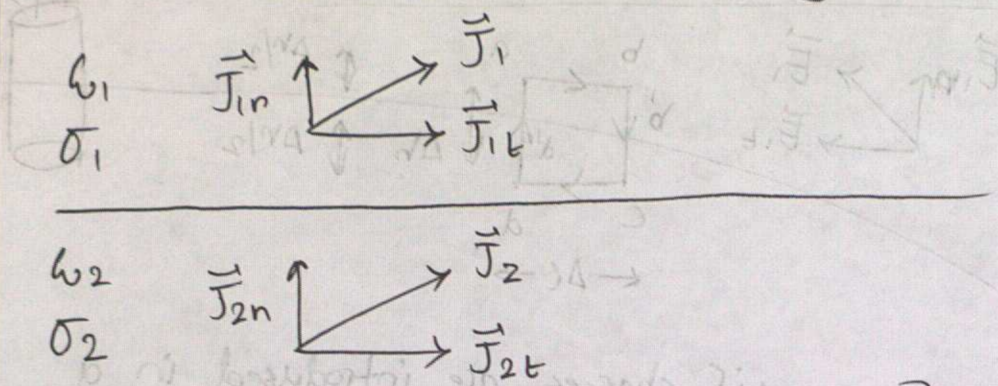
inside dielectric: $\cancel{\Delta s}\,\vec{D}_{1n}\cdot\hat{n} = \rho_s \cancel{\Delta s}$

Summary: $[C/m^2] \rightarrow \boxed{D_{1n} = \rho_s} \rightarrow [C/m^2]$

note that perfect conductor, $E=0$, $D=0$

then $\begin{cases} E_{1t} = D_{1t} = 0 \\ D_{1n} = \epsilon_1 E_{1n} = \rho_s \end{cases}$ (Table 4-3, pg 202)

## 4-8.2 Conductor-Conductor Boundary




Already know $\underline{\vec{E}_{1t} = \vec{E}_{2t}}$ (i) and $\underline{\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s}$ (ii)

Since conductor, then applying $\vec{E}$, a current will flow

Recall $\vec{J} = \sigma \vec{E} \Rightarrow \vec{E} = \frac{\vec{J}}{\sigma}$

$\left\{ \begin{array}{l} \vec{J}_{1t} = \sigma_1 \vec{E}_{1t} \\ \vec{J}_{2t} = \sigma_2 \vec{E}_{2t} \end{array} \right\}$ divide $\Rightarrow \frac{\vec{J}_{1t}}{\vec{J}_{2t}} = \frac{\sigma_1 \cancel{\vec{E}_{1t}}}{\sigma_2 \cancel{\vec{E}_{2t}}}$ since same $\vec{E}$ applied

$\therefore \boxed{\frac{\vec{J}_{1t}}{\sigma_1} = \frac{\vec{J}_{2t}}{\sigma_2}}$

$\left\{ \begin{array}{l} \vec{J}_{1n} = \sigma_1 \vec{E}_{1n} \\ \vec{J}_{2n} = \sigma_2 \vec{E}_{2n} \end{array} \right. \Rightarrow \begin{array}{l} \vec{E}_{1n} = \frac{\vec{J}_{1n}}{\sigma_1} \\ \vec{E}_{2n} = \frac{\vec{J}_{2n}}{\sigma_2} \end{array}$ plug in eq (ii)

$\therefore \boxed{\epsilon_1 \frac{\vec{J}_{1n}}{\sigma_1} - \epsilon_2 \frac{\vec{J}_{2n}}{\sigma_2} = \rho_s}$ if $J_{1n} \neq J_{2n}$

$\rho_s$ bitws $\neq \rho_s$ leaves conclude $\vec{J}_{1n} = \vec{J}_{2n}$

$\boxed{J_{1n} \left( \frac{\epsilon_1}{\sigma_1} - \frac{\epsilon_2}{\sigma_2} \right) = \rho_s}$

Week 4: Lecture 3

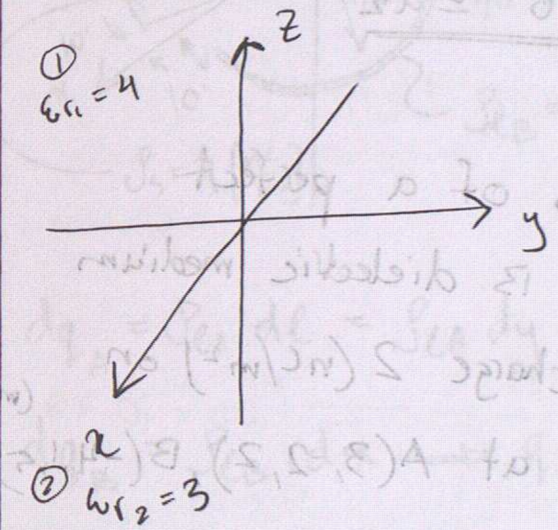
Ex 1: Two extensive homogenous isotropic dielectrics meet on plane $z=0$ with $\rho_s = 0$

for $\left\{ \begin{array}{ll} z > 0 & \epsilon_{r_1} = 4 \\ z < 0 & \epsilon_{r_2} = 3 \end{array} \right.$ and a uniform $\vec{E}$ is for $z \geq 0$

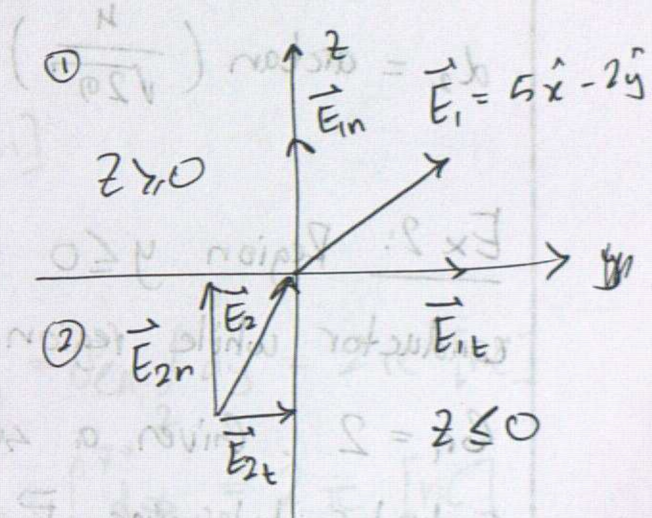
$\vec{E}_1 = 5\hat{x} - 2\hat{y} + 3\hat{z}$ [kV/m]

a) $\vec{E}_2$ for $z \leq 0$

b) The angles $E_1$ and $E_2$ make with interface




$\Rightarrow$




$\vec{E}_1 = 5\hat{x} - 2\hat{y} + 3\hat{z} \left\{ \begin{array}{l} \vec{E}_{1t} = 5\hat{x} - 2\hat{y} \\ \vec{E}_{1n} = 3\hat{z} \end{array} \right.$

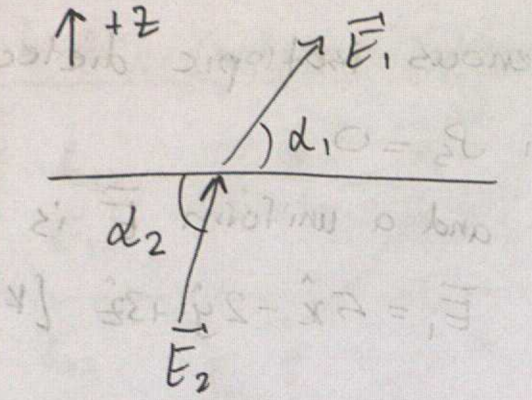
$\boxed{\vec{E}_{1t} = \vec{E}_{2t}} \Rightarrow \vec{E}_{2t} = 5\hat{x} - 2\hat{y}$ //

Knowing $D_{1n} - D_{2n} = \rho_s \Rightarrow \rho_s = 0$, $D_{1n} = D_{2n}$

and $\epsilon_1 E_{1n} = \epsilon_2 E_{2n} \Rightarrow \vec{E}_{2n} = \frac{\epsilon_1}{\epsilon_2} E_{1n} = \frac{4}{3}(3\hat{z}) = 4\hat{z}$ //

$\therefore \vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n} = \underline{5\hat{x} - 2\hat{y} + 4\hat{z}}$

b)




$|\vec{E}_{1n}| = 3$

$|\vec{E}_{1t}| = \sqrt{5^2+2^2} = \sqrt{29}$

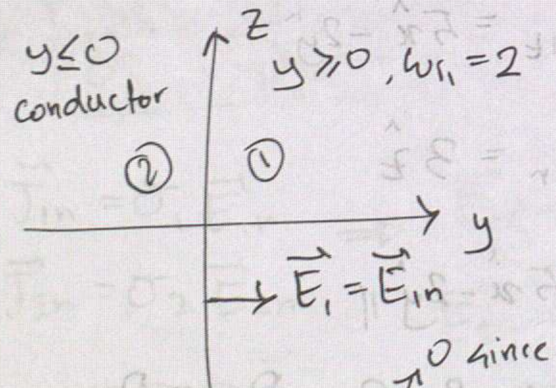
$\alpha_1 = \arctan\left(\frac{3}{\sqrt{29}}\right)$

$\underline{\alpha_1 \simeq 29.1^\circ}$

$|\vec{E}_{2t}| = |\vec{E}_{1t}| = \sqrt{29}$ , $|\vec{E}_{2n}| = 4$

$\alpha_2 = \arctan\left(\frac{4}{\sqrt{29}}\right) = \underline{36.6^\circ \simeq \alpha_2}$

Ex 2: Region $y \leq 0$ consists of a perfect conductor while region $y \geq 0$ is dielectric medium $\varepsilon_{r_1} = 2$. Given a surface charge $2\ (nC/m^2)$ on conductor, determine $\vec{E}$ and $\vec{D}$ at $A(3,-2,2)$, $B(-4,1,5)$ (m)




a) Find $\vec{E} = ?$ in conductor (point A)

$\vec{E} = 0 \quad \vec{D} = 0$

b) Find $\vec{E}$ inside dielectric (Point B)

Knowing $D_{1n} - \underbrace{D_{2n}}_{0 \text{ since } \vec{E}=0} = \rho_s \Rightarrow D_{1n} = \rho_s = 2\ [nC/m^2]$

$\vec{E}_{1t} = \vec{E}_{2t} = 0$

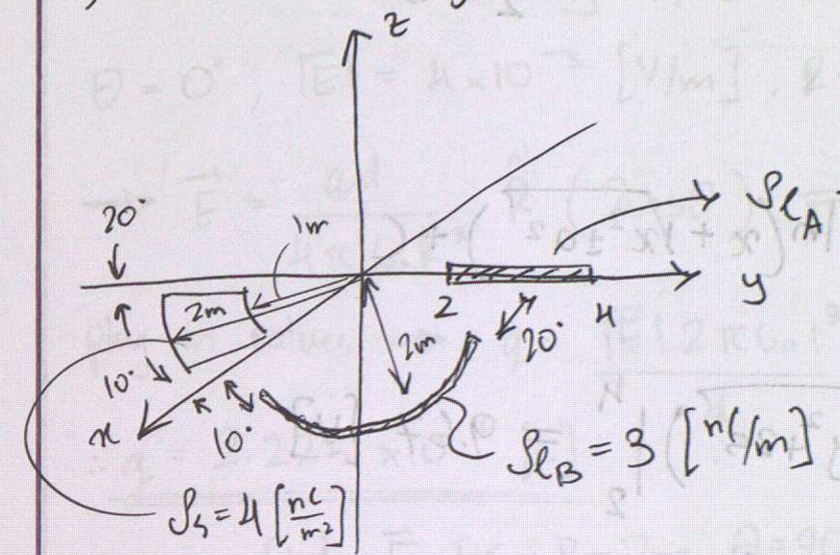
$E_{1n} = \frac{D_{1n}}{\varepsilon_{r_1}} = \frac{2\ [nC/m^2]}{2\varepsilon_0} = 36\pi$

$\varepsilon_0 = 8.854\times10^{-12}$

$\underline{\therefore \vec{E}_1 = 36\pi\hat{y}\ [V/m]}$

Ex 3: Consider 3 separate charge distributions in free space.

a) find total charge for each distribution




$dq_A = \rho_{lA}\,dl = \rho_{lA}\,dy \Rightarrow q_A = \rho_{lA}\int_2^4 dy = 2\pi\ [nC]$

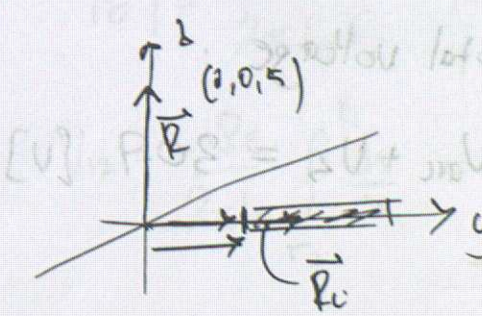
$dq_B = \rho_{lB}\,r\,d\phi \Rightarrow q_B = \rho_{lB}\,r\int_{10^\circ}^{70^\circ} d\phi = 2\pi\ [nC]$

$dq_s = \rho_s\,ds = \rho_s\,r\,d\phi\,dr \Rightarrow q_s = \rho_s\int_1^2 r\,dr\int_0^{\pi/3} d\phi = 2\pi\ [nC]$

b) Find potential at $P(0,0,5)$ [m] caused by each of these three distributions acting alone.

$dV = \frac{dq}{4\pi\varepsilon_0|\vec{R}-\vec{R}_i|}$

$|\vec{R}-\vec{R}_i| = \sqrt{5^2+y^2} = \sqrt{y^2+25}$




$\vec{R} = 5\hat{z}$

$\vec{R}_i = y\hat{y}$

→ straight line:

$$dV = \frac{\rho_{\ell A}\, dy}{4\pi\varepsilon_0 \sqrt{y^2+25}} \Rightarrow V = \frac{\rho_{\ell A}}{4\pi\varepsilon_0}\int_2^4 \frac{dy}{\sqrt{y^2+25}}$$

integral table:

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left(x + \sqrt{x^2 \pm a^2}\right) + C$$

$$V = \frac{\pi}{4\pi\varepsilon_0}\ln\left(y + \sqrt{y^2+25}\right)\Big|_2^4 = 9.67\ [V]$$

→ arc: $|\vec{R} - \vec{R}_i| = |5\hat{z} - 2\hat{r}|$

$$V_{arc} = \frac{\rho_{\ell B}}{4\pi\varepsilon_0}\int_{10^\circ}^{70^\circ} \frac{r\, d\phi}{\sqrt{5^2 + r^2}} = \frac{3(2)}{4\pi\varepsilon_0}\frac{1}{\sqrt{29}}\int_0^{\pi/3} d\phi = 10.49\ [V]$$

→ surface: $|\vec{R} - \vec{R}_i| = |5\hat{z} - r\hat{r}|$, $dq_s = \rho_s\, r\, dr\, d\phi$

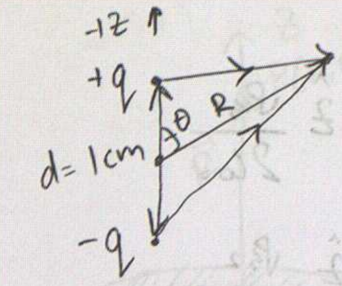
$$V = \frac{\rho_s}{4\pi\varepsilon_0}\int_{10^\circ}^{70^\circ}\int_1^2 \frac{r\, dr\, d\phi}{\sqrt{r^2+25}} = \frac{1}{\pi\varepsilon_0}\left(\frac{\pi}{3}\right)\int \ldots = 10.77\ [V]$$

c) Find total voltage:

$$V = V_\ell + V_{arc} + V_s = 30.9\ [V]$$

Week 5 Tutorial 4 Feb 3, 2025

4.35:




$$\vec{E} = \frac{qd}{4\pi\varepsilon_0 R^3}\left[\hat{R}\, 2\cos\theta + \hat{\theta}\sin\theta\right] \text{ —— eq. 4.56}$$

• for any $\phi$, E is same (in this configuration)

$\theta = 0^\circ$, $|\vec{E}| = 4\times10^{-3}$ [V/m], $R = 1$ m

$$\Rightarrow \vec{E} = \frac{qd}{4\pi\varepsilon_0 R^3}\hat{R}\,(2\cos\theta) \Rightarrow |\vec{E}| = \frac{qd}{\cancel{4}_2\pi\varepsilon_0 R^3}(\cancel{2}\cos\theta)$$

plug in values $\Rightarrow q = \frac{|\vec{E}|\, 2\pi\varepsilon_0\, 1^3}{d} = \frac{(4\times10^{-3})(2\pi\varepsilon_0)}{0.01}$

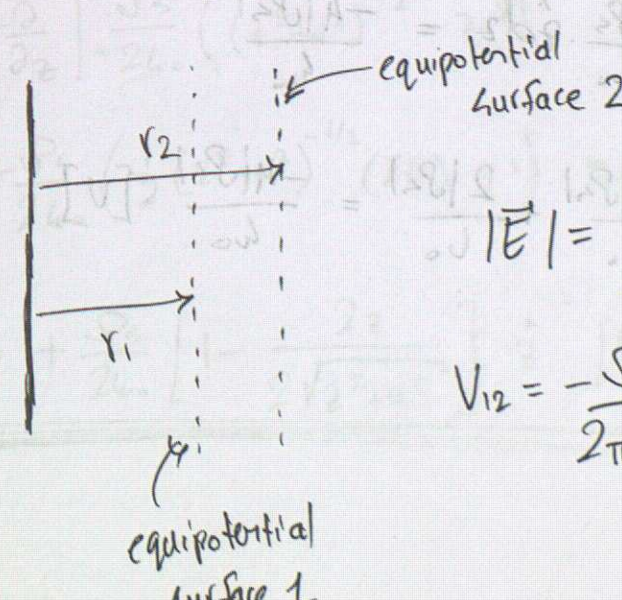
$\therefore q = 2.225\times10^{-11}$ [C]

⇒ now find $\vec{E}$ for $R = 2m$, $\theta = 90^\circ$

$$|\vec{E}| \Rightarrow \frac{qd}{4\pi\varepsilon_0 R^3}\left[\hat{R}\, \underbrace{2\cos\theta}_{\cos 90 = 0} + \hat{\theta}\sin\theta\right] = \vec{E}$$

$$|\vec{E}| = \frac{(2.225\times10^{-11})(0.01)}{4\pi(8.854\times10^{-12})(2)^3} = 2.5\times10^{-4}\ [V/m] = |\vec{E}|$$

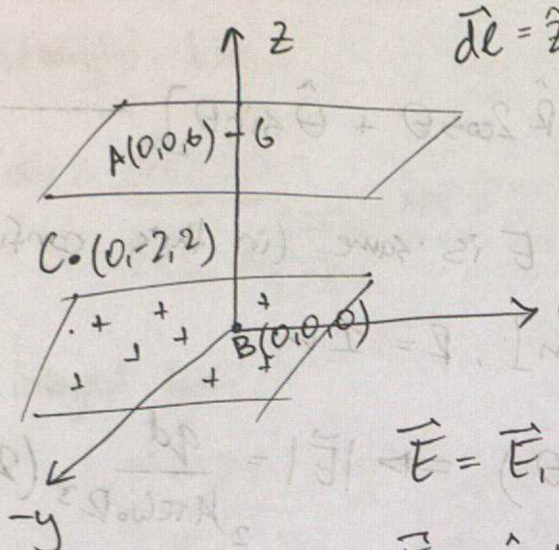
4.33 (review)




$$|\vec{E}| = \frac{\rho_\ell}{2\pi\varepsilon_0 r}, \quad V_{12} = -\int_{r_2}^{r_1} \vec{E}\cdot d\vec{l}$$

$$V_{12} = -\frac{\rho_\ell}{2\pi\varepsilon_0}\int_{r_2}^{r_1}\frac{dx}{x} = \frac{\rho_\ell}{2\pi\varepsilon_0}\ln\left[\frac{r_2}{r_1}\right]$$

4.40




$\vec{dl} = \hat{z}\,dz$

$\vec{E}_1 = \hat{z}\frac{\rho_{s1}}{2\epsilon_0}$

$\vec{E}_2 = \hat{z}\frac{\rho_{s2}}{2\epsilon_0}$

$\vec{E} = \vec{E}_1+\vec{E}_2 = \hat{z}\frac{1}{2\epsilon_0}(\rho_{s1}+\rho_{s2})$

$\rho_s$

$\vec{E} = \hat{z}\frac{\rho_s}{\epsilon_0}$

$V_{AB} = -\int_B^A \vec{E}\cdot\vec{dl} = V_A - V_B$

$= -\int_0^6 \hat{z}\frac{\rho_s}{\epsilon_0}\cdot\hat{z}\,dz = -\frac{\rho_s}{\epsilon_0}\int_0^6 dz = -\frac{6|\rho_s|}{\epsilon_0}$

$V_{BC} = -\int_C^B \vec{E}\cdot\vec{dl} = -\int_2^0 \hat{z}\frac{\rho_s}{\epsilon_0}\cdot\hat{z}\,dz = +\frac{2|\rho_s|}{\epsilon_0}$

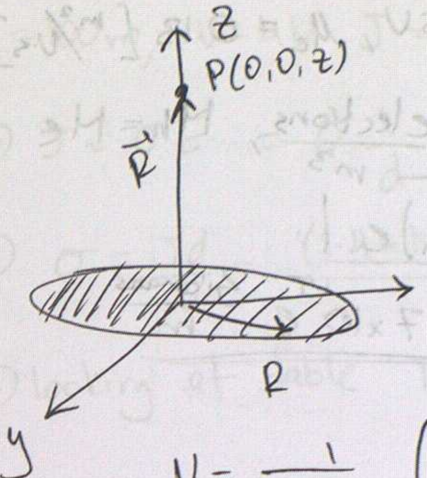
$V_{AC} = -\int_C^A \vec{E}\cdot\vec{dl} = -\int_2^6 \hat{z}\frac{\rho_s}{\epsilon_0}\cdot\hat{z}\,dz = \frac{-4|\rho_s|}{\epsilon_0}$

or

$V_{AC} = V_{AB}+V_{BC} = \frac{-6|\rho_s|}{\epsilon_0}+\frac{2|\rho_s|}{\epsilon_0} = \frac{-4|\rho_s|}{\epsilon_0}$ [V]

4.31

a)




$ds = r\,dr\,d\phi$

$dq = \rho_s\,ds = \rho_s\,r\,dr\,d\phi$

$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{R'}$ ← $\vec{R}'' = \vec{R}-\vec{R}'$

$R = \sqrt{r^2+z^2}$

$V = \frac{1}{4\pi\epsilon_0}\int_0^a\int_0^{2\pi}\frac{\rho_s r}{\sqrt{r^2+z^2}}\,\cancel{dr\,d\phi}\,d\phi\,dr$

$V = \frac{\rho_s}{2\epsilon_0}\int_0^a\frac{r}{\sqrt{r^2+z^2}}dr \Longrightarrow u = r^2+z^2,\ du = 2r\,dr \rightarrow dr = \frac{1}{2r}du$

$= \frac{\rho_s}{2\epsilon_0}\int_{z^2}^{a^2+z^2}\frac{r}{\sqrt{u}}\cdot\frac{1}{2r}du = \frac{\rho_s}{4\epsilon_0}\int_{z^2}^{a^2+z^2}(u)^{-\frac{1}{2}}du \Rightarrow$ integral

$V = \frac{\rho_s}{2\epsilon_0}\left[\sqrt{z^2+a^2}-z\right]$

b) $\vec{E} = -\nabla V = -\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)V = \frac{\partial V}{\partial z}\hat{z}$

$= -\frac{\partial}{\partial z}\left[\frac{\rho_s}{2\epsilon_0}\left((z^2+a^2)^{1/2}-z\right)\right]$

$= -\frac{\rho_s}{2\epsilon_0}\left(\frac{1}{2}(z^2+a^2)^{-1/2}(2z)-1\right) = -\frac{\rho_s}{2\epsilon_0}\left[\frac{2z}{2\sqrt{z^2+a^2}}-1\right] = E$

$\vec{E} = +\frac{\rho_s}{2\epsilon_0}\left[1-\frac{2z}{2\sqrt{z^2+a^2}}\right]\hat{z}$ [V/m]

4.41 a) radius = 4mm, length = 8cm, $V_{app} = 5V$, $\mu_e = 0.13\ [m^2/V\cdot s]$

$\mu_h = 0.05\ [m^2/V\cdot s]$, $N_e = 1.5\times10^{16}\ \frac{\text{electrons}}{m^3}$, $N_h = N_e$

eq. 4.65 — $\boxed{\sigma = (N_e \mu_e + N_h \mu_h)e}$

$\sigma = (1.5\times10^{16})(0.13+0.05)e = \underline{2.7\times10^{15}\,e\ \frac{\text{Siemens}}{m}}$

$\underline{\sigma = 4.32\times10^{-4}\ S/m}$

b) $I = JA = \sigma EA$, but $Ed = V \Rightarrow E = \frac{V}{d}$

$I = \frac{\sigma V A}{d} = \frac{(4.32\times10^{-4})(5)(\pi(4\times10^{-3})^2)}{(\cancel{0.008})(0.08)} = \underline{1.36\times10^{-6}\ A}$

c) $u_e = -\mu_e E = -\frac{\mu_e V}{d} = -\frac{(0.13)(5)}{0.08} = \underline{-8.125\ m/s\ \hat{E}}$

$u_h = +\mu_h E = \frac{-(0.05)(5)}{0.08} = \underline{3.125\ m/s\ \hat{E}}$

d) $R = \frac{V}{I} = \frac{5}{1.36\ \mu A} = \underline{3.68\ M\Omega}$

e) $P = IV = (5)(1.36\ \mu A) = \underline{6.8\ \mu W}$

4.42 repeat Q4.41 for Germanium (Ge)

a) 2.3 S/m

b) $I = 7.225$ mA

c) $u_e = -25\ m/s\ \hat{E}$ $u_h = 12.5\ m/s\ \hat{E}$

d) $R = 0.69\ k\Omega$

e) $P = 36.125$ mW

4.43 $l = 100m$, $V = 4V$, $J = 1.4\times10^6\ [A/m^2]$ — what material?

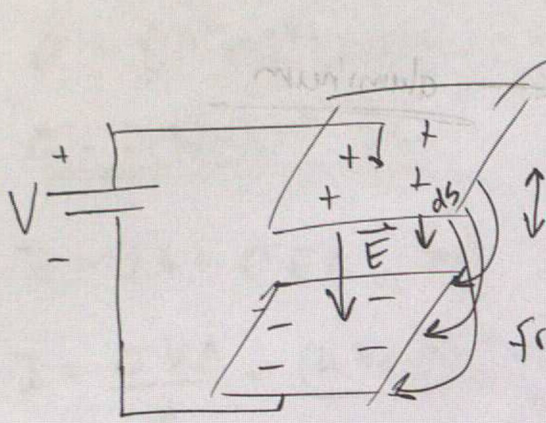
① $J = \sigma E = \sigma \frac{V}{d}$

② $\sigma = \frac{Jd}{V} = \frac{(1.4\times10^6)(100)}{4} = 35\ MS/m$ ← $\frac{\text{megasiemens}}{m}$

③ looking at table T-B2 ← $\underline{\text{aluminum}}$

## 4.9 Capacitance

- two conductors (any shape/geometry) connected to voltage (V)





→ Define Capacitance:

$$\boxed{C \triangleq \frac{Q}{V} \quad [C/V] \text{ or } [F]}$$

conductor - dielectric:

$$\begin{cases} E_t = 0 \\ D_n = \rho_s \end{cases}$$

- Tangential $\vec{E}$ ($E_t$) vanishes at the surface and $\vec{E}$ is always $\perp$ to the surface

$$\Rightarrow \vec{E}\cdot\hat{n} = E_n = \frac{\rho_s}{\epsilon} \rightarrow \rho_s = \epsilon \vec{E}_n \cdot \hat{n}$$

$$Q = \int_s \rho_s ds$$

$$= \int_s \epsilon \vec{E}\cdot\hat{n} ds = \int_s \epsilon \vec{E}\cdot d\vec{s}$$

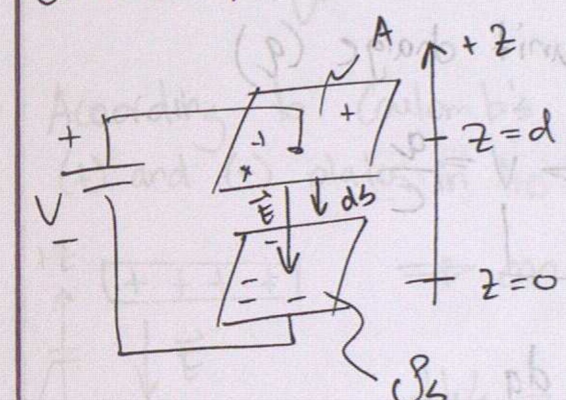
Knowing that $V = -\int_l \vec{E}\cdot dl \Rightarrow \boxed{C = \frac{\int_s \epsilon \vec{E}\cdot d\vec{s}}{-\int_l \vec{E}\cdot d\vec{l}} \quad [F]}$

- Recall: $\cancel{R}$ $\cancel{\frac{-\int_l \vec{E}\cdot d\vec{l}}{\int_s E}}$ $\quad R = \frac{-\int_l \vec{E}\cdot d\vec{l}}{\int_s \sigma \vec{E}\cdot d\vec{s}}$

$$\Rightarrow RC = \frac{-\cancel{\int_l \vec{E}\cdot d\vec{l}}}{\int_s \sigma \vec{E}\cdot d\vec{s}} \cdot \frac{\int_s \epsilon \vec{E}\cdot d\vec{s}}{-\cancel{\int_l \vec{E}\cdot d\vec{l}}} \Rightarrow \boxed{RC = \frac{\epsilon}{\sigma} = \tau \ [s]}$$

(time constant)

- Consider Parallel Plates: find C from geometry




Knowing: $C = \frac{Q}{V}$

$$C = \frac{\int_s \epsilon \vec{E}\cdot d\vec{s}}{-\int_l \vec{E}\cdot d\vec{l}}$$

$$Q = \int_s \epsilon \vec{E}\cdot d\vec{s} = \int_s \epsilon E \underbrace{(-\hat{z})\cdot(-\hat{z})}_{=1} ds = \epsilon E \int_s ds = \epsilon E A$$

$$\therefore \boxed{Q = \epsilon E A}$$

$$V = -\int \vec{E}\cdot d\vec{l} = -\int_0^d E(-\hat{z})\cdot\hat{z}\, ds = Ed \Rightarrow \boxed{V = Ed}$$

$$C = \frac{Q}{V} = \frac{\epsilon \cancel{E} A}{\cancel{E} d} \rightarrow \boxed{C = \frac{\epsilon A}{d}} \ [F]$$

## 4.10: Electrostatic Energy

- consider ideal capacitor with no ohmic losses

$$\begin{cases} \sigma \to \infty & \text{plates (conductor)} \\ \sigma \to \infty & \text{dielectric} \end{cases}$$

then, in charging a capacitor, the energy is stored as <u>electrostatic potential energy</u>

electric potential energy per unit charge (q)

$V = \frac{dW_e}{dq}$, knowing $C = \frac{q}{V} \Rightarrow V = \frac{q}{C}$

plug

$\to \frac{q}{C} = \frac{dW_e}{dq} \Rightarrow dW_e = \frac{q}{C}dq$

- if charging from 0 to Q   $q = CV$

$$W_e = \int_0^Q \frac{q}{C}dq = \frac{1}{2}\frac{Q^2}{C}$$

$$\boxed{W_e = \frac{1}{2}CV^2 \quad [J]}$$

- so far, we know $C = \frac{\epsilon A}{d}$, $V = Ed \Rightarrow E = \frac{V}{d}$ [V/m]

and $W_e = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\epsilon A}{d}(Ed)^2 = \frac{1}{2}\epsilon E^2 Ad$   $Ad$ = Volume $\mathcal{V}$

or $\boxed{W_e = \frac{1}{2}\int_{\mathcal{V}} \epsilon E^2 d\mathcal{V}}$   $W_e = \frac{1}{2}\int$

- connecting E and W and others:

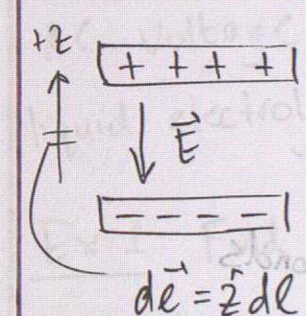
$$\boxed{W_e = \frac{1}{2}\int_{\mathcal{V}} \epsilon E^2 d\mathcal{V}} \ [J] \qquad \boxed{W_e = \frac{1}{2}\int_{\mathcal{V}} \frac{D^2}{\epsilon} d\mathcal{V}}$$

- Electrostatic Energy density ($w_e$)

$\to$ is defined as energy $W_e$ per unit volume

$$w_e = \frac{W_e}{\mathcal{V}} \Rightarrow w_e = \frac{W_e}{\mathcal{V}} \Rightarrow \boxed{w_e = \frac{1}{2}\epsilon E^2 \quad [J/m^3]}$$

- According to Coulomb's Law, there is a force b/n (+) and (−) plates in the same direction of $\vec{E}$




$\Rightarrow$ can write $\vec{F_e} = -F_e\hat{z}$

$dW_e = \vec{F_e}\cdot d\vec{l}$

$= -F_e\hat{z}\cdot\hat{z}\,dl = -F_e\,dz$

$F_e = -\frac{dW_e}{dz}$

- note: we can generalize dz (dl) along any direction

$\Rightarrow \boxed{\vec{F_e} = -\nabla W_e}$ $\leftarrow$ similar to $\vec{E} = -\nabla V$

- recall (d=z) $W_e = \frac{1}{2}\frac{\epsilon A V^2}{z} \xrightarrow{\text{can write}} dW_e = \frac{1}{2}\epsilon A V^2\left(-\frac{1}{z^2}dz\right)$

$dW_e = -\frac{1}{2}\frac{\epsilon A}{z^2}V^2 dz$

sub in $F_e = -\frac{dW_e}{dz}$

$$F_e = -\frac{1}{dz}\left(-\frac{1}{2}\frac{\varepsilon A V^2}{z^2}\, dz\right)$$

$$\vec{F_e} = \frac{1}{2}\frac{\varepsilon A V^2}{z^2}\,\hat{z} \longrightarrow z = d$$

$$\boxed{\vec{F_e} = \frac{1}{2}\varepsilon A \frac{V^2}{d^2}\,\hat{z}} \ [N]$$

- <u>supercapacitor</u>
- suppose 1 Farad, $d = 1mm$, $\varepsilon_r = 4$

$$C = \frac{\varepsilon_0 \varepsilon_r A}{d} \Longrightarrow A = 28.3 \times 10^6 \ [m^2]$$

- suppose $25V$, $C = 1F$

$$W = \frac{1}{2} CV^2 = \frac{1}{2}(1)(25)^2 = 312.5 \ J$$

means can have 100W light bulb $\simeq$ 3 seconds

2W LED for 2.6 min

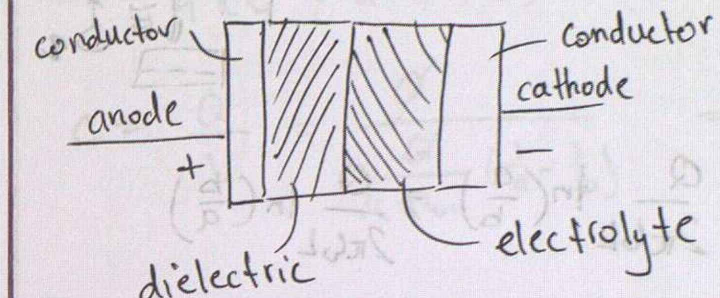
Amazon: $20 \times 3000F$ @ 2.7 [V] $1540

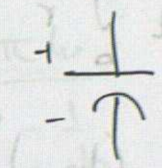
$1659

Week 5: Lecture ~~3~~ 2 Feb 5, 2025

<u>Electrolytic Capacitor</u> → asymmetric, so must take care the orientation

- has an <u>asymmetrical</u> construction, hence, has positive (anode) and negative (cathode) polarities



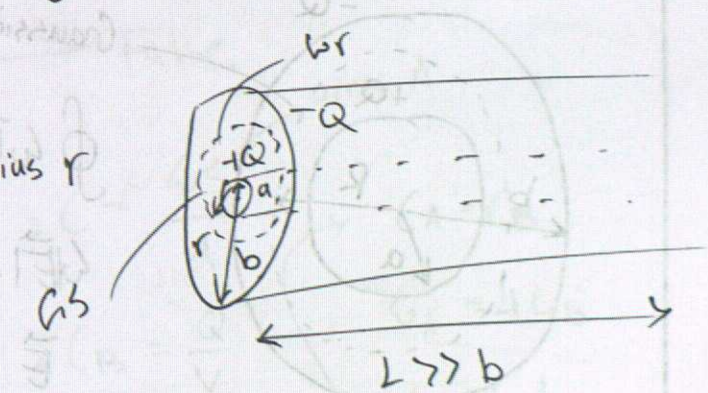

- has high capacitance
- good for low frequency filtering and smoothing rectified AC voltage
- liquid electrolyte can dry out over time

<u>Ex 1</u>: Find capacitance of coaxial capacitor filled with dielectric $\varepsilon_r$ and charge $+Q$ and $-Q$ where $L \gg b$

Knowing $C = \frac{Q}{V}$

Use cylinder G.S. with radius r

Apply Gauss Law




→ ignore fringing

$$\oint \varepsilon \vec{E} \cdot \vec{ds} = Q_{enc} = Q$$

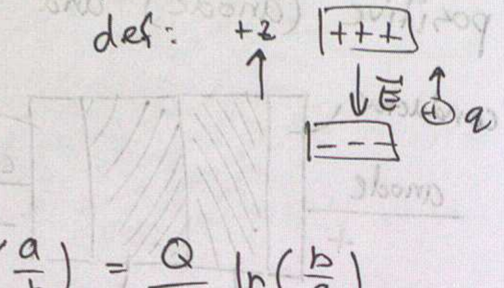
$$\varepsilon E 2\pi r L = Q$$

$$\vec{E} = \frac{Q}{2\pi \varepsilon L r}\,\hat{r}$$

* coaxial shape maybe on mid.

$$V = -\int_l \vec{E}\cdot d\vec{l} = -\int_b^a \frac{Q}{2\pi\epsilon L r}\,\hat{r}\cdot\hat{r}\,dr$$

go against the field as per def:



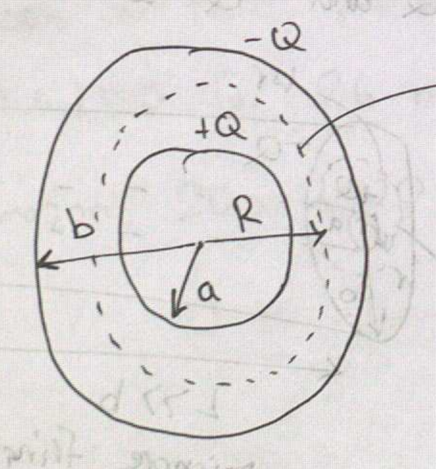

$$= \frac{-Q}{2\pi\epsilon L}\int_b^a \frac{1}{r}\,dr$$

$$= \frac{-Q}{2\pi\epsilon L}\left(\ln r\right)_b^a = -\frac{Q}{2\pi\epsilon L}\ln\left(\frac{a}{b}\right) = \frac{Q}{2\pi\epsilon L}\ln\left(\frac{b}{a}\right)$$

Q: what Q do we take? →

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{2\pi\epsilon L}\ln\left(\frac{b}{a}\right)} = \frac{2\pi\epsilon L}{\ln(b/a)} \quad [F]$$

Ex2: Find capacitance of spherical capacitor: two concentric spherical conductors separated by dielectric with $\epsilon$ and having charges $+Q$ and $-Q$




$$\oint \epsilon\vec{E}\cdot d\vec{s} = Q_{enc}$$

$$\epsilon|\vec{E}|4\pi R^2 = Q$$

$$\vec{E} = \frac{Q}{4\pi\epsilon R^2}\hat{R}$$

$$V = -\int \vec{E}\cdot d\vec{l} = -\int_b^a \frac{Q}{4\pi\epsilon R^2}\hat{R}\cdot\hat{R}\,dR$$

$$V = -\frac{Q}{4\pi\epsilon}\int_b^a \frac{dR}{R^2} = \frac{-Q}{4\pi\epsilon}\left(\frac{-1}{R}\Big|_b^a\right)$$

$$= \frac{Q}{4\pi\epsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$C = \frac{Q}{V} = \frac{\cancel{Q}}{\frac{\cancel{Q}}{4\pi\epsilon}\left(\frac{1}{a}-\frac{1}{b}\right)} = \frac{4\pi\epsilon}{\frac{1}{a}-\frac{1}{b}} \quad [F]$$

Ex 3: Capacitor

| | Area | Spacing | charge | Voltage |
|---|---|---|---|---|
| Cap A | S | d | Q | V |
| Cap B | 2S | d/2 | Q | V |

Is it true that cap B ~~have~~ has a dielectric more easily polarized? ⟶ False

$\chi_e = \epsilon_r + 1$, $q = CV$, $C = \frac{\epsilon S}{d}$ ← area A

$C = \frac{Q}{V} \Rightarrow C_A = \frac{Q}{V}$, $C_B = \frac{Q}{V}$

$C_A = \frac{\epsilon_A S}{d}$, $C_B = \frac{\epsilon_B 2S}{d/2} = \frac{\epsilon_B 4S}{d}$

$C_A = C_B$

$\epsilon_A = 4\epsilon_B$

$\epsilon = \epsilon_0\epsilon_r$

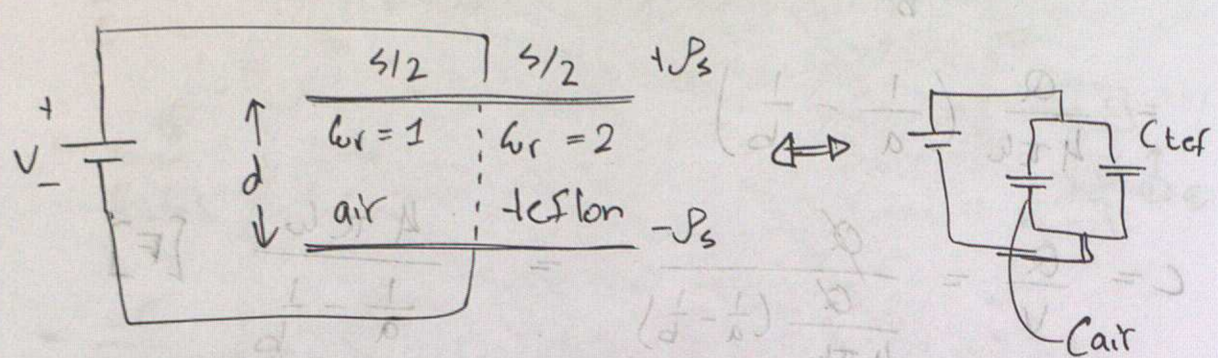
$\Rightarrow \epsilon_{rA} = 4\epsilon_{rB}$

$\Rightarrow \chi_{eA} = \epsilon_{rA} + 1 > \chi_{eB} = \frac{1}{4}\epsilon_{rA} + 1$

$\boxed{\text{∴ Cap A more receptive than Cap B ... ∴ False}}$

go on page/note back and do for exam

## Ex 4: Compare surface charge densities $\rho_{s\,air}$ and $\rho_{s\,tef}$




$d_{air} = \cancel{d_{tef}} = d_{tef}$

$V_{air} = V_{tef} \rightarrow E_{air} = E_{tef}$

$D_{air} = \varepsilon_0 \varepsilon_r \vec{E}$

$D_{tef} = \varepsilon_0 \varepsilon_r \vec{E}$

from boundary $= \begin{cases} E_t = 0 \\ D_n = \rho_s \rightarrow D = \rho_s \end{cases}$

$D_{air} = \varepsilon_0 \varepsilon_r E = \varepsilon_0 E = \rho_{s\,air}$

$D_{tef} = \varepsilon_0 \varepsilon_r E = 2\varepsilon_0 E = \rho_{s\,tef}$

$\therefore \rho_{s\,tef} > \rho_{s\,air}$

$\rho_{s\,tef} = 2\rho_{s\,air}$

Compare total charge $Q_{air}$ and $Q_{tef}$

$C = \dfrac{\varepsilon_0 \varepsilon_r S}{d}$, $C_{air} = \dfrac{\varepsilon_0 \, S/2}{d}$, $C_{tef} = \dfrac{2\varepsilon_0 \, S/2}{d}$

$C_{tef} = 2C_{air}$, $Q = CV \rightarrow Q_{air} = C_{air} V$

$Q_{tef} = C_{tef} V = 2 C_{air} V$

$\rightarrow Q_{tef} = 2Q_{air}$

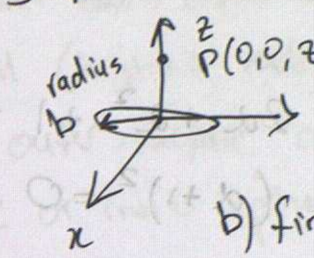
Week 5 lecture 3

## Midterm 2014

Q3 Consider a ring on xy-plane: a) sketch variation of $\rho_l$

$\rho_l = \rho_{l0} |\sin\phi|$

positive constant

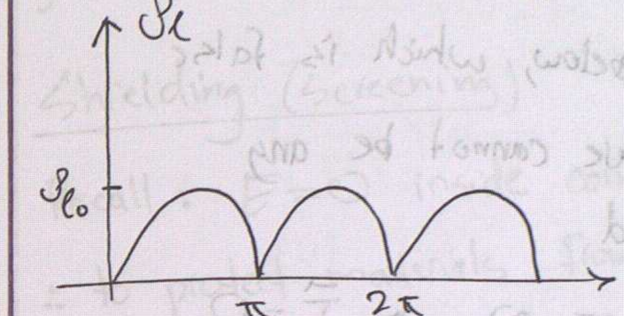



b) find $\vec{D}$ and $\vec{V}$ at P(0,0,z)

$dl = b\,d\phi \rightarrow dq = \rho_l b \, d\phi$

$$V = \int \frac{dq}{4\pi\varepsilon_0 |\vec{R} - \vec{R}'|}$$

a)




b)

$$V = \int_0^{2\pi} \frac{\rho_{l0}|\sin\phi|\, b\,d\phi}{4\pi\varepsilon_0 \sqrt{b^2+z^2}} d\phi = 2\int_0^{\pi} \frac{\rho_{l0}\sin\phi\, b\,d\phi}{4\pi\varepsilon_0\sqrt{b^2+z^2}}$$

$$= \frac{\rho_{l0} b}{2\pi\varepsilon_0\sqrt{b^2+z^2}} \int_0^{\pi} \sin\phi\, d\phi = \frac{\rho_l b}{2\pi\varepsilon_0\sqrt{b^2+z^2}} \underbrace{(-(\cos\pi - \cos 0))}_{+2}$$

$$= \frac{\rho_{l0} b}{\pi\varepsilon_0\sqrt{b^2+z^2}} \; [V] \rightarrow \text{to find D, knowing } \vec{E} = -\nabla V$$

$\vec{D} = \varepsilon_0 \vec{E} \Rightarrow \vec{D} = \varepsilon_0(-\nabla V)$

$$\Rightarrow \vec{D} = \varepsilon_0 \left(\frac{\partial}{\partial z}[V]\right) = \varepsilon_0 \frac{\partial}{\partial z}\left[\frac{\rho_{l0} b}{\pi\varepsilon_0\sqrt{b^2+z^2}}\right]\hat{z}$$

$$\vec{D} = -\frac{\rho_{l0} b}{\pi} \hat{z} \left(-\frac{1}{2}(b^2+z^2)^{-\frac{3}{2}}(2z)\right)$$

$$\vec{D} = \frac{\rho_{l0} b}{\pi} \frac{z}{(b^2+z^2)^{3/2}} \hat{z} \quad [C/m^2]$$

Ex 2: Given that $\vec{A} = 2\alpha\hat{x} + \hat{y} + \hat{z}$ and $\vec{B} = \hat{x} + \alpha^2\hat{y} + \hat{z}$. If $\vec{A}$ and $\vec{B}$ are normal to each other, then what is $\alpha$?

$\rightarrow \vec{A}\cdot\vec{B} = 0 \Rightarrow 2\alpha + \alpha^2 + 1 = 0$

$(\alpha+1)^2 = 0 \Rightarrow \underline{\alpha = -1}$

Ex 3: From the statements below, which is false

A) In a perfect dielectric, there cannot be any current or any electric field

$\vec{J} = \sigma\vec{E}$ dielectric $\rightarrow \sigma = 0 \Rightarrow \vec{J} = 0$

but $E \neq 0$

T B) A perfectly conducting material requires no applied field for current to flow inside it.

$\vec{J} = \sigma\vec{E} \rightarrow \sigma \rightarrow \infty \Rightarrow \vec{J} \rightarrow \infty$ no E required

$E = \frac{J}{\sigma} = \frac{0}{\sigma} = 0 \longrightarrow E = \frac{J}{\sigma} = \frac{\text{something}}{\infty} = 0$

T c) A perfect dielectric always has a zero current inside it

T d) A perfect conductor always has a zero electric field inside it

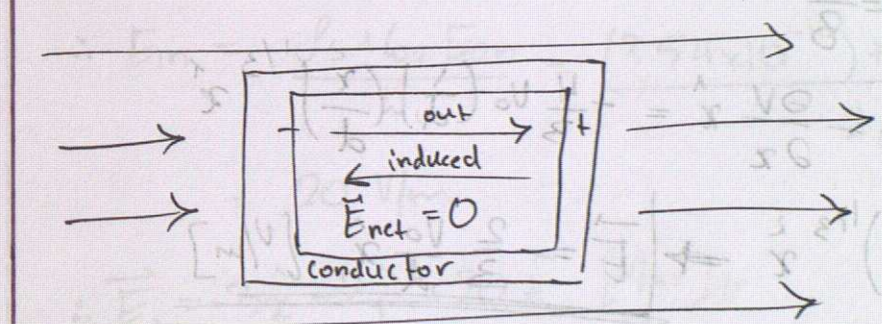
Ex 4:

- the charges are same sign, they don't like each other and want to be as far away from each other as possible $\rightarrow$ only on outer surface of shell (highest radius, surface)
- charge ∴ stays on outer surface of hollow sphere, doesn't go to small sphere or inner sphere

Shielding (screening)

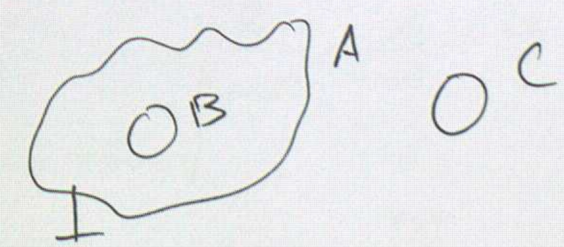
Recall: $E = 0$ inside conductor

∴ to protect materials from external $\vec{E}$, then ~~by hollow~~ surround them by hollow conductors




$\vec{E}_{external}$ causes charges to rearrange, then the resulting $E_{net}$ becomes zero

Ex 5: Consider hollow A conductor at potential zero then A screens B from outside and C from B

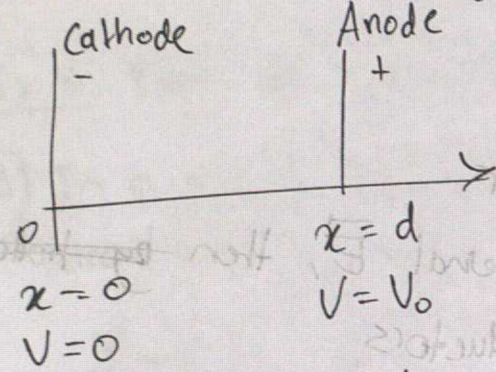


Ex 4: Consider a 2-plate cap with

$V = V_0 \left(\frac{x}{d}\right)^{4/3}$ [V] where $V_0$ is constant and

$d$ is spacing with $0 \le x \le d$

a) Find $V = ?$ @ $x = \frac{d}{\sqrt{8}}$




$$V = V_0\left(\frac{d}{\sqrt{8}d}\right)^{4/3} = V_0\left(\frac{1}{2^{3/2}}\right)^{4/3} = \frac{V_0}{4} \text{ [V]}$$

b) Find E at $x = \frac{d}{8}$

$$\vec{E} = -\nabla V \xrightarrow{1D} -\frac{\partial V}{\partial x}\hat{x} = -\frac{4}{3}V_0\left(\frac{1}{d}\right)\left(\frac{x}{d}\right)^{1/3}\hat{x}$$

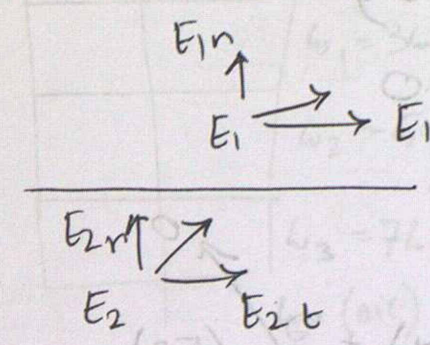
$$\vec{E} = -\frac{4}{3}\frac{V_0}{d}\left(\frac{d}{8d}\right)^{1/3}\hat{x} \Rightarrow \boxed{\vec{E} = -\frac{2}{3}\frac{V_0}{d}\hat{x} \text{ [V/m]}}$$

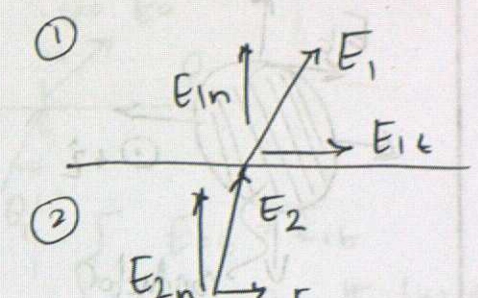
↑ plug $x$



4.48: $\vec{E}_2 = 3\hat{x} - 2\hat{y} + 2\hat{z}$ [V/m]

a)




$E_{2t} = 3\hat{x} - 2\hat{y}$ and $\boxed{E_{1t} = E_{2t}} = 3\hat{x} - 2\hat{y}$

$E_{2n} = 2\hat{z} = E_{2z}$ $\boxed{(\vec{D}_1 - \vec{D}_2)\cdot\hat{n} = \rho_s}$

$\hat{z}(\epsilon_1 E_{1n}\hat{z} - \epsilon_2 E_{2n}\hat{z}) = \rho_s$

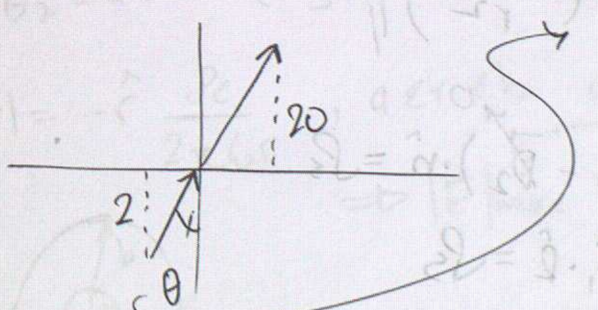
$\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s$

$$\therefore E_{1n} = \frac{\rho_s + \epsilon_2 E_{2n}}{\epsilon_1} = \frac{(3.54\times10^{-11}) + (18)(8.85\times10^{-12})(2)}{2(8.85\times10^{-12})}$$

$= 20$ V/m

$\therefore \vec{E}_1 = 3\hat{x} - 2\hat{y} + 20\hat{z}$ [V/m]

b)

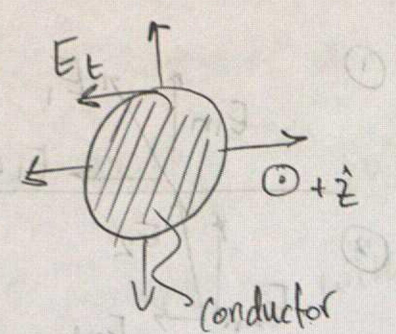



$|\vec{E}_2|\cos\theta = E_{2z} = 2$

$\theta = \arccos\left(\frac{2}{\sqrt{3^2+2^2+2^2}}\right)$

$\theta \simeq 61°$

**4.49:**




$$\vec{E} = \hat{r} E_r + \hat{\varphi} E_\varphi$$

and $\nabla \cdot \vec{D} = 0$

$$\vec{D} = \varepsilon_0 \vec{E}$$

$$\nabla \cdot \vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(r E_r) + \frac{1}{r}\frac{\partial}{\partial \phi}(E_\phi) + \cancel{\frac{\partial}{\partial z}(E_z)} \xrightarrow{} 0$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(r E_r) + \frac{1}{r}\frac{\partial}{\partial \phi}(E_\phi) = 0$$

$$\frac{\partial}{\partial r}(r E_r) + \frac{1}{r^2}(+\sin\phi) = 0 \qquad E_r = \frac{\sin\phi}{r^2}$$

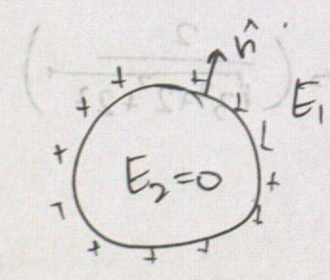
$$\frac{\partial}{\partial r}(r E_r) = -\frac{1}{r^2}\sin\phi$$

$$\int \partial\, r E_r = \int -\frac{1}{r^2}\sin\phi \, \partial r$$

$$r E_r = -\sin\phi \int \frac{1}{r^2}\partial r = \frac{-\sin\phi\,(-1)}{r}$$

$$\therefore \vec{E} = \hat{r}\left(\frac{\sin\phi}{r^2}\right) - \hat{\phi}\left(\frac{\cos\phi}{r^2}\right)$$

**4.50:**




$$(\vec{D}_1 - \cancel{\vec{D}_2}) \cdot \hat{n} = \rho_s \qquad (\vec{D}_2 \to 0)$$

$$\vec{D}_1 \cdot \hat{R} = \rho_s$$

$$\varepsilon_0 E_{1n} = \rho_s$$

$$E_{1n} = \frac{\rho_s}{\varepsilon_0} = \frac{Q}{4\pi\varepsilon_0 R^2} \qquad R = 5\text{cm}$$

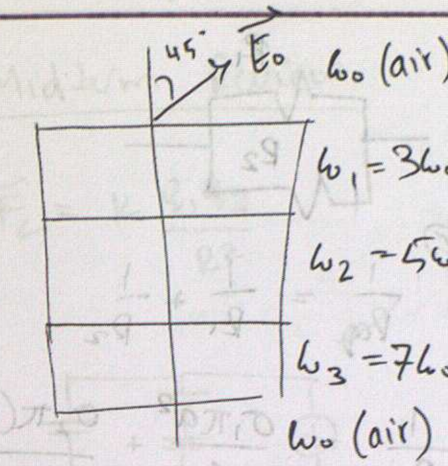
$$Q = \int_s \rho_s \, ds$$

$$\rho_s = \frac{Q}{4\pi R^2}$$

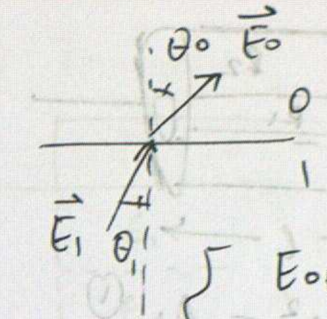
$$\therefore Q = (150)(4\pi\varepsilon_0)(0.05)^2$$

$$Q = 4.2\times10^{-11}\ [C]$$

**4.51:**




A




$$\begin{cases} E_{0t} = E_{1t} \\ |E_0|\sin\theta_0 = |E_1|\sin\theta_1 \quad \text{— ①} \end{cases}$$

$$D_{0n} - D_{1n} = 0$$

Since no charges on surface:

$$\Rightarrow D_{0n} = D_{1n}$$

$$\varepsilon_0 E_{0n} = \varepsilon_1 E_{1n}$$

$$\varepsilon_0 |E_0|\cos\theta_0 = \varepsilon_1 |E_1|\cos\theta_1 \quad \text{— ②}$$

divide ① ÷ ②

$$\frac{\tan\theta_0}{\varepsilon_0} = \frac{\tan\theta_1}{\varepsilon_1}$$

$\theta_\phi = \arctan\left(\frac{\varepsilon_1}{\varepsilon_0}\tan\theta_0\right)$ ← use this for every part

$\theta_2 = \arctan\left(\frac{\varepsilon_2}{\varepsilon_1}\tan\theta_1\right)$ and work your way backwards

$\theta_3 = \arctan\left(\frac{\varepsilon_3}{\varepsilon_2}\tan\theta_2\right)$

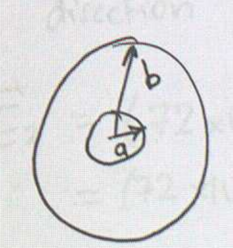
$$\theta_1 = \arctan\left(3\arctan\left(\frac{\pi}{4}\right)\right) = 1.249 \text{ rad}$$

$$\theta_2 = \arctan\left(\frac{5}{3}\arctan(\cdots)\right)\cdots\cdots$$

**4.53:** $|E| = -\hat{r}\dfrac{\rho_l}{2\pi\varepsilon r}$, $a < r < b$ and max $|\vec{E}|$ at $r = a$

$$\Rightarrow |E|_{max} = \frac{\rho_l}{2\pi\varepsilon a} = 200 \text{ MV}$$

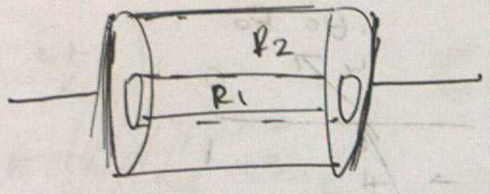
$\varepsilon$: from data table



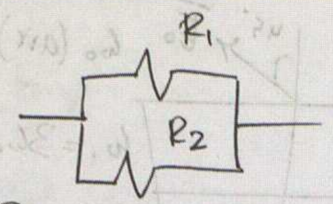

$$\rho_l = (200\times10^{6})(2\pi)(8.85\times10^{-12})(0.01)(6) = 667.6\times10^{-6}\ C/m$$

$$V_{ab} = \frac{\rho_l}{2\pi\varepsilon_r\varepsilon_0}\ln\left(\frac{b}{a}\right) = \frac{(667.6\times10^{-6})}{2\pi(6)(8.85\times10^{-12})}\ln\left(\frac{2}{1}\right) = 1.39\times10^{6}\ [V]$$

4.44




≡




Inner : $\pi a^2 = A_1$ (1)

outer : $\pi(b^2-a^2) = A_2$ (2)

$R_1 = \frac{l_1}{\sigma_1 A_1}$ and $R_2 = \frac{l_1}{\sigma_2 A_2}$

(3) $= \frac{l_1}{\sigma_1 \pi a^2}$ (4) $= \frac{l_1}{\sigma_2 \pi (b^2-a^2)}$

(5) $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$

(6) $\frac{1}{R_{eq}} = \frac{\sigma_1 \pi a^2}{l_1} + \frac{\sigma_2 \pi (b^2-a^2)}{l_1}$

$\therefore R_{eq} = \frac{l}{\pi[\sigma_1 a^2 + \sigma_2(b^2-a^2)]}$ //

4.54. a) $\frac{V}{d} = E = 1.6 \times 10^{-16}$ N

b) $F_{net} = ma \Rightarrow a = 1.76 \times 10^{14}$ m/s$^2$

c) $d = \underset{0}{v_0 t} + \frac{1}{2} a t^2 \Rightarrow t = \sqrt{\frac{2d}{a}} \Rightarrow 10.7 \times 10^{-9}$ s

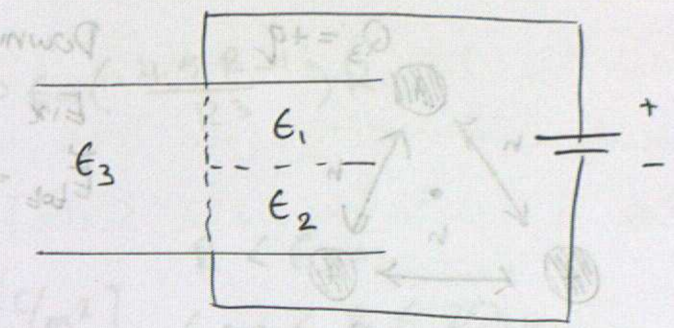
4.55: $W_e = \frac{1}{2} \int_0^3 \int_0^2 \int_{-1}^{1} \epsilon \left[ (x^2+2z)^2 + x^4 + (y+z)^2 ) \right] dx\,dy\,dz$

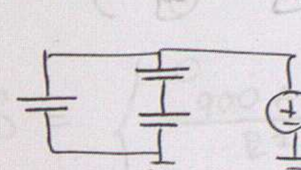
4.57: a) $C = C_1 + C_2 = \frac{\epsilon_1 A_1}{d} + \frac{\epsilon_2 A_2}{d}$

Week 6: Lecture 1 Feb 11, 2025

## Midterm Review

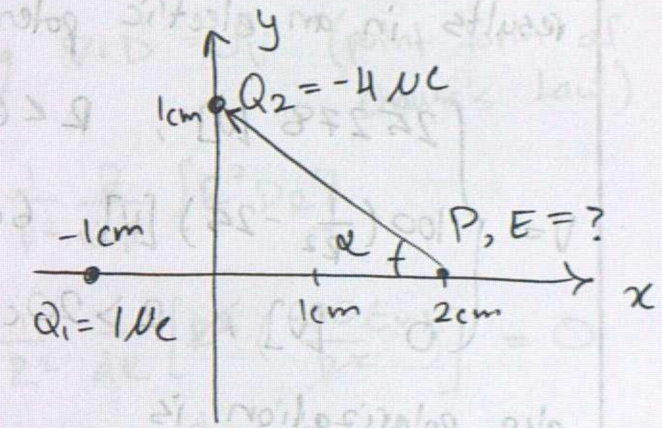
$\vec{F}_e = k \frac{q_1 q_2}{R^2}$



Ex 1: Find $\vec{E}$ @ P(2cm, 0, 0)

Apply superposition




$\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2$

$\vec{E}_1 = k \frac{Q_1}{R_1^2} \hat{x}$

$= \frac{(9\times10^9)(1\times10^{-6})}{(0.03)^2} \hat{x} = 10^7 \hat{x}$ [V/m]

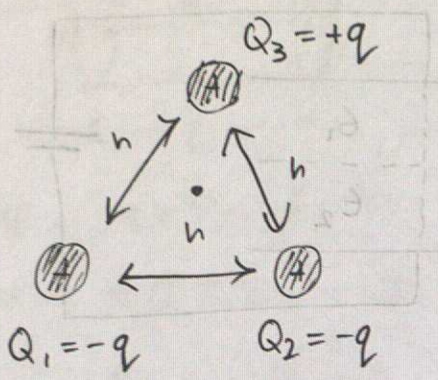
$E_2 = k \frac{Q_2}{R_2^2} = \frac{(9\times10^9)(4\times10^{-6})}{(\sqrt{1^2+2^2} \times 10^{-2})^2} = 72 \times 10^6$ [N/C]

direction $\alpha = \tan^{-1}(\frac{1}{2}) \simeq 26.6°$

$\vec{E}_2 = (72\times10^6)(-\cos 26.6° \hat{x} + \sin 26.6 \hat{y})$
$= (72\times10^6)(-0.89\hat{x} + 0.45\hat{y})$

$\vec{E}_{tot} = -54.4 \times 10^6 \hat{x} + 32.3 \times 10^6 \hat{y}$ [N/C]

Ex 2: Find direction of E at the center




Downward:

$E_{1x} = E_{2x} \longrightarrow$ cancel

$\vec{E}_{tot} = \vec{E}_3 + \vec{E}_{1y} + \vec{E}_{2y}$

Ex 3: A certain collection of charge distribution results in an electric potential field given by:

$$V = \begin{cases} 25\,278\ [V], & R < 6\text{ cm} \\ 100\left(\frac{1}{R^2} - 25\right)[V], & 6\text{ cm} \le R \le 20\text{ cm} \\ 0\ [V], & R > 20\text{ cm} \end{cases}$$

also, polarization is

$$\vec{P} = \begin{cases} 0\ [C/m^2] & R < 6\text{cm} \\ 200\epsilon_0\left(\frac{4.5R - 1}{R^3}\right)\hat{R}\ [C/m^2] & 6\text{cm} \le R \le 20\text{cm} \\ 0 & R > 20\text{cm} \end{cases}$$

a) Find $\vec{E}$: $\vec{E} = -\nabla V$

$$\vec{E} = \begin{cases} 0 \\ -\frac{d}{dR}\left(100\left(\frac{1}{R^2} - 25\right)\right) \\ 0 \end{cases} = \begin{cases} 0 \\ -100\left(-\frac{2}{R^3}\right)R^2 \\ 0 \end{cases}$$

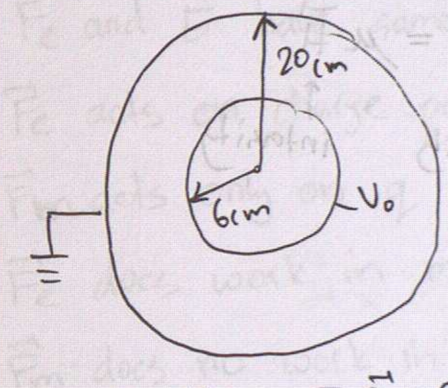
$$\vec{E} = \begin{cases} 0 & R < 6\text{ cm} \\ \frac{200}{R^3}\hat{R} & 6\text{cm} \le R \le 20\text{cm} \\ 0 & R > 6\text{cm} \end{cases}$$

b) Find $\vec{D}$: $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

$$\vec{D} = \begin{cases} 0 \\ \frac{200}{R^3}\hat{R} + 200\epsilon_0\left(\frac{4.5R - 1}{R^3}\right)\hat{R} \\ 0 \end{cases}$$

$$\vec{D} = \begin{cases} 0 & R < 6\text{cm} \\ \frac{900\epsilon_0}{R^2}\hat{R}\ [C/m^2] & 6\text{cm} \le R \le 20\text{cm} \\ 0 & R \geqslant 20\text{cm} \end{cases}$$

c) Describe charge distribution




knowing $\nabla\cdot\vec{D} = \rho_v$ (point form of Gauss's Law)

$$\nabla\cdot\vec{D} = \frac{1}{R^2}\frac{\partial}{\partial R}\left[R^2 D_R\right]$$

$$= \frac{1}{R^2}\frac{d}{dR}\left[R^2\left(\frac{900\epsilon_0}{R^2}\right)\right] = 0$$

$\Rightarrow$ since $\nabla\cdot\vec{D} = 0$, $\therefore$ no volume charge

$\therefore$ field must be due to surface charge

Q: why $\rho_s = D$?

$$\rho_s\big|_{6cm} = D\big|_{6cm} = \frac{900\epsilon_0}{(6\times10^{-2})^2} = 2.21\times10^{-6}\ [C/m^2]$$

$$\rho_s\big|_{20cm} = -D\big|_{20cm} = \frac{-900\epsilon_0}{(20\times10^{-2})^2} = -0.199\times10^{-6}\ [C/m^2]$$

d) Find C and Energy:

$$C = \frac{Q}{V} = \frac{\rho_s|_{6cm}\,4\pi R^2}{V} = \frac{(2.21\times10^{-6})(4\pi)(6\times10^{-2})^2}{25\,278} = 3.96\text{ pF}$$

$$W_e = \frac{1}{2}CV^2 = \frac{1}{2}(3.96\text{ pF})(25\,278)^2 = 1.26\text{ mJ}$$

Week 6 : Lecture 2

## Chapter 5 : Magnetostatics

- stationary charge (Q) $\longrightarrow \vec{E}$ (static electric field)
- steady ~~non-varying~~ (non-time varying) current (I)

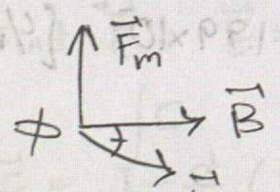
  $\hookrightarrow \vec{H}$ (static magnetic field)

| Electrostatic | Magnetostatic |
|---|---|
| $\nabla \cdot \vec{D} = \rho_v$ | $\nabla \cdot \vec{B} = 0$ |
| $\nabla \times \vec{E} = 0$ | $\nabla \times \vec{H} = \vec{J}$ |
| $\vec{D} = \varepsilon \vec{E}$ | $\vec{B} = \mu \vec{H}$ |
| $\vec{D}$ ↑ density, $\vec{E}$ ↑ intensity | $\vec{B}$ ↑ density, $\vec{H}$ ↑ intensity |

### 5.1: Magnetic Force & Torque

→ define magnetic flux density ($\vec{B}$)

- fire a test charge (+q) with velocity $\vec{u}$ through point P
- if a force $\vec{F}_m$ acts on moving +q, then a magnetic field $\vec{B}$ is present at point P such that:




$\left.\begin{array}{l}\phi = 0 \Rightarrow F_m = 0 \\ \phi = 90^\circ \Rightarrow F_m \text{ is max}\end{array}\right\}$ observation

$F_m = quB\sin\phi$

$$\boxed{\vec{F}_m = q\vec{u} \times \vec{B}}$$

$\hookrightarrow$ [T] Tesla $= \left[\frac{N}{C \cdot \frac{m}{s}}\right] = \left[\frac{N}{A \cdot m}\right]$

other unit: (non SI) Gauss [G] $\Rightarrow$ T

$1T = 10^4 G$

Notes:

- direction of Fm is given by R.H. rule.
- if both $\vec{E}$ and $\vec{B}$ are present, then electromagnetic force is $\vec{F} = \vec{F}_e + \vec{F}_m$

$$= q\vec{E} + q\vec{u} \times \vec{B}$$

$$\boxed{\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})} \longleftarrow \text{known as Lorentz Force}$$

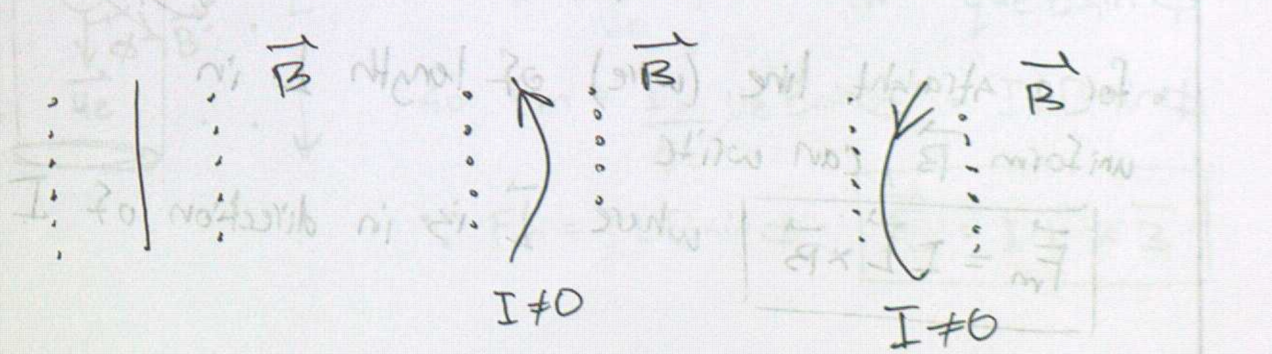
Differences between Fe and Fm

- $\vec{F}_e$ and $\vec{E}$ have same direction, but $\vec{F}_m$ and $\vec{B}$ are $\perp$
- $\vec{F}_e$ acts on charge particle both in motion and stationary
- $\vec{F}_m$ acts only on q in motion ($\vec{u} \neq 0$)
- $\vec{F}_e$ does work in moving q
- $\vec{F}_m$ does no work in moving q ; it only changes direction

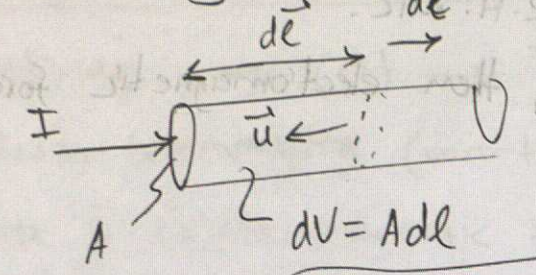
### 5.1.1: Magnetic Force on Current-Carrying Conductor (wire)

- Note: $\vec{B}$ exerts a sideway force on a moving charge
- current (I) is moving charges

$\Rightarrow \vec{B}$ should also exert a sideway force on the current-carrying conductor.



* For conducting wire




can write $\rho_{ve} = -N_e e$ — # of carriers per unit volume

↳ free electron charge density per unit volume

then $dq = \rho_{ve} A dl$

$= -N_e e A dl$

Note: direction of current I is opposite to flow of e (-q)

but $d\vec{F}_m = dq\, \vec{u}_e \times \vec{B}$

$= -N_e e A dl\, \vec{u}_e \times \vec{B}$

then can write $dl\,\vec{u}_e = -d\vec{l}\, u_e$

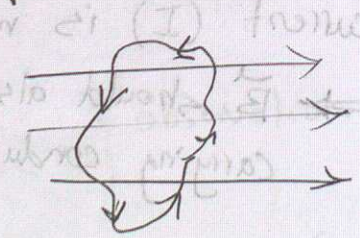
sub that: $\Rightarrow d\vec{F}_m = \underbrace{N_e e A u_e}_{I} d\vec{l} \times \vec{B}$

$\therefore d\vec{F}_m = I\, d\vec{l} \times \vec{B}$

$\Rightarrow \boxed{\vec{F}_m = I\int d\vec{l} \times \vec{B}}$ Note that vector sum of $d\vec{l}$ over a closed path equals zero

$\Rightarrow \vec{F}_m = I\oint_l d\vec{l} \times \vec{B} = 0$

$= 0$

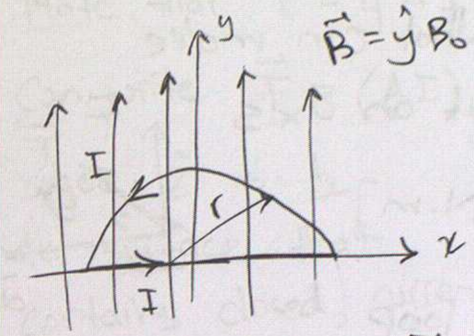


* for a straight line (wire) of length L in uniform $\vec{B}$, can write

$\boxed{\vec{F}_m = I\vec{L} \times \vec{B}}$ where $\vec{L}$ is in direction of I

Week 6: Lecture 3 — Feb 14, 2025

Ex 1: Find force on wire knowing $\vec{F} = I\int d\vec{l} \times \vec{B}$




$\vec{B} = \hat{y} B_0$

straight line:

$\vec{F}_1 = I\int \underbrace{2r\hat{x} \times \hat{y}}_{\hat{z}} B_0$

$= I2rB_0\hat{z} \Rightarrow \vec{F}_1 = 2IrB_0\hat{z}$

curved half-loop: $\vec{F}_2 = I\int d\vec{l} \times B$

$dl = r\,d\phi$, and by RHR, direction is $-\hat{z}$

$|\vec{F}_2| = I\int dl \sin\phi\, B_0 = I\int_0^{\pi} r\sin\phi\, B_0\, d\phi = IrB_0(2)$

$\vec{F}_2 = 2rIB_0(-\hat{z})$

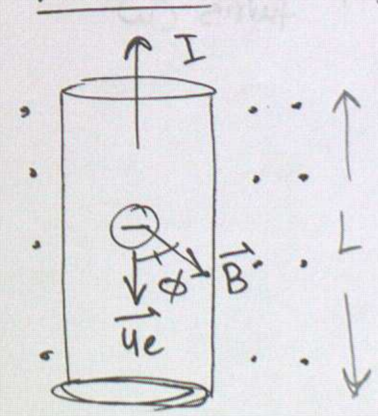
☆ take notes

## 5-1.2: Magnetic Torque on current-carrying Loop

Torque: is a measurement of force that can make an object to rotate about an axis

---

Method 2




time for e to travel distance L is

$t = \frac{L}{u_e}$, but $I = \frac{dq}{dt} \rightarrow q = It$

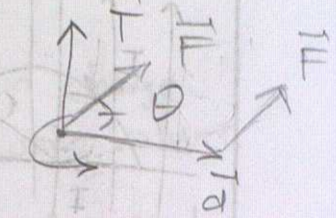
$\therefore q = \frac{IL}{u_e}$, but $F_m = q u_e B\sin\phi$

so $F_m = \frac{IL}{u_e} u_e B\sin\phi = ILB\sin\phi$

$\therefore F_m = ILB\sin\phi$, $\boxed{\vec{F}_m = I\vec{L} \times \vec{B}}$

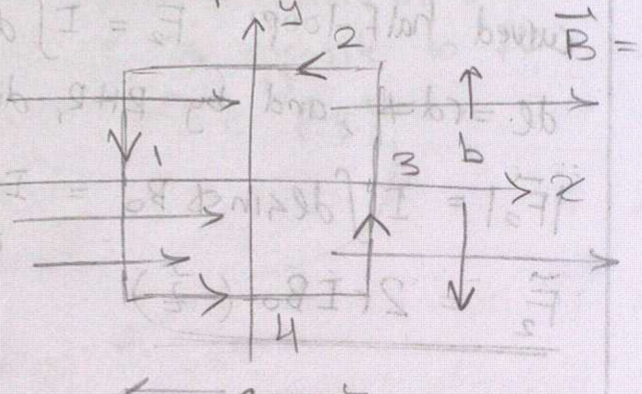
## 5-1.2 : Magnetic Torque on Current-carrying loop

Torque: Is a measure of force that can make an object to rotate about an axis




$T = dF\sin\theta$ or $\vec{T} = \vec{d}\times\vec{F}$ [N.m]

→ Magnetic field in the plane of the loop




Knowing: $\vec{F} = I\vec{l}\times\vec{B}$

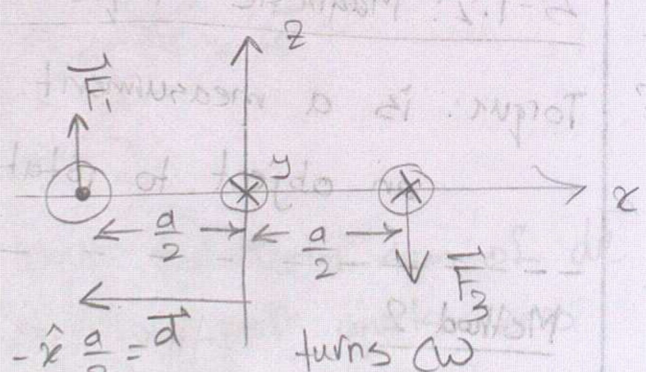
and $\vec{T} = \hat{x}B_0$

$\vec{F_1} = I(-\hat{y}b)\times\hat{x}B_0 = \hat{z}IbB_0$

$\vec{F_3} = I(\hat{y}b)\times\hat{x}B_0 = -\hat{z}IbB_0$

and $\vec{F_2} = \vec{F_4} = 0$

Knowing $\vec{T} = d\vec{x}\times\vec{F}$




$\vec{T} = \vec{d_1}\times\vec{F_1} + \vec{d_3}\times\vec{F_3}$

$= -\hat{x}\frac{a}{2}\times\hat{z}IbB_0$

$+\hat{x}\frac{a}{2}\times -\hat{z}IbB_0$

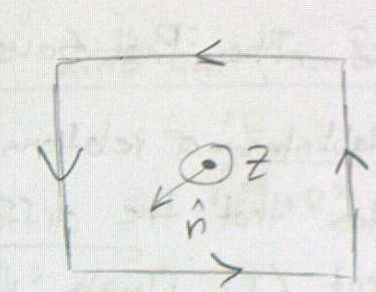
$\vec{T} = \frac{a}{2}IbB_0(-\hat{x}\times\hat{z} - \hat{x}\times\hat{z})$

$= \frac{a}{2}IbB_0(-2(\underbrace{\hat{x}\times\hat{z}}_{-y}))$

area

$\boxed{\vec{T} = \hat{y}AIB_0}$

## Magnetic Moment ($\vec{m}$)




Note that $\vec{T} = \hat{y}AIB_0$

Can write $\vec{T} = (AI)\underbrace{\hat{z}\times\hat{x}}_{\hat{y}}B_0$

where $\hat{z}$ is $\perp$ to surface that contains closed, current-carrying loop (R.H.R)

⟹ Define magnetic moment as $\vec{m} = AI\hat{z}$

⟹ $\vec{T} = \vec{m}\times\vec{B}$

Can also generalize $\vec{m} = AI\hat{z}$ as $\boxed{\vec{m} = AI\hat{n} \quad [A\cdot m^2]}$

where $\hat{n}$ is unit vector normal to surface containing loop.

* if loop has N turns, then $\boxed{\vec{m} = NAI\hat{n} \quad [A\cdot m^2]}$

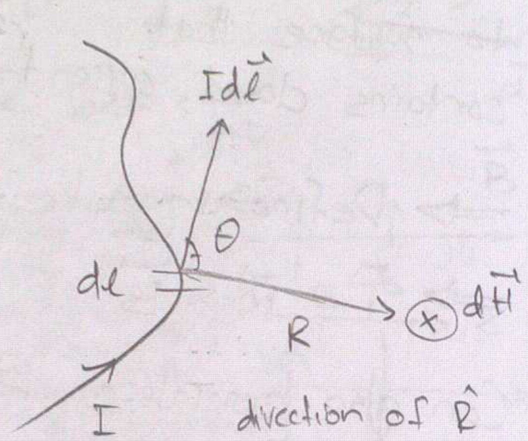
## 5.2: The Biot-Savart Law (experimental)

- establishes a relationship b/n I and H
- states that the differential $\vec{H}$ ($d\vec{H}$) generated by a steady current (I) flowing through $dl$ in:

$$dH = \frac{1}{4\pi}\frac{I\,dl\sin\theta}{R^2} \quad [A/m]$$

$$d\vec{H} = \frac{I}{4\pi}\frac{d\vec{l}\times\hat{R}}{R^2} \quad [A/m]$$

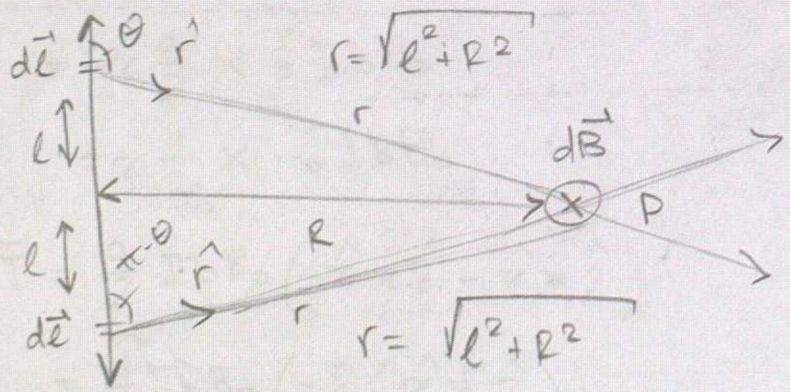
$$\boxed{\vec{H} = \frac{I}{4\pi}\int_l \frac{d\vec{l}\times\hat{R}}{R^2} \quad [A/m]}$$




direction of $\hat{R}$ is wire to point $d\vec{H}$

calculating integral is hard and depends on the path.

## Magnetic field due to current in a long straight wire




$D = \varepsilon E$

$\boxed{B = \mu\vec{H}}$

density  intensity

Apply Biot-Savart Law

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl\sin\theta}{r^2}$$



Note Symmetries:

$\sin\theta = \sin(\pi-\theta) = \frac{R}{\sqrt{l^2+R^2}}$

$$\rightarrow B = 2\int_0^\infty \frac{\mu_0}{4\pi}\frac{I\,dl\sin\theta}{r^2}$$

$$= \frac{\mu_0 I}{2\pi}\int_0^\infty \frac{\sin\theta\,dl}{r^2} \qquad r=\sqrt{l^2+R^2}$$

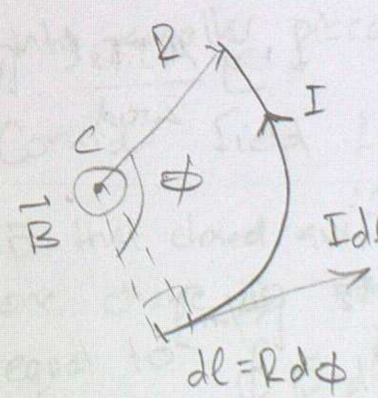
$$= \frac{\mu_0 I}{2\pi}\int_0^\infty \frac{\frac{R}{\sqrt{l^2+R^2}}}{l^2+R^2}\,dl$$

$$= \frac{\mu_0 I}{2\pi}\int_0^\infty \frac{R\,dl}{(l^2+R^2)^{3/2}} \quad \leftarrow \text{solve using tables or substitution}$$

$$\boxed{B = \frac{\mu_0 I}{2\pi R} \quad [T]}$$

magnetic field due to current in a long straight wire (memorize this equation)

## Magnetic field due to current in a circular arc of wire. Find $\vec{B}$ at center C due to I




I is fingers
thumb is B

Apply Biot-Savart:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l}\times\hat{r}}{r^2}$$

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl\sin\theta}{r^2}$$

direction of $\vec{B}$ is outwards

no matter where $dl$ is, $d\vec{l}$ ⊥ to $\hat{r}$

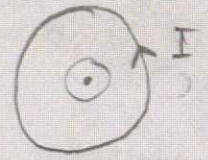
$\Rightarrow \theta = 90°$ and $dl = R d\phi$

$$B = \frac{\mu_0 I}{4\pi} \int \frac{dl}{R^2} = \frac{\mu_0 I}{4\pi} \int_0^{\phi} \frac{R d\phi}{R^2} = \frac{\mu_0 I}{4\pi R} \int_0^{\phi} d\phi$$

$$\boxed{B = \frac{\mu_0 I \phi}{4\pi R} \; [T]}$$ magnetic field at center of arc

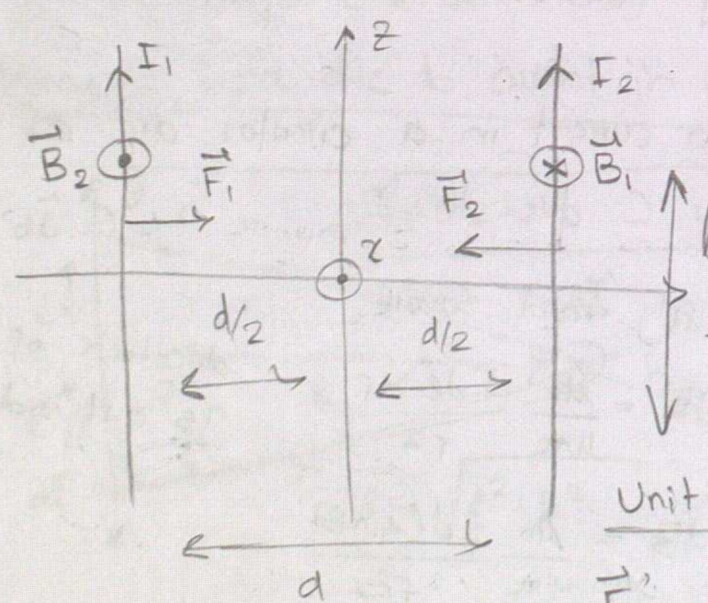
Special Case: Full circle:

full circle, set $\phi = 2\pi$




$$B = \frac{\mu_0 I}{4\pi R} \int_0^{2\pi} d\phi \rightarrow \boxed{B = \frac{\mu_0 I}{2R} \; [T]}$$ magnetic field at center of a full circle

5-2.3: Magnetic force between 2 parallel conductors (currents)




$$\vec{B}_1 = -\hat{x} \frac{\mu_0 I_1}{2\pi d}$$

$$\vec{F}_2 = I_2 l \hat{z} \times \vec{B}_1 = -\hat{y} \frac{\mu_0 I_1 I_2 l}{2\pi d} \; [N]$$

Unit length:

$$\vec{F}_2' = -\hat{y} \frac{\mu_0 I_1 I_2}{2\pi d} \; [N/m]$$

parallel attract

opposite repel

$$\vec{F}_1' = \hat{y} \frac{\mu_0 I_1 I_2}{2\pi d} \; [N/m]$$

Week 7: Lecture 2 Feb 26, 2025

## 5.3: Maxwell's Magnetostatic Equations

Recall electrostatic:

Gaus's Law for electricity:

$$\nabla \cdot \vec{D} = \rho_v \longleftrightarrow \oint_S \vec{D} \cdot d\vec{s} = Q_{enc}$$

point form integral form

Magnetostatics

The counterpart is:

$$\nabla \cdot \vec{B} = 0 \longleftrightarrow \oint_S \vec{B} \cdot d\vec{s} = 0$$

Point form integral form

→ called Gaus's Law for magnetism

note: zero reflects that the magnetic equivalent of electric $\rho_v$ (or $Q_{enc}$) does not exist

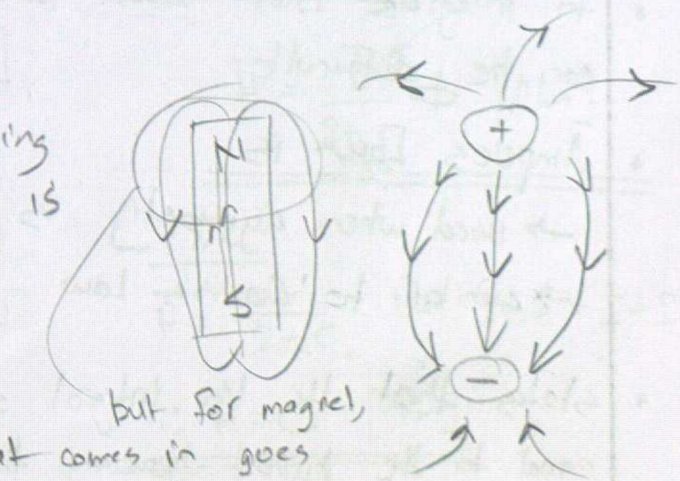
* since no matter how many times a magnet is divided into smaller pieces, still have magnetic dipole

Consider field lines

* $\vec{D}$ thru closed surface surrounding one charge is non-zero and is equal to $\oint \vec{D} \cdot d\vec{s} = Q_{enc}$

* magent case

$$\oint_S \vec{B} \cdot d\vec{s} = 0$$




but for magnet, what comes in goes out, so net flux is zero

Integration

$$B = \frac{\mu_0 I}{2\pi} \int_0^{\infty} \frac{R\,dl}{(l^2+R^2)^{3/2}}$$

Let $l = R\tan\theta$

$dl = R\sec^2\theta\, d\theta$

$\Rightarrow l^2 = R^2\tan^2\theta$

$l^2 + R^2 = R^2\tan^2\theta + R^2$

$= R^2(1+\tan^2\theta)$

$= R^2\sec^2\theta$

$l = 0 \Rightarrow \theta = 0$

$l = \infty \Rightarrow \theta = \pi/2$

$$B = \frac{\mu_0 I}{2\pi} \int_0^{\pi/2} \frac{R\,R\sec^2\theta\, d\theta}{(R^2\sec^2\theta)^{3/2}}$$

$$= \frac{\mu_0 I}{2\pi}\frac{R^2}{R^3} \int_0^{\pi/2} \frac{\sec^2\theta}{\sec^3\theta}\, d\theta$$

$$= \frac{\mu_0 I}{2\pi R}\int_0^{\pi/2} \frac{1}{\sec\theta}\, d\theta = \frac{\mu_0 I}{2\pi R}\int_0^{\pi/2}\cos\theta\, d\theta$$

$$\rightarrow B = \frac{\mu_0 I}{2\pi R} \quad [T]$$

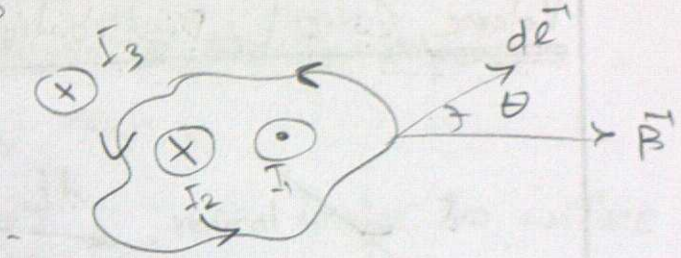
## 5-3.2 Ampere's Law

- to integrate Biot-Savart law to find magnetic field can be difficult

$$\boxed{\oint_c \vec{H}\cdot d\vec{l} = I_{enc}}$$

- Ampere's Law is
  - → used when symmetry is present
  - → similar to Gauss's Law ↳ $\oint_s \vec{D}\cdot d\vec{s} = Q_{enc}$
- states that the line integral of $\vec{H}$ around a closed path is equal to the current traversing the surface bound by the path

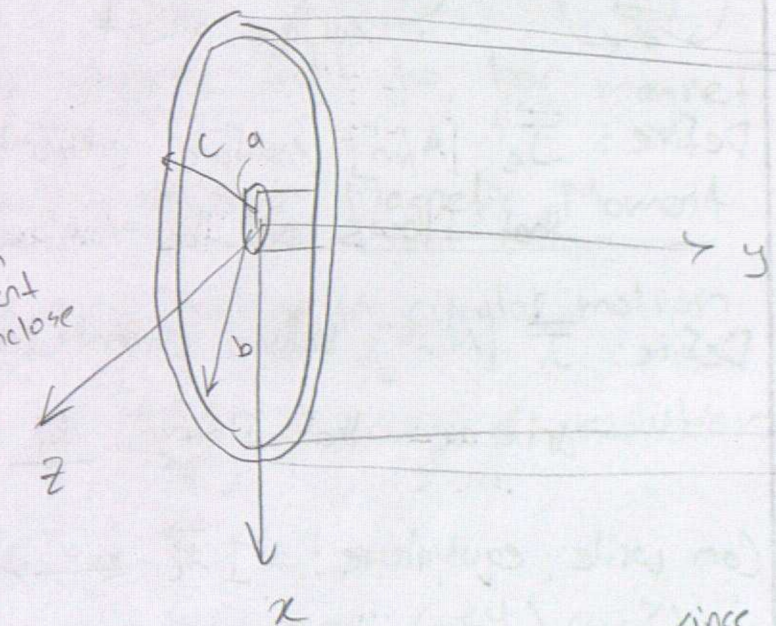
- the closed loop is called Amperian loop
- $d\vec{l}$ is along tangent to loop




$I_{enc} = I_1 - I_2$

- direction of current is determined by R.H.R.
- contribution of current outside of loop is zero

**Ex 1:** find magnetic field for coaxial cable everywhere




Apply Ampere's Law

$$\oint_c \vec{H}\cdot d\vec{l} = I_{enc}$$

$0 < r < a$

$H 2\pi r = I_{enc}$ — fraction of current we enclose in loop

$I_{enc} = I\frac{\pi r^2}{\pi a^2}$

$= I\frac{r^2}{a^2}$

$\therefore \vec{H} = \frac{I}{2\pi a^2} r\,\hat{\phi}$ [A/m] or [Wb]

$a \le r \le b$ — since $I_{enc}$ is $I$

$\vec{H} = \frac{I}{2\pi r}\hat{\phi}$ [A/m]

$b \le r \le c$ — $I_{enc} = I - I\frac{\pi(r^2-b^2)}{\pi(c^2-b^2)}$

$\vec{H} = \frac{I_{enc}}{2\pi r}$

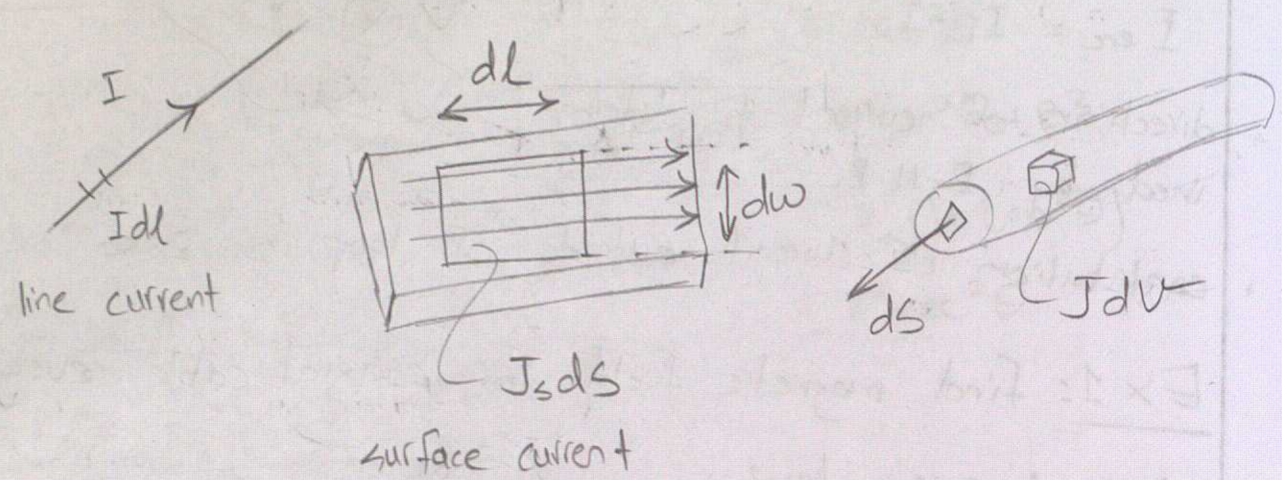
$\vec{H} = \frac{I}{2\pi r}\left(1 - \frac{r^2-b^2}{c^2-b^2}\right)\hat{\phi}$ [A/m]

$r \ge c$

$\vec{H} = \frac{I_{enc}}{2\pi r}$, but $I_{enc} = I - I = 0$

$\therefore \vec{H} = 0$ [A/m]

## 5-2.1: Magnetic Field Due to Surface and Volume Current Distribution




* Define: $\vec{J}_s$ [A/m] surface current density as current that flows on the surface of conductor

* Define: $\vec{J}$ [A/m²] volume current density as current per unit area that flows ⊥ to surface

* Can write equivalence: $I\vec{dl} \longleftrightarrow \vec{J}_s dS \longleftrightarrow \vec{J} dV$

  $A \cdot m$ $\qquad [\frac{A}{m}] \cdot [m^2] = A \cdot m \qquad [\frac{A}{m^2}] \cdot [m^3] = A \cdot m$

* Biot-Savart Law can be written:

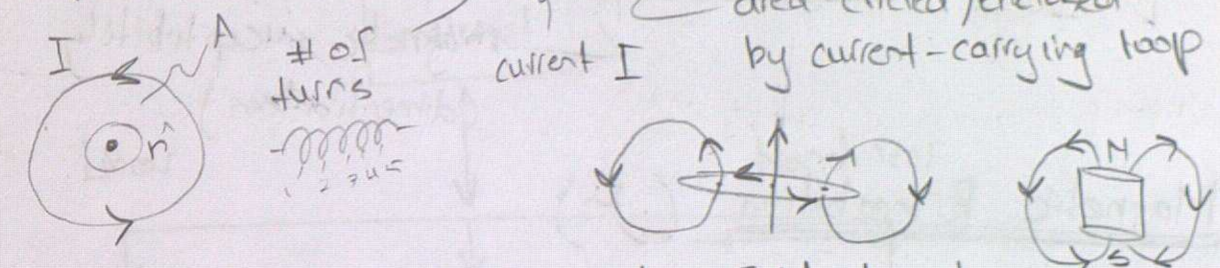
$$\vec{H} = \int_l \frac{I\vec{dl} \times \hat{R}}{4\pi R^2} \qquad \vec{H} = \iint_S \frac{\vec{J}_s dS \times \hat{R}}{4\pi R^2} \qquad \vec{H} = \iiint_V \frac{\vec{J} dV \times \hat{R}}{4\pi R^2}$$

## 5-5: Magnetic Properties of Materials

* Recall Magnetic Dipole: analogous to electric dipole and is a current-carrying loop such:

$$\boxed{\vec{m} = N I A \hat{n}}$$

magnetic moment; # of turns; current I; area circled/enclosed by current-carrying loop; normal vector to surface



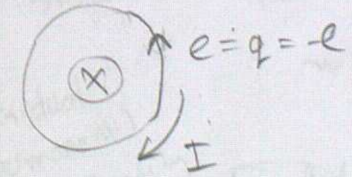

* magnetic dipole similar magnetic field to bar magnet

### 5-5.1: Electron Orbital and Spin Magnetic Moment

* Consider electron with $q = e$ moving in circular motion with speed $u$ $\Rightarrow T = \frac{2\pi r}{u}$ [seconds] 1 revolution

$$I = \frac{dq}{dt} = \frac{-e}{T} = \frac{-eu}{2\pi r}$$

* orbital magnetic moment: $m_o = IA = \left(\frac{-eu}{2\pi r}\right)(\pi r^2) = -\frac{eur}{2}$




$$\boxed{m_o = -\frac{eur}{2}}$$

# 5-5.2 : Magnetic Permeability $\mu$

* magnetization vector: is a measure of how receptive material is to magnetization and (for some mater.) given by:

magnetization $[A/m]$ $\boxed{\vec{M} = \chi_m \vec{H}}$ magnetic field

magnetic suceptibility (dimentionless)

## Magnetic Permeability ($\mu$)

Recall: $\vec{D} = \varepsilon_0 \vec{E} \longrightarrow \vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

Likewise: $\vec{B} = \mu_0 \vec{H} \longrightarrow \vec{B} = \mu_0 \vec{H} + \mu \vec{M}$

$$\Rightarrow \vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(\vec{H} + \chi_m \vec{H})$$

$$= \mu_0 \underbrace{(1 + \chi_m)}_{\mu_r} \vec{H}$$

$$\vec{B} = \mu_0 \mu_r \vec{H} \qquad \mu_r = \frac{\mu}{\mu_0} = 1 + \chi_m$$

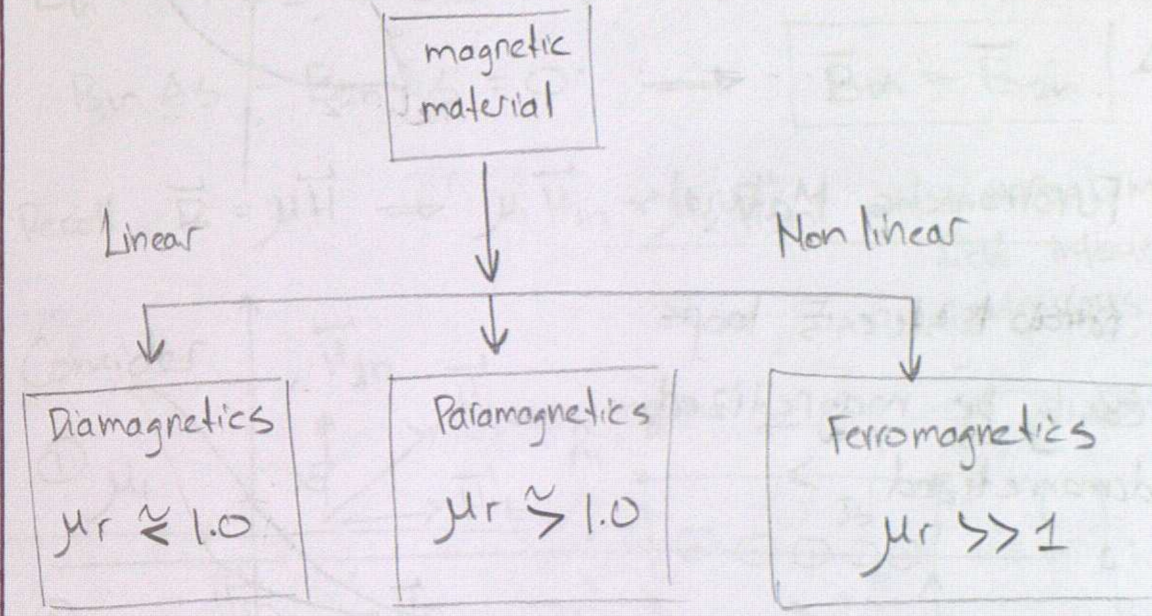
$$= \mu \vec{H}$$

$$\mu_0 = 4\pi \times 10^{-7} \; [H/m]$$

Week 8: Lecture 1 Mar 2, 2025

## Classification of Materials

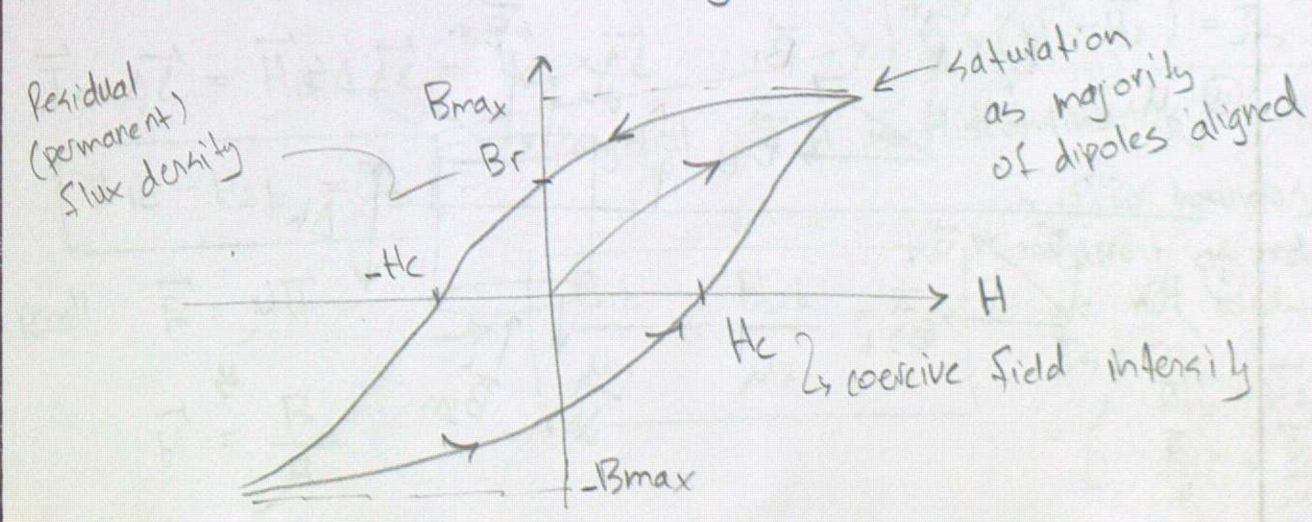
* is based on $\chi_m$
* nature of magnetic behaviour depends on the crystaline structure of material and is used to classify materials




## 5-5.3 Magnetic Hysteresis of Ferromagnets

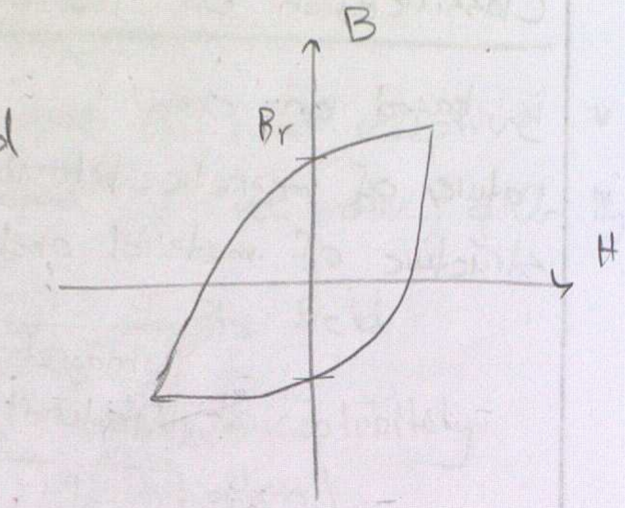
* "hysterisis" means "lag behind" ..
* eg: iron, nickel, cobalt, alloys
* is shown in terms of B-H magnetization curve

magnetization process depends on Both → H, → magnetic history



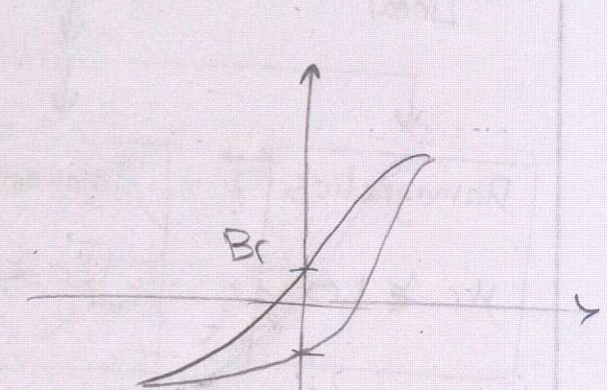
## Hard Ferromagnetic Materials

* have wide hysteresis loops
* not easily magnetized/demagnetized



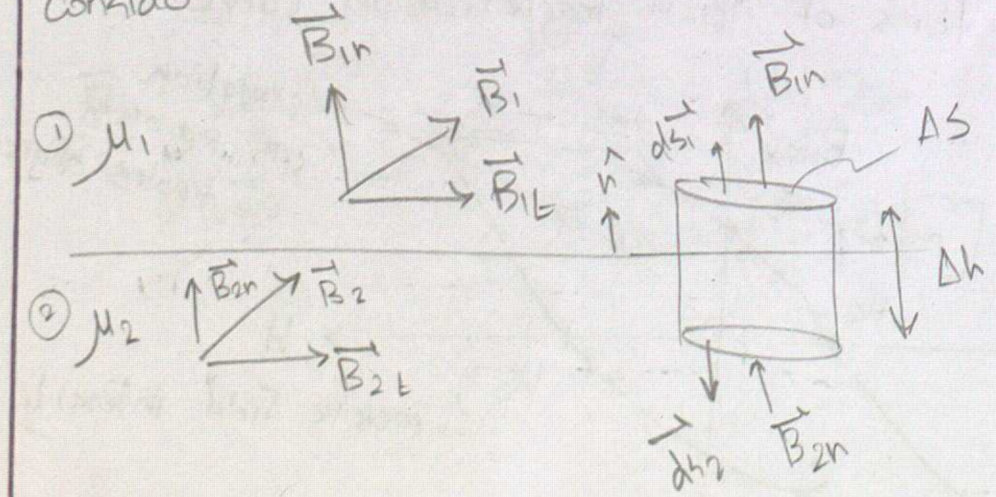

## Soft Ferromagnetic Materials

* have narrow hysteresis loops
* can easily be magnetized or demagnetized




## 5-6 Magnetic Boundary Conditions

* are the conditions $\vec{H}$ / $\vec{B}$ field must satisfy @ boundary b/n 2 different media
* consider




* Apply Gauss Law for magnetism

$$\oint_S \vec{B}\cdot d\vec{s} = 0$$ and let $\Delta h \rightarrow 0$ (side contribution goes to zero) so only contributions from top & bottom remain

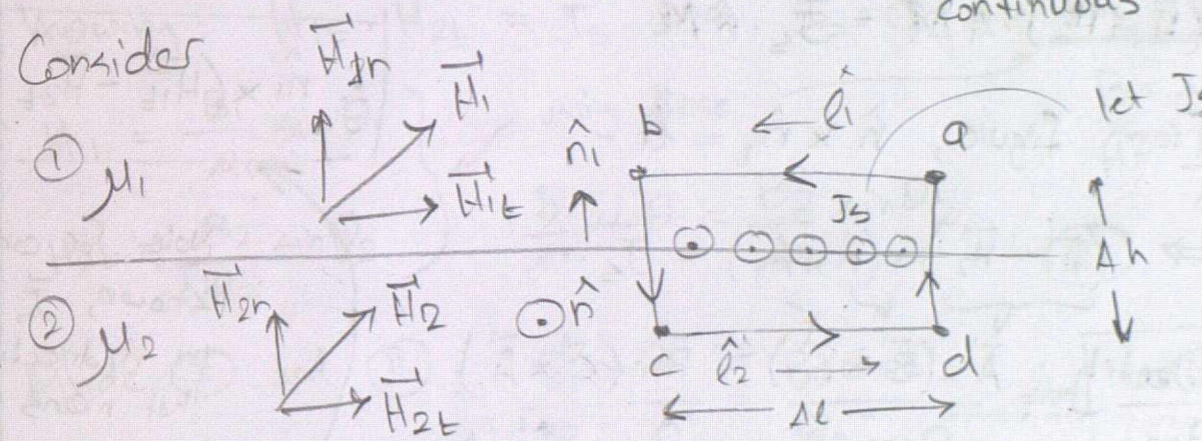
$$B_{1n}\,\hat{n}\cdot\hat{n}\,\Delta S + B_{2n}\,\hat{n}\cdot(-\hat{n})\,\Delta S = 0$$

$$B_{1n}\,\Delta S - B_{2n}\,\Delta S = 0 \Longrightarrow \boxed{\vec{B_{1n}} = \vec{B_{2n}}}$$

$\therefore$ normal B is cont @ boundary

+ Recall $\vec{B} = \mu\vec{H} \Rightarrow \mu_1 \vec{H_{1n}} = \mu_2 \vec{H_{2n}} \Rightarrow$ $\therefore$ normal magnetic field intensity is not continuous

* Consider




* Apply Ampere's Law: $\oint_l \vec{H}\cdot d\vec{l} = I_{enc}$

$$\boxed{\hat{n}_1 \times (\vec{H_{1t}} - \vec{H_{2t}}) = \vec{J_s}}$$

* and $I = \vec{J_s}\cdot\Delta\vec{l}$
* let $J_s$ be surface current also $\Delta h \rightarrow 0$

$$\vec{J_s}\cdot\Delta\vec{l} = \vec{H_{1t}}\cdot\Delta\vec{l} - \vec{H_{2t}}\cdot\Delta\vec{l}$$

$$\boxed{\hat{n}_1 \times (\vec{H_1} - \vec{H_2}) = J_s}$$

$$\boxed{H_{1t} - H_{2t} = J_s}$$

$\rightarrow$ tangential of H is discontinuous

* recall $\vec{B} = \mu\vec{H}$

$$\Downarrow$$

$$\vec{H} = \frac{B}{\mu}$$

$$\Rightarrow \boxed{\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = J_s}$$

$\rightarrow$ if no boundary current or media are not conductors, $J_s = 0$

$$= \left[\begin{array}{l} \vec{H_{1t}} = \vec{H_{2t}} \\ \dfrac{\vec{B_{1t}}}{\mu_1} = \dfrac{\vec{B_{2t}}}{\mu_2} \end{array}\right.$$

Week 8: Lecture 2 Mar 5, 2025

## Another Method

* Apply Ampere's Law $\oint_l \vec{H}\cdot d\vec{l} = I_{enc}$ and let $\Delta h \to 0$

$$\oint_l \vec{H}\cdot d\vec{l} = \int_a^b \vec{H}_1 \cdot \hat{l}_1\, dl + \int_c^d \vec{H}_2 \cdot \hat{l}_2\, dl$$

0 otherwise dot product is zero

* Can write $\vec{H}_1 \cdot \hat{l}_1 \Delta l + \vec{H}_2 \cdot \hat{l}_2 \Delta l = I$ → for $\Delta l$ → $I = \vec{J}_s \cdot \Delta \vec{l}$ $= \vec{J}_s \cdot \hat{n} \Delta l$

* Also, $\hat{l}_2 = -\hat{l}_1 \Rightarrow (\vec{H} - \vec{H}_2)\cdot \hat{l}_1 \Delta l = I$

$$(\vec{H}_1 - \vec{H}_2)\cdot \hat{l}_1 \cancel{\Delta l} = \vec{J}_s \cdot \hat{n} \cancel{\Delta l}$$

$$\boxed{\therefore \hat{n}_1 \times (\vec{H}_{1t} - \vec{H}_{2t}) = \vec{J}_s}$$

* From figure, $\hat{n} \times \hat{n}_1 = l_1$

$\Rightarrow (\vec{H}_1 - \vec{H}_2)\cdot(\hat{n}\times\hat{n}_1) = \vec{J}_s \cdot \hat{n}$

* Identity: $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A})$

$\rightarrow \hat{n}\cdot(\hat{n}_1 \times (\vec{H}_1 - \vec{H}_2)) = \vec{J}_s \cdot \hat{n}$

Note: for config shown, $\vec{J}_s$ is in $\hat{n}$ direction, that means $H_{1t} < H_{2t}$

if $H_{1t} < H_{2t} \Rightarrow \hat{n}$

$H_{1t} > H_{2t} \Rightarrow -\hat{n}$

$H_{1t} = H_{2t} \Rightarrow J_s = 0$

* Identity: $\vec{K}\cdot\vec{L} = \vec{L}\cdot\vec{K}$

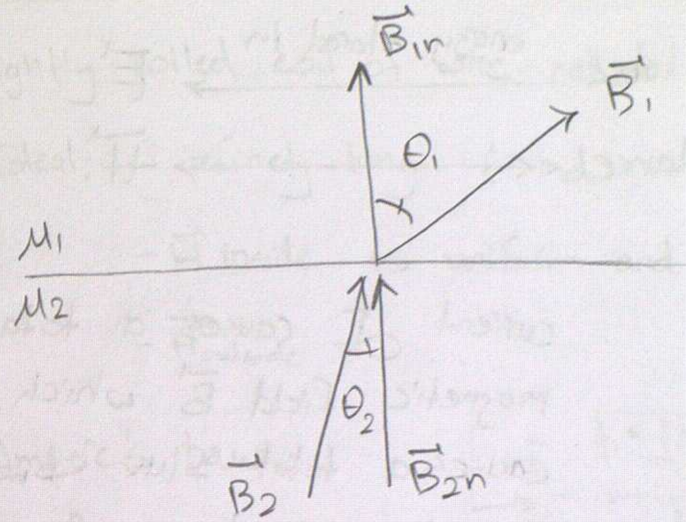
$\Rightarrow (\hat{n}_1 \times (\vec{H}_1 - \vec{H}_2))\cdot\cancel{\hat{n}} = \vec{J}_s \cdot \cancel{\hat{n}}$

$\hat{n}_1 \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s$ ← but $\vec{H}_1 = \vec{H}_{1n} + \vec{H}_{1t}$, $\vec{H}_2 = \vec{H}_{2n} + \vec{H}_{2t}$

$\Rightarrow \hat{n}_1 \times (\vec{H}_{1n} + \vec{H}_{1t} - (\vec{H}_{2n} + \vec{H}_{2t})) = \vec{J}_s$

note $\hat{n}_1 \times \vec{H}_{1n} = 0$ and $\hat{n}_1 \times \vec{H}_{2n} = 0$

Ex: Find relation b/n $\mu_1$ and $\mu_2$ given figure:




* We know $\vec{B}_{1n} = \vec{B}_{2n}$

can write:

$$\begin{cases} B_{1n} = B_1 \cos\theta_1 \\ B_{2n} = B_2 \cos\theta_2 \end{cases}$$

$\therefore B_1 \cos\theta_1 = B_2 \cos\theta_2$ — I

* Knowing: $H_{1t} - H_{2t} = J_s$ and $J_s = 0 \Rightarrow H_{1t} = H_{2t}$

$B = \mu H$

$H = \frac{B}{\mu}$

$H_{1t} = \frac{B_1}{\mu_1}\sin\theta_1$

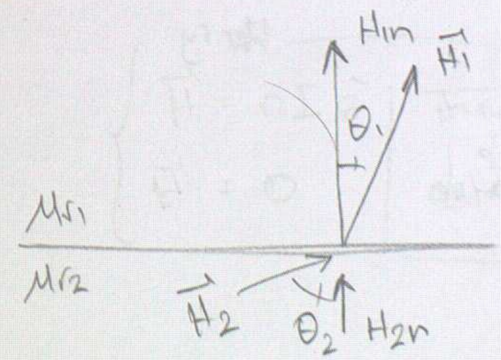
$H_{2t} = \frac{B_2}{\mu_2}\sin\theta_2$

using these

$\frac{B_1}{\mu_1}\sin\theta_1 = \frac{B_2}{\mu_2}\sin\theta_2$ — II

Divide I and II: $\frac{\cancel{B_1}\cos\theta_1}{\frac{\cancel{B_1}}{\mu_1}\sin\theta_1} = \frac{\cancel{B_2}\cos\theta_2}{\frac{\cancel{B_2}}{\mu_2}\cos\theta_2} \Rightarrow \boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}}$

Ex: for magnetic field shown, how does $\mu_{r1}$ relate to $\mu_{r2}$




$B_{1n} = B_{2n}$ $B = \mu H$

$\mu_{r1} H_{1n} = \mu_{r2} H_{2n}$

$H_{1n} = \frac{\mu_{r2}}{\mu_{r1}} H_{2n}$

from figure drawn to scale $H_{1n} > H_{2n}$

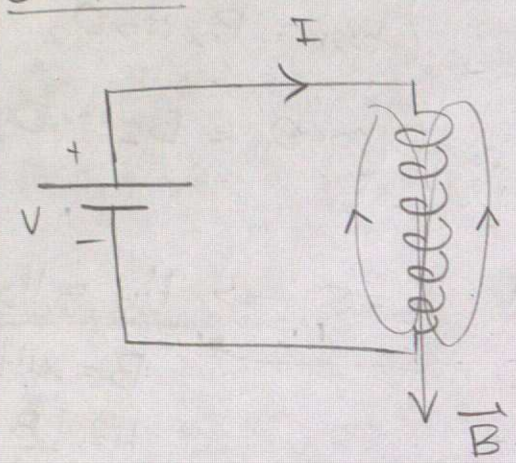
☆

## 5-7: Inductance

capacitor $\xrightarrow{\text{has}}$ capacitance $\xrightarrow{\text{energy stored in}}$ $\vec{E}$

Inductor $\longrightarrow$ inductance $\longrightarrow$ $\vec{H}$

* Consider




current I causes a total magnetic field $\vec{B}$ which causes a total flux $\Phi_m$ through each turn of the coil given by:

$$\boxed{\Phi_B = \int_S \vec{B} \cdot d\vec{s}}$$

* For N identical and tightly wrapped coils, can define magnetic flux linkage as

$$\boxed{\Lambda = N \Phi_B \quad [Wb]}$$

capital lambda

* then inductance is defined as the ratio of $\Lambda$ to current [I]

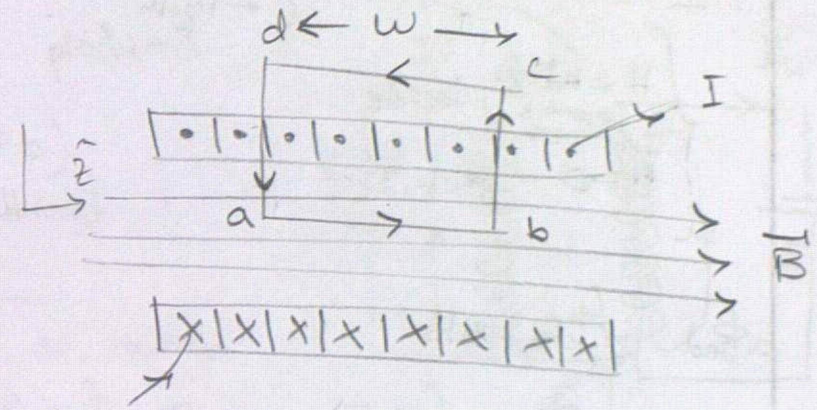
Henry

$$\boxed{L = \frac{\Lambda}{I} = \frac{N\Phi}{I} \quad [H] \text{ or } \left[\frac{Wb}{A}\right]}$$

## 5-7.1 Magnetic Field in a Solenoid

* Tightly coiled coil of wire called a "solenoid"
* Ideal: - infinetly long $l >> d$
  - $\vec{H}$ inside is uniform and parallel to solenoid axis
  - $\vec{H}_{outside} = 0$
* Ampere's Law:

$$\oint_l \vec{H} \cdot d\vec{l} = I_{enc}$$




$$\oint_l \vec{H} \cdot d\vec{l} = \int_a^b \vec{H} \cdot d\vec{l} + \int_b^c \vec{H} \cdot d\vec{l} + \int_c^d \vec{H} \cdot d\vec{l} + \int_d^a \vec{H} \cdot d\vec{l}$$

$\theta = 0$ $\therefore \neq 0$ ; $\theta = 90°$ $\therefore = 0$ ; outside $\therefore = 0$ ; $\theta = 90°$ $\therefore = 0$

$$\oint_l \vec{H} \cdot d\vec{l} = Hw + 0 + 0 + 0 = I_{enc}$$

$Hw = I n w$ → # of turns per length

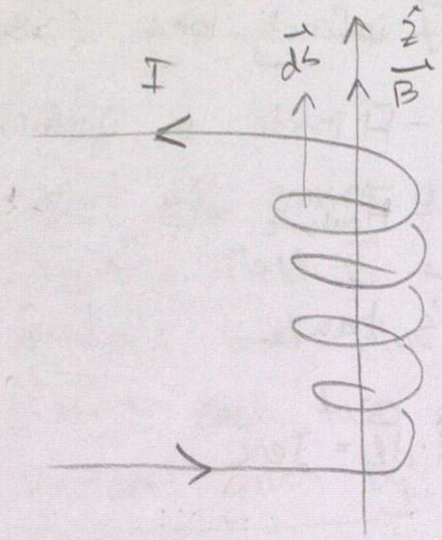
$$\begin{cases} \vec{H} = nI\hat{z} & \text{inside} \\ \vec{H} = 0 & \text{outside} \end{cases} \qquad \vec{B} = \mu\vec{H}$$

## 5-7-2: Self Inductance of a Solenoid

- Self Inductance: is when the inductance is produced by the inductor itself
- Consider:

$n = \frac{N}{l}$ ← total # of turns, ← length

$$\begin{cases} H = nI\hat{z} & \text{inside} \\ \vec{H} = 0 & \text{outside} \end{cases}$$




and
$$\begin{cases} B = \mu H \\ \Phi_m = \int_S \vec{B}\cdot d\vec{s} \implies \Phi = \int_S \hat{z}\left(\frac{N}{l}\mu\right)\cdot \hat{z}\, ds \end{cases}$$

$\implies \Phi_m = \mu \frac{N}{l} I S$ → $S$ is cross-sectional area

- Can write in terms of magnetic flux linkage

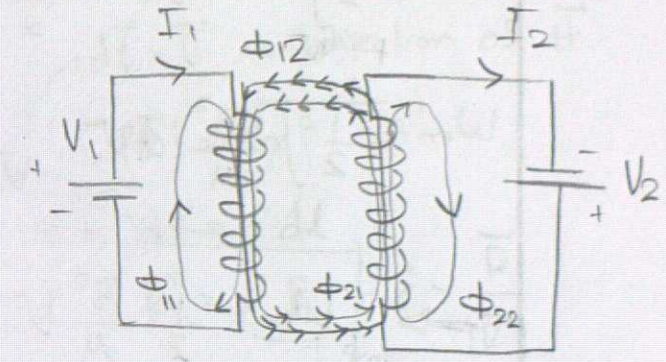
$$\Lambda = N\Phi_m = N\left(\mu \frac{N}{l} I S\right) = \mu \frac{N^2}{l} I S \quad [Wb]$$

Self inductance $L = \frac{\Lambda}{I} = \frac{\mu N^2 \cancel{I} S}{l \cancel{I}} = \frac{\mu N^2 S}{l}$

$$\boxed{\therefore L = \frac{\mu N^2 S}{l} \quad [H]} \iff C = \frac{\varepsilon A}{d} \rightarrow S \quad [F]$$

## 5-7-4: Mutual Inductance

- magnetic coupling b/n two conducting structures is called /or described as mutual inductance
- consider a pair of coils with currents $I_1$ and $I_2$
- then $\Phi_{11}, \Phi_{12}, \Phi_{22}, \Phi_{21}$ produced
- some of the flux ($\Phi_{12}$) produced by $I_1$ links through coil 2
- This flux is common or mutual to both coils




$$\implies M_{12} = \frac{\Lambda_{12}}{I_1} = \frac{N_2\Phi_{12}}{I_1} = \frac{N_2}{I_1}\int \vec{B}_1\cdot d\vec{S}_2$$

Note self inductance is $L_1 = \frac{\Lambda_{11}}{I_1} = \frac{N_1\Phi_{11}}{I_1}$ — $\Phi_1$

Similarly $M_{21} = \frac{\Lambda_{21}}{I_2} = \frac{N_1\Phi_{21}}{I_2}$

$$L_2 = \frac{\Lambda_{22}}{I_2} = \frac{N_2\Phi_{22}}{I_2}$$

## 5-8: Magnetic Energy in Inductor

* Recal Cap: $W_e = \frac{1}{2} C V^2$
* likewise L: $W_m = \frac{1}{2} L I^2$

$$W_m = \frac{1}{2}\int_V \vec{B}\cdot\vec{H}\,dV = \frac{1}{2}\int \mu H^2\,dV \quad \text{since } B = \mu H$$

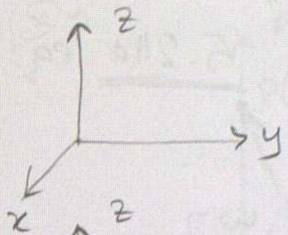
$$W_m = \frac{1}{2}\int \frac{B^2}{\mu}\,dV$$

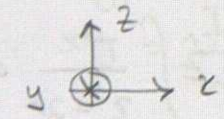
V: volume

$$\Rightarrow \boxed{W_m = \frac{1}{2}\frac{B^2}{\mu}V, \quad W_m = \frac{1}{2}\mu H^2 V \quad [J]}$$



5.10

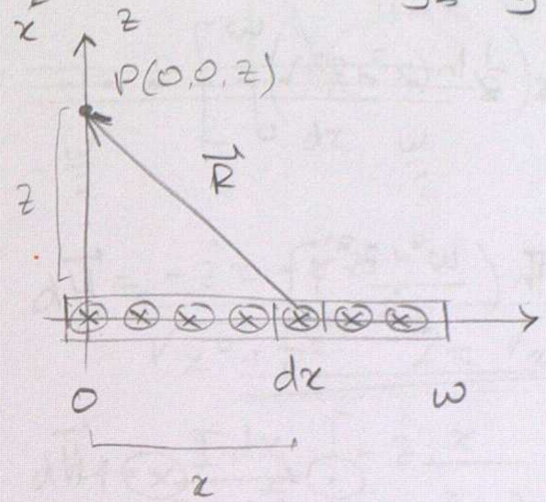



rotate so $\hat{y}$ into page

$$\boxed{\vec{H} = \frac{I}{2\pi d}\hat{\phi}}$$

$\vec{J}_s = \hat{y}5$ [A/m]




Biot-Savart: $d\vec{l}\times\vec{R} = dl(\vec{a}\times\vec{R})$

$\vec{R} = -x\hat{x} + z\hat{z}$

$d\vec{l}\times\vec{R}$ ← direction of $\vec{H}$

$|\vec{R}| = \sqrt{x^2+z^2}$

$d\vec{l} = \hat{a}\,dl$

$\vec{R} = \hat{r}R$, $\hat{r} = \frac{\vec{R}}{|\vec{R}|}$

$$d\vec{l}\times\vec{R} = dlR(\vec{a}\times\vec{r})$$

$$= dlR\left(\hat{y}\times\frac{-x\hat{x}+z\hat{z}}{\sqrt{x^2+z^2}}\right)$$

$$\therefore d\vec{H} = \frac{\hat{z}x+\hat{x}z}{\sqrt{x^2+z^2}}\cdot\frac{I}{2\pi R}$$

R found

$$\hat{\phi} = \hat{y}\times\frac{-x\hat{x}+z\hat{z}}{\sqrt{x^2+z^2}} = \frac{\hat{z}x+\hat{x}z}{\sqrt{x^2+z^2}}$$

$$d\vec{H} = \frac{\hat{z}x+\hat{x}z}{\sqrt{x^2+z^2}}\cdot\frac{I}{2\pi\sqrt{x^2+z^2}} = \frac{I}{2\pi}\left[\hat{z}\frac{x}{x^2+z^2} + \hat{x}\frac{z}{z^2+z^2}\right]$$

and $I = |J_s|\,dx = 5\,dx$

$$d\vec{H} = \frac{5\,dx}{2\pi}\left[\hat{z}\frac{x}{z^2+\hat{z}} + \hat{x}\frac{z}{x^2+z^2}\right]$$

$$\vec{H} = \int_0^w \frac{5\,dx}{2\pi}\left[\hat{z}\frac{x}{x^2+z^2} + \hat{x}\frac{z}{z^2+z^2}\right]$$

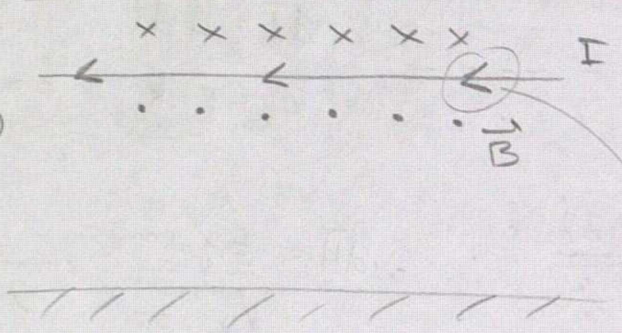
☆ why $\vec{H}$ vs $\vec{B}$? what about Biot Savart

$$\vec{H} = \frac{|\vec{J}_s|}{2\pi}\left[\hat{x}\int_0^w \frac{z}{x^2+z^2}\,dx + \hat{z}\int_0^w \frac{x}{x^2+z^2}\,dx\right]$$ 5.24a eq:

$$= \frac{|\vec{J}_s|}{2\pi}\left[\hat{x}\left(\frac{1}{z}\arctan\left(\frac{x}{z}\right)\right)_0^w + \hat{z}\left(\frac{1}{2}\ln(x^2+z^2)\right)_0^w\right]$$

$$= \frac{|\vec{J}_s|}{2\pi}\left[\frac{\hat{x}}{z}\arctan\left(\frac{w}{z}\right) + \frac{\hat{z}}{2}\ln\left(\frac{w^2+z^2}{z^2}\right)\right]$$

5.13




⊙S ⊗N

$B = (20-15)\ \mu T$

$B = \frac{\mu_0 I}{2\pi d}$

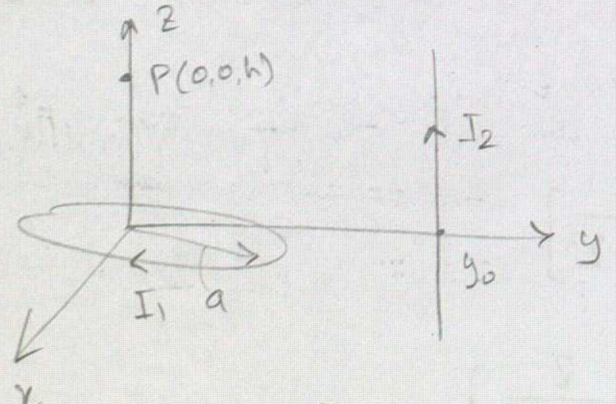
direction not relevant

$\therefore I = \frac{2\pi d B}{\mu_0} = \frac{2\pi(8)(5)}{\mu_0}$

$= 200$ A current

5.15 a)

eq.5.34



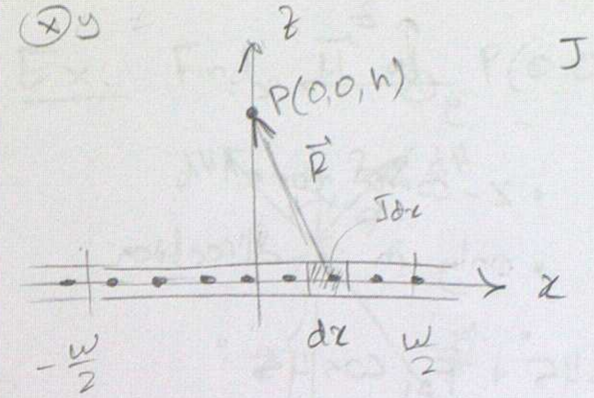

5.34 $\vec{H} = \hat{z}\frac{Ia^2}{2(a^2+z^2)^{3/2}}$

$\therefore \vec{H}_1 = -\hat{z}\frac{I_1 a^2}{2(a^2+h^2)^{3/2}}$

$\vec{H}_2 = \hat{x}\frac{I_2}{2\pi y_0}$

$\therefore \vec{H}(0,0,h) = \vec{H}_1 + \vec{H}_2$

5.18 a)




$J = \frac{I}{w} \Rightarrow I_x = J\,dx = \frac{I}{w}dx$

$\vec{R} = -x\hat{x} + z\hat{z}$

$R = \sqrt{x^2+z^2}$

$H = \frac{I}{2\pi R}$

$\hat{a}\times\hat{R} = -\hat{y}\times\frac{\vec{R}}{|\vec{R}|} = -\hat{y}\times\frac{-x\hat{x}+z\hat{z}}{\sqrt{x^2+z^2}} = \frac{-\hat{z}x - \hat{x}z}{\sqrt{x^2+z^2}}$

x z y

$$d\vec{H} = \frac{-\hat{z}x - \hat{x}z}{\sqrt{x^2+z^2}}\cdot\frac{I_x}{2\pi\sqrt{x^2+z^2}}$$

☆

$$d\vec{H} = \frac{I\,dx}{2\pi w}\left[-\hat{z}\frac{x}{x^2+z^2} - \hat{x}\frac{z}{z^2+z^2}\right]$$

$$\vec{H} = \frac{-I}{2\pi w}\left[\hat{z}\int_{-w/2}^{w/2}\frac{x}{x^2+z^2}dx + \hat{x}z\int_{-w/2}^{w/2}\frac{1}{x^2+z^2}dx\right]$$

$$= \frac{-I}{2\pi w}\left[\hat{x}z\left(\frac{1}{z}\arctan\left(\frac{x}{z}\right)\right)_{-\frac{w}{2}}^{w/2} + \hat{z}\frac{1}{2}\ln(x^2+z^2)_{-w/2}^{w/2}\right]$$

$$= \frac{-I}{2\pi w}\left[\hat{x}\left(\arctan\frac{w}{2z} + \arctan\frac{w}{2z}\right) + \hat{z}(\ \dots\right.$$

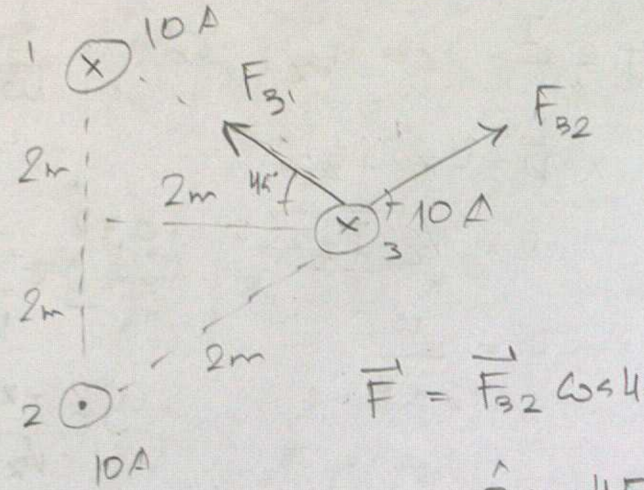
$$\vec{H} = -\hat{x}\frac{I}{\pi w}\text{atan}\left(\frac{w}{2z}\right) \xrightarrow{\text{plug } z=h} \vec{H} = -\hat{x}\frac{I}{\pi w}\arctan\left(\frac{w}{2h}\right)$$

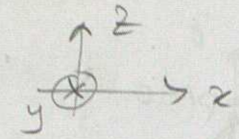
b) $\vec{B} = -\hat{x}\frac{I\mu_0}{\pi w}\arctan\left(\frac{w}{2h}\right)$, $\vec{F} = I\vec{l}\times\vec{B}$, $\frac{d\vec{F}}{dl} \Rightarrow d\vec{F} = I\,d\vec{l}\times\vec{B}$

$dl = -\hat{y}\,dl$, $d\vec{F} = I\,dl(-\hat{y})\times\vec{B}$

$$\frac{d\vec{F}}{dl} = -I(\hat{y}\times\vec{B}) = -I\left[\hat{y}\times -\hat{x}\frac{\mu_0 I}{\pi w}\arctan\left(\frac{w}{2h}\right)\right] = -\hat{z}\frac{\mu_0 I^2}{\pi w}\arctan\left(\frac{w}{2h}\right)$$

5.19:




• x-comp cancel

• only in z-direction

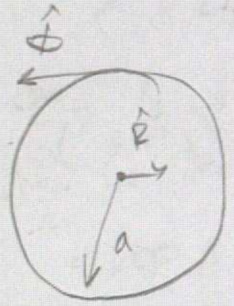
$$\vec{F} = \vec{F}_{32}\cos 45^\circ + \vec{F}_{31}\cos 45^\circ$$

$$= \hat{z}\cos 45 \times 2\left(\frac{\mu_0 I I_3}{2\pi R}\right), \quad R = R_1 = R_2 \; (2\sqrt{2}), \quad I = I_1 = I_2 \; (10)$$

$$\vec{F} = \hat{z}\cos 45^\circ\left(\frac{\mu_0 I^2}{2\pi R} + \frac{\mu_0 I^2}{2\pi R}\right)$$

$$= \frac{\hat{z}\cos 45^\circ \mu_0 I^2}{\pi R} = \underline{\hat{z}\,(10)^{-5} \; [N/m]}$$

5.25




$$\oint \vec{H}\cdot d\vec{l} = I_{enc}$$

$$d\vec{l} = a\,d\phi\,\hat{\phi}$$

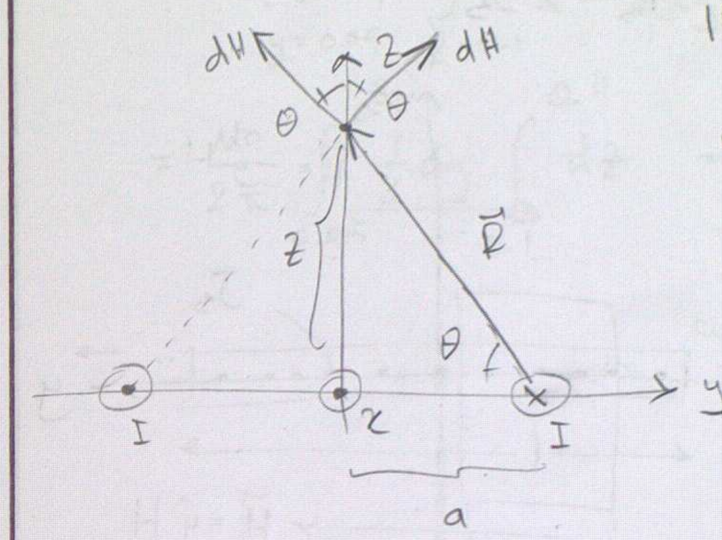
$$I_{enc} = \int_0^{2\pi} \hat{\phi}\,\frac{2}{r}\left[1-(4r+1)e^{-4r}\right]\cdot\hat{\phi}\,a\,d\phi$$

$$= \frac{4\pi a}{r}\left[1-(4r+1)e^{-4r}\right] \quad \text{for } r = 0.05$$

$$I = 4\pi\left(1-(4\times 0.05+1)e^{-4(0.05)}\right) = 0.22A$$

Week 9: Lecture 1 — Mar 11, 2025

Ex: Find $\vec{H}$ at $P(0,0,z)$




$$|\vec{R}| = \sqrt{a^2+z^2} \qquad \sin\frac{\pi}{2} = 1$$

$$d\vec{H} = \frac{I\,d\vec{l}\times\hat{R}}{4\pi R^2}, \quad dH = \frac{I\,dl\,(\sin\alpha)}{4\pi R^2}$$

$$dH = \frac{I\,dl}{4\pi(a^2+z^2)}$$ , also, radial (x-y plane) comp cancel out

$$d\vec{H} = \hat{z}\,dH_z = \hat{z}\,dH\cos\theta$$

$$d\vec{H} = \hat{z}\,\frac{I\cos\theta}{4\pi(a^2+z^2)}\,dl \leftarrow dl = a\,d\phi$$

$$\vec{H} = \hat{z}\int\frac{I\cos\theta}{4\pi(a^2+z^2)}\,dl = \hat{z}\,\frac{I\cos\theta}{4\pi(a^2+z^2)}\int_0^{2\pi} a\,d\phi$$

$$= \hat{z}\,\frac{Ia}{2(a^2+z^2)}\cos\theta, \quad \text{but } \cos\theta = \frac{a}{\sqrt{a^2+z^2}} = \frac{a}{R}$$

$$= \underline{\hat{z}\,\frac{Ia^2}{2(a^2+z^2)^{3/2}} \; [A/m]} \quad @\ P(0,0,z)$$

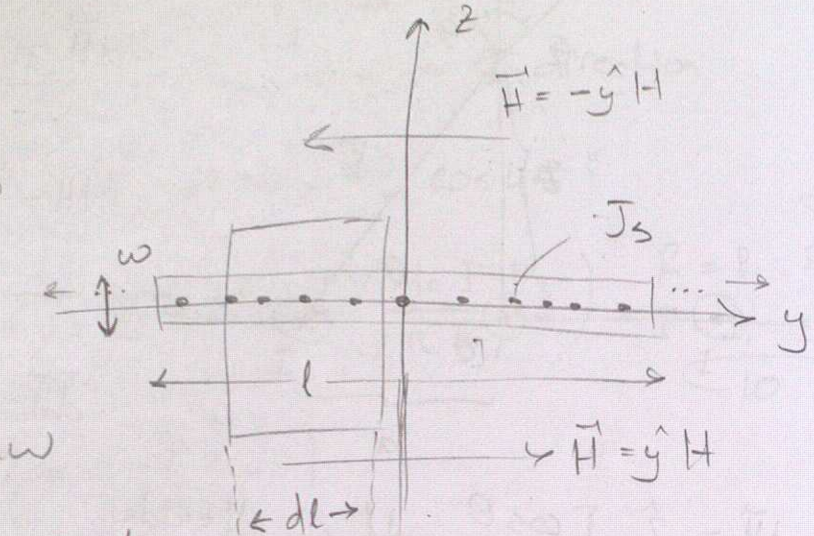
@ (0,0,0): $\underline{\vec{H} = \hat{z}\,\dfrac{I}{2a} \; [A/m]}$

@ $z^2 >> a^2$ (ie very far)

$$\underline{\vec{H} = \hat{z}\,\frac{Ia^2}{2|z|^3} \; [A/m]}$$

Ex: Find magnetic field of infinite current sheet where $\vec{J}_s = \hat{x} J_s$, $\vec{H} = ?$

Direction of $\vec{H}$ found by R.H.R




$$\vec{H} = \begin{cases} -\hat{y} H, & z > 0 \\ +\hat{y} H, & z < 0 \end{cases}$$

Apply Ampere's law

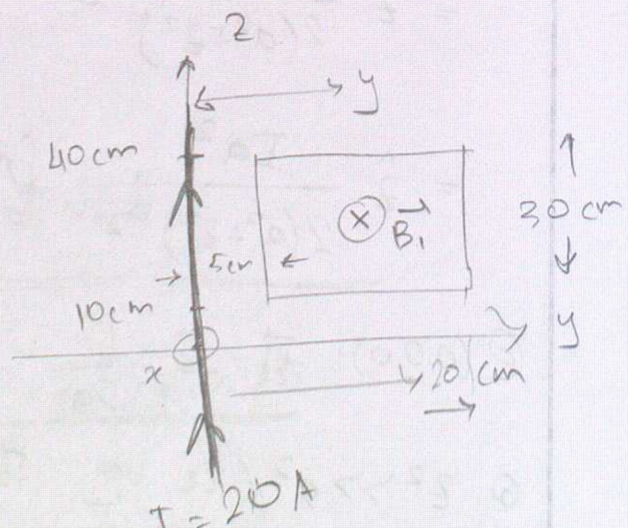
$$\oint \vec{H} \cdot d\vec{l} = I_{enc} = J_s l$$

$$H(2l) = J_s l \Rightarrow \vec{H} = \begin{cases} -\hat{y} \dfrac{J_s}{2}, & z > 0 \\ +\hat{y} \dfrac{J_s}{2}, & z < 0 \end{cases}$$

Ex: Find mutual inductance b/n wire & loop

Let ① be wire
② be loop

$N_2 = 1$
$I_1 = I = 20A$




$$M_{12} = \frac{N_2}{I_1} \int_{S_2} \vec{B}_1 \cdot d\vec{s}_2$$

$$\vec{B}_1 = -\hat{x} \frac{\mu_0 I_1}{2\pi y}$$

$$d\vec{s}_2 = -\hat{x}\, dy\, dz$$

$$M_{12} = \frac{1}{I_1} \int_{y=0.05}^{0.20} \int_{z=0.1}^{z=0.4} \left(-\hat{x} \frac{\mu_0 I_1}{2\pi y}\right) \cdot \left(-\hat{x}\, dy\, dz\right)$$

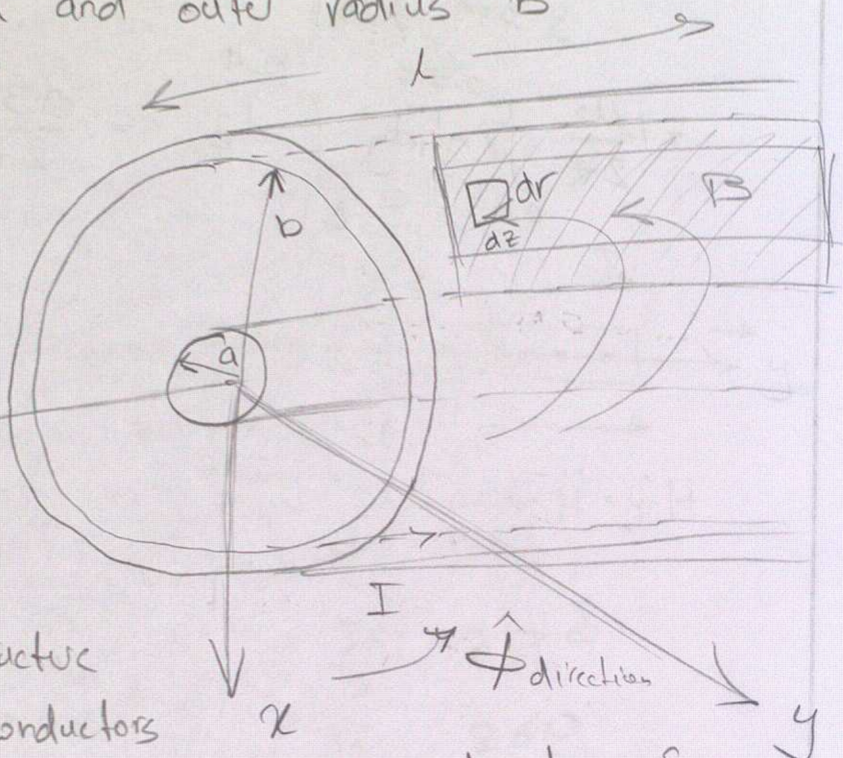
$$= \frac{\mu_0}{2\pi} \int_{0.05}^{0.2} \frac{1}{y} dy \int_{0.1}^{0.4} dz = \frac{0.3 \mu_0}{2\pi} \left(\ln y\right)_{0.05}^{0.2} = \underline{83\ nH}$$

Week 9: Lecture 2 Mar 12, 2025

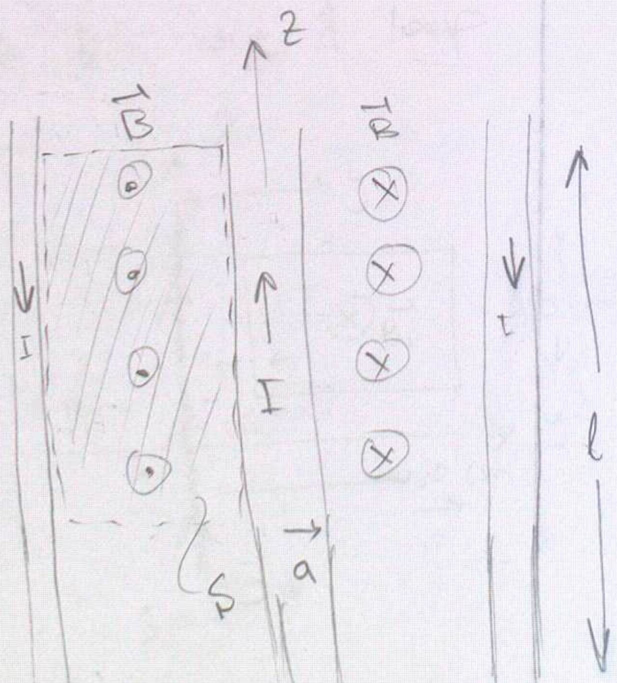
Ex: Determine self-inductance of a coaxial cable of inner radius a and outer radius b

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I}$$

$$\Phi = \int \vec{B} \cdot d\vec{s}$$

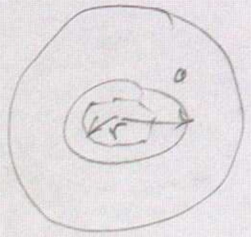



Note that the flux linkage for the structure with two separate conductors relates to the flux $\Phi_m$ through a closed surface between 2 conductors




① Apply Ampere's Law

$$\oint \vec{H} \cdot d\vec{l} = I_{enc}$$




* for $0 \le r \le a$ :1

$$I_{enc} = I \frac{\pi r^2}{\pi a^2} = I \frac{r^2}{a^2}$$

$$\vec{H} = \frac{I r^2}{a^2} \frac{1}{2\pi r} = \frac{I}{2\pi a^2} r \phi$$

$$\therefore \vec{B}_1 = \frac{\mu I}{2\pi a^2} r \hat{\phi} \ [T]$$

$$d\Phi_1 = B_1 ds = B_1 dr dz = \frac{\mu I}{2\pi a^2} r dr dz$$

* Flux linkage is $d\Phi_1$ multiplied by the ratio of the area within the path enclosing the flux to the total area

$$d\Lambda_1 = d\Phi_1 \frac{I_{enc}}{I} = d\Phi_1 \frac{\cancel{\pi} r^2}{\cancel{\pi} a^2} = \frac{\mu I}{2\pi a^2} r dr dz \frac{r^2}{a^2}$$

$$\Lambda_1 = \int_{r=0}^{a} \int_{z=0}^{l} \frac{\mu I}{2\pi a^4} r^3 dr dz = \frac{\mu I}{2\pi a^4}\left[\frac{1}{4} a^4 l\right] = \frac{\mu I}{8\pi} l \ [Wb]$$

$$L_{in} = \frac{\Lambda_1}{I} = \frac{\mu l}{8\pi} \ [H]$$

* $a \le r \le b$ : $\oint \vec{H} \cdot d\vec{l} = I_{enc} \rightarrow H(2\pi r) = I \rightarrow \vec{H} = \frac{I}{2\pi r} \hat{\phi} \ [A/m]$

$$\therefore \vec{B}_2 = \frac{\mu I}{2\pi r} \hat{\phi} \ [T]$$

$$d\Phi_2 = B_2 ds = \frac{\mu I}{2\pi r} dr dz$$

$$d\Lambda_2 = \frac{\mu I}{2\pi r} dr dz \rightarrow \Lambda_2 = \int_{r=a}^{b} \int_{0}^{l} \frac{\mu I}{2\pi r} dr dz = \frac{\mu I l}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$L_{ext} = \frac{\Lambda_2}{I} = \frac{\mu l}{2\pi} \ln\left(\frac{b}{a}\right) \ [H]$$

$\therefore$ total $L = L_{in} + L_{ext}$

$$\boxed{L = \frac{\mu l}{2\pi}\left[\frac{1}{4} + \ln\frac{b}{a}\right] \ [H]}$$

## Method 2

Knowing: $W_m = \frac{1}{2} L I^2 = \frac{1}{2}\int \vec{B}\cdot\vec{H}\, dv = \frac{1}{2}\int \frac{B^2}{\mu} dv$

$$L_{in} = \frac{2W_m}{I^2} = \frac{2}{I^2}\cdot\frac{1}{2}\int \frac{B_1^2}{\mu} dv$$

$$= \frac{1}{I^2\mu}\int_0^{l}\int_0^{2\pi}\int_0^{a}\left(\frac{\mu I r}{2\pi a^2}\right)^2 r\,dr\,d\phi\,dz$$

$$= \frac{\mu l}{8\pi} \quad [H]$$

$$L_{ext} = \frac{2}{I^2}\int \frac{B_2^2}{2\mu} dv = \frac{1}{\mu I^2}\int_0^{l}\int_0^{2\pi}\int_a^{b}\left(\frac{\mu I}{2\pi r}\right)^2 r\,dr\,d\phi\,dz$$

$$L_{ext} = \frac{\mu}{4\pi^2}(l)(2\pi)\ln(b/a) = \frac{\mu l}{2\pi}\ln(b/a)$$

$$L = \frac{\mu l}{2\pi}\left(\frac{1}{4} + \ln(b/a)\right) \quad [H]$$

irl, $a \ll b$, so ln term dominates and we usually ignore contribution of inner

Week 9: Lecture 3 — Mar 14, 2025

## Ch. 6: Dynamic Fields

* so far, concentrated on static EM fields where electrostatic fields ($\vec{E}$) and magnetostatic fields ($\vec{H}$) are independent of each other

$Q \rightarrow \vec{E}$ if $\frac{dQ}{dt}$ ⇒ $\vec{E}$ changes ⇒ then they couple together and travel as electromagnetic waves

$I \rightarrow \vec{H}$ if $\frac{dI}{dt}$ ⇒ H changes

## Maxwell's Equations

| | Point Form | Integral Form |
|---|---|---|
| Gauss's Law: | $\nabla\cdot\vec{D} = \rho_v$ | $\oint \vec{D}\cdot d\vec{s} = Q_{enc}$ |
| Gauss's Law for mag: | $\nabla\cdot\vec{B} = 0$ | $\oint \vec{B}\cdot d\vec{s} = 0$ |
| Faraday's Law: | $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ | $\oint \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{s}$ |
| Ampere's Law: | $\nabla\times\vec{H} = \vec{J}_c + \frac{\partial\vec{D}}{\partial t}$ | $\oint \vec{H}\cdot d\vec{l} = \int_s \left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)\cdot d\vec{s}$ |

Lorentz Force Equation: $\vec{F} = q(\vec{E} + \vec{u}\times\vec{B})$

Constitutive Relations: $\begin{cases}\vec{D} = \varepsilon\vec{E}\\ \vec{B} = \mu\vec{H}\\ \vec{J} = \sigma\vec{E}\end{cases}$

* stationary charges $\longrightarrow$ electrostatic field
* steady current $\longrightarrow$ magnetostatic field
* time varying currents $\longrightarrow$ electromagnetic fields

## 2 Major Concepts

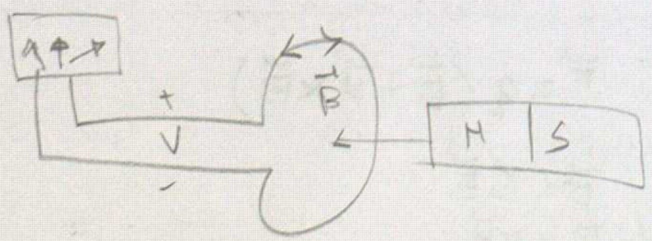
1) Electromotive force (emf) based on Faraday's experiments
2) Displacement current that resulted from Maxwell's hypothesis

## 6-1: Faraday's Law

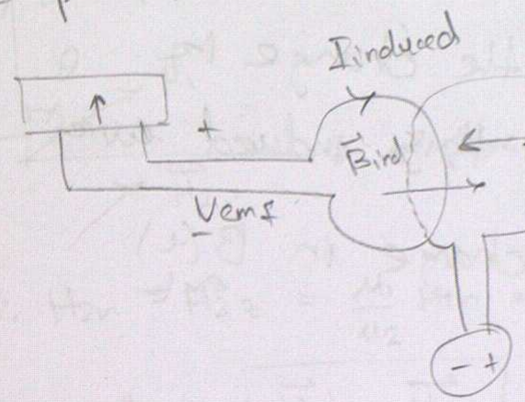
* connection b/n electr. and magn.
* Orstead established connection b/n electr. and magn.
* Faraday hypothesized that if I produces B, then B should produce I called $I_{induced}$

### Experiment

* magnetic field can produce I in a closed loop only if B changes, ie: $\frac{\partial B}{\partial t}$



* A voltage is detected called electromotive force (emf)
* the process is called electromagnetic induction



* a time-varying magnetic field produces an induced voltage called emf in a closed circuit that causes a current $I_{induced}$ to flow

$$\boxed{V_{emf} = -N\frac{\partial \Phi}{\partial t} = -N\frac{\partial}{\partial t}\int_S \vec{B}\cdot d\vec{s} \quad [V]}$$

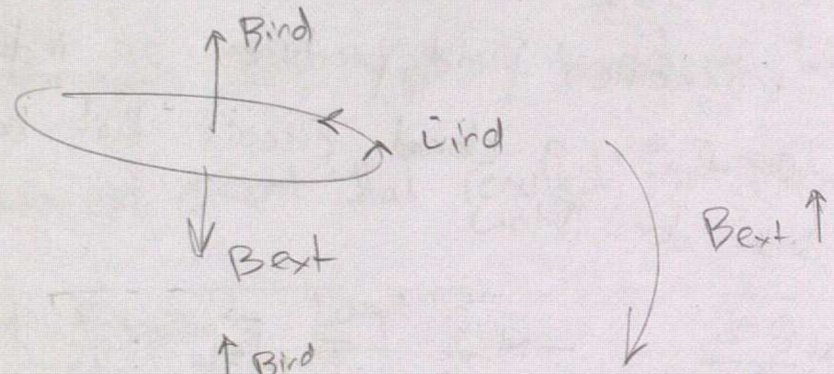
* negative sign shows the resulting emf opposes the generatory magnetic field (comes from Lenz's law)
* variation of flux with time may be achieved in 3 ways:

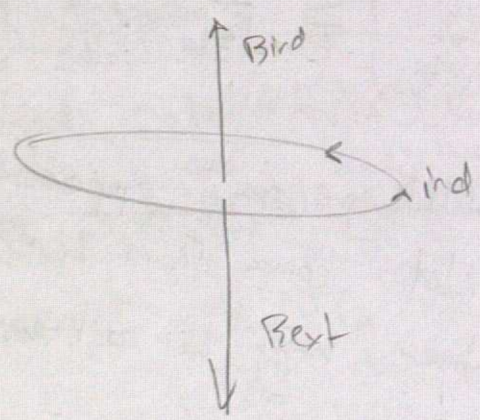
1) by having a stationary loop in a time-varying $\vec{B}$ called transformer emf $\longrightarrow$ shown $V_{ems}^{tr}$
2) by having a time-varying loop area in static $\vec{B}$ called motional emf $\longrightarrow$ shown $V_{ems}^{m}$
3) by having a time-varying loop area in a time varying $\vec{B}$

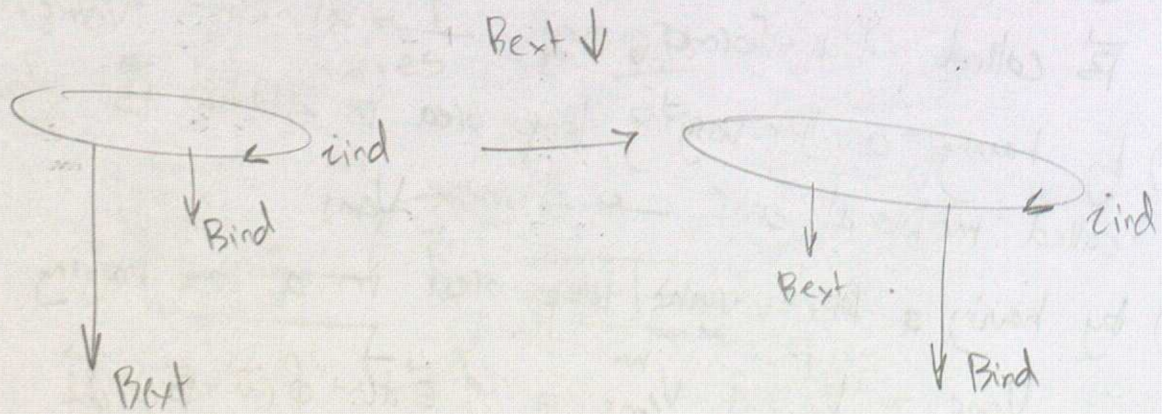
$$V_{emf} = V_{ems}^{tr} + V_{ems}^{m} = \oint_C \vec{E}\cdot d\vec{l} + \oint_C (\vec{u}\times\vec{B})\cdot d\vec{l}$$

## Lenz's Law

* induced current direction is s.t. its magnetic field opposes the change in magnetic flux inducing this induced current
* Note: $B_{ind}$ opposes the change in $B_{ext}(t)$

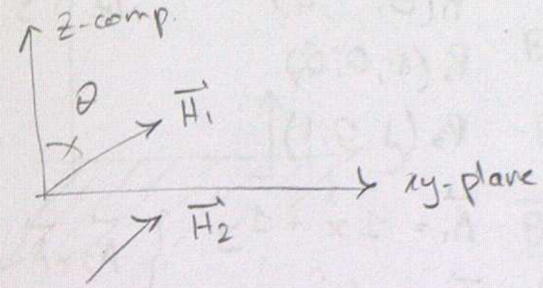



## Problem 5.32

a)




$\vec{H}_{1n} = H_{1z}\hat{z} \Rightarrow H_{1n} = H_{1z}$

$H_{1t} = H_{1x}\hat{x} + H_{1y}\hat{y} \Rightarrow H_{1t} = \sqrt{H_{1x}^2 + H_{1y}^2}$

from theory: $\boxed{\vec{B}_{1n} = \vec{B}_{2n}}$

$\mu_1 \vec{H}_{1n} = \mu_2 \vec{H}_{2n}$

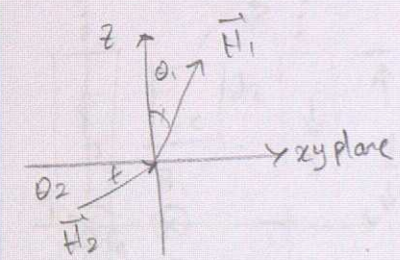
$\therefore H_{2n} = H_{2z} = \frac{\mu_1}{\mu_2} H_{1n} = \frac{\mu_1}{\mu_2} H_{1z}$ //

from theory: $\boxed{\vec{H}_{1t} = \vec{H}_{2t}}$ since no $J_s$

$\therefore \hat{x}H_{1x} + \hat{y}H_{1y} = \hat{x}H_{2x} + \hat{y}H_{2y} \Rightarrow H_{2x} = H_{1x},\ H_{2y} = H_{1y}$ //

$\therefore \vec{H}_2 = \hat{x}H_{1x} + \hat{y}H_{1y} + \hat{z}\frac{\mu_1}{\mu_2}H_{1z}$

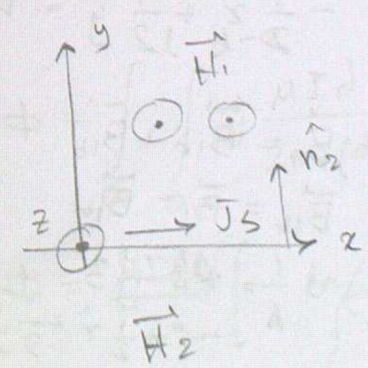
b)




$\tan\theta_1 = \frac{|\vec{H}_{1t}|}{|\vec{H}_{1n}|} = \frac{\sqrt{H_{1x}^2 + H_{1y}^2}}{H_{1z}}$ //

$\tan\theta_2 = \frac{|\vec{H}_{2t}|}{|\vec{H}_{2n}|} = \frac{\sqrt{H_{1x}^2 + H_{1y}^2}}{\frac{\mu_1}{\mu_2}H_{1z}}$ //

## Problem 5.33




from theory: $\boxed{\hat{n}_2 \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s}$

⊥ to interface, from mat 2 to mat 1 $\therefore \hat{n}_2 = \hat{y}$

$\vec{J}_s = \hat{x}8$

$\Rightarrow \hat{y}\times(\hat{z}11 - \vec{H}_2) = \hat{x}8$

$\hat{x}11 - \hat{y}\times\vec{H}_2 = \hat{x}8$

$\hat{x}3 = \hat{y}\times\vec{H}_2$

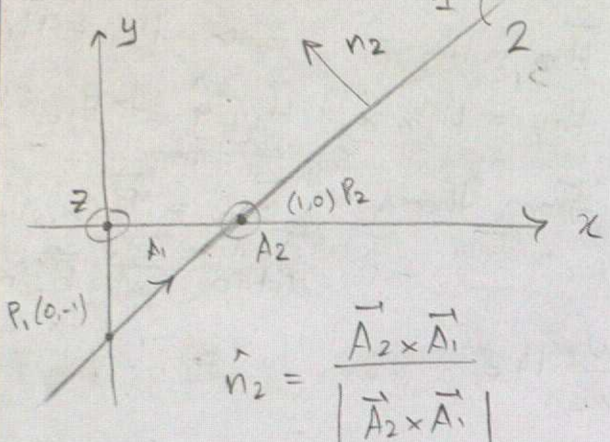
$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 1 & 0 \\ H_{2x} & H_{2y} & H_{2z} \end{vmatrix} = \hat{x}H_{2z} - \hat{z}H_{2x}$

$\hat{x}3 = \hat{x}H_{2z} - \hat{z}\cancelto{0}{H_{2x}}$

$\therefore H_{2z} = 3$

$\therefore \vec{H}_2 = \hat{z}3$

## Problem 5.34




$P_1(0,-1,0)$

$P_2(1,0,0)$

$P_3(1,0,1)$

$$\left.\begin{aligned}\vec{A}_1 &= 1\hat{x} + 1\hat{y}\\ \vec{A}_2 &= 1\hat{z}\end{aligned}\right\}\ \vec{A}_2\times\vec{A}_1 = \vec{n}_2$$

$$\hat{n}_2 = \frac{\vec{A}_2\times\vec{A}_1}{|\vec{A}_2\times\vec{A}_1|}$$

$$\vec{A}_2\times\vec{A}_1 = \hat{z}\times(\hat{x}+\hat{y}) = \hat{y}-\hat{x} \Rightarrow \hat{n}_2 = -\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y}\ //$$

$$|\vec{A}_2\times\vec{A}_1| = \frac{1}{\sqrt{2}}$$

from theory: $\boxed{\vec{B}_{1n} = \vec{B}_{2n}}$ ① $\boxed{\vec{H}_{1t} = \vec{H}_{2t}}$ ② no $J_s$

① $\Rightarrow B_{1n} = B_{2n}$ dot with dir. to get that comp. of $\vec{B}_1$

$$B_{1n} = \hat{n}_2\cdot\vec{B}_1 = B_{2n}$$

$$= \left(-\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y}\right)\cdot(\hat{x}2+\hat{y}3)$$

$$= -\frac{2}{\sqrt{2}} + \frac{3}{\sqrt{2}} = \frac{1}{\sqrt{2}} = B_{2n}$$

also, $\hat{n}_2 = \frac{\vec{B}_{2n}}{|\vec{B}_{2n}|} \Rightarrow \vec{B}_{2n} = \hat{n}_2|\vec{B}_{2n}| = -\frac{1}{2}\hat{x} + \frac{1}{2}\hat{y} = \vec{B}_{2n}$ //

② $\vec{H}_{1t} = \vec{H}_{2t}$

$$\frac{\vec{B}_{1t}}{\mu_1} = \frac{\vec{B}_{2t}}{\mu_2} \rightarrow \vec{B}_{2t} = \frac{\mu_2}{\mu_1}\vec{B}_{1t}$$

also, $\vec{B}_1 = \vec{B}_{1n} + \vec{B}_{1t}$

$$\vec{B}_{1t} = \vec{B}_1 - \vec{B}_{1n}$$

$$= \hat{x}2 + \hat{y}3 - \left(-\frac{1}{2}\hat{x} + \frac{1}{2}\hat{y}\right)$$

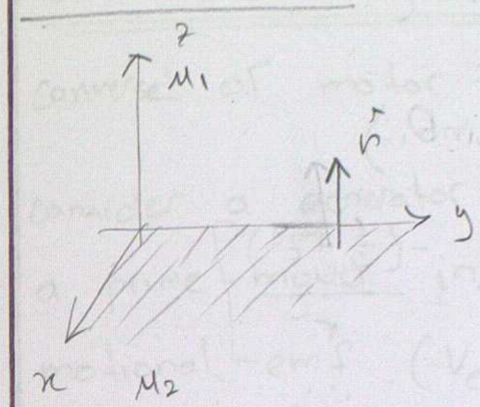
$$= 2.5\hat{x} + 2.5\hat{y}$$

$$\vec{B}_{2t} = \frac{\mu_2}{\mu_1}(2.5\hat{x} + 2.5\hat{y})$$

$$\vec{B}_{2t} = \frac{1}{2}\hat{x} + \frac{1}{2}\hat{y}\ //$$

$$\underline{\vec{B}_2 = \vec{B}_{2n} + \vec{B}_{1n} = 1\hat{y}}$$

## Problem 5.35




$\vec{B}_{1n} = \vec{B}_{2n}$, $\vec{H}_{1t} = \vec{H}_{2t}$

$B_{1n} = B_{1z}$

$B_{2n} = B_{2z} = B_{1z} = B_{1n} = 8 = B_{2n}$ //

$$\frac{\vec{B}_{1t}}{\mu_1} = \frac{\vec{B}_{2t}}{\mu_2} \Rightarrow \vec{B}_{2t} = \frac{\mu_2}{\mu_1}\vec{B}_{1t}$$

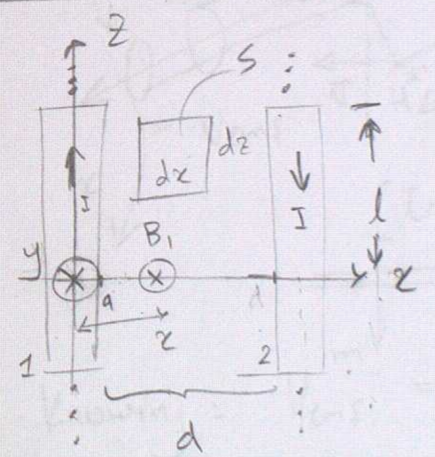
$$= \frac{\mu_2}{\mu_1}(\hat{x}4 - \hat{y}6)$$

$$\vec{B}_{2t} = 20k\hat{x} - 30k\hat{y}\ //$$

$$\vec{B}_2 = \vec{B}_{2n} + \vec{B}_{2t}$$

$$\underline{\vec{B}_2 = 20{,}000\hat{x} - 30{,}000\hat{y} + 8\hat{z}}$$

## Problem 5.37




$$\vec{B}_1 = \hat{y}\frac{\mu I}{2\pi x} \qquad \vec{B}_2 = \hat{y}\frac{\mu I}{2\pi(d-x)}$$

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \hat{y}\frac{\mu I}{2\pi}\left[\frac{1}{x} + \frac{1}{d-x}\right]$$

$$\vec{B} = \hat{y}\frac{\mu I d}{2\pi x(d-x)} \hookrightarrow \Phi = \iint_S \vec{B}\cdot d\vec{s} = ?$$

$$d\vec{s} = \hat{y}\,dx\,dz$$

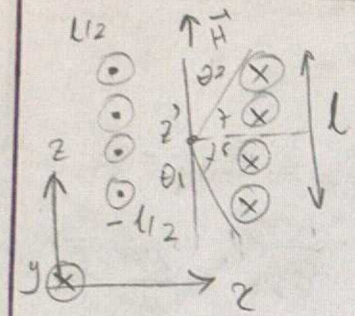
PFD

$$\Phi = \int_0^l\int_a^{d-a}\hat{y}\frac{\mu I d}{2\pi x(d-x)}\cdot\hat{y}\,dx\,dz = \frac{\mu I d l}{2\pi}\int_a^{d-a}\frac{1}{x(d-x)}dx$$

$$\Phi = \frac{\mu I d l}{2\pi}\left[\frac{1}{d}\ln\left(\frac{x}{d-x}\right)\right]_a^{d-a} = \frac{\mu I l}{\pi}\ln\left(\frac{d-a}{a}\right)\ //$$

$$L = \frac{\Phi}{I} = \frac{\mu l}{\pi}\ln\left(\frac{d-a}{a}\right) \Rightarrow L' = \frac{L}{l} = \frac{\mu}{\pi}\ln\left(\frac{d-a}{a}\right)\ [\mathrm{H/m}]$$

Problem 5.38 (eq. 5.89)




$$\vec{H} = \hat{z}\,\frac{nI}{2}\left(\sin\theta_2 - \sin\theta_1\right)$$

$$= \hat{z}\,\frac{nI}{2}\left[\frac{\frac{l}{2}-z'}{r} - \frac{-\left(\frac{l}{2}+z'\right)}{r}\right]$$

$$|\vec{H}| = \frac{nI}{2}\left[\frac{\frac{l}{2}-z'}{r} + \frac{\frac{l}{2}+z'}{r}\right] \Rightarrow$$

Problem 5.39 (eq. 5.99)

$$L' = \frac{\mu_0}{2\pi}\ln\left(\frac{l}{a}\right) = \frac{L}{l} \Rightarrow L = \frac{\mu_0 l}{2\pi}\ln\left(\frac{l}{a}\right)$$

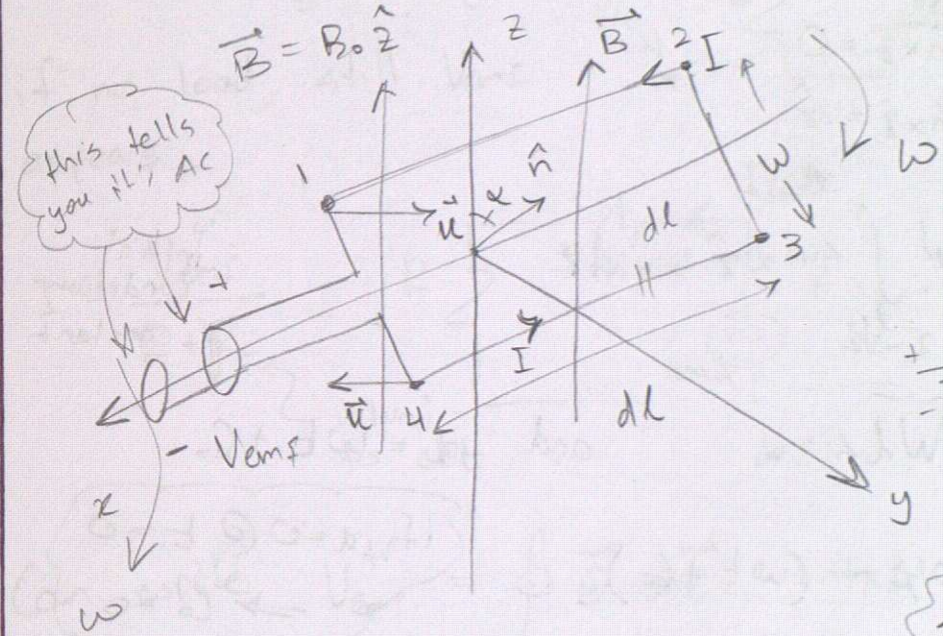
$L = 416\,nH$ and energy $= \frac{1}{2}LI^2$

$$W = \frac{1}{2}(416n)(I^2) = 208\,I^2\ [nJ]$$

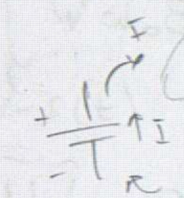
Problem 5.40

## 6.5 : Electromagnetic Generators

- Converse of motor
- consider a generator where a loop is rotating by a prime-mover in a magnetic field c.w., then motional emf ($V^m_{emf}$) is produced in the loop




$\alpha$ is angle b/n $\hat{n}$ and $\hat{z}$




current goes from (-)ve to (+)ve sive this is generator

segments 2-3 and 1-4 do not contribute

Knowing : $V^m_{emf} = \oint_L (\vec{u}\times\vec{B})\cdot d\vec{l}$

segments $\begin{cases} 1\text{-}2 \\ 3\text{-}4 \end{cases} \Rightarrow \begin{matrix} \vec{u} = \hat{n}\,\omega\frac{W}{2} \\ \vec{u} = -\hat{n}\,\omega\frac{W}{2} \end{matrix}$

$\vec{V} = r\omega$

$\vec{u} = \omega\frac{W}{2}\hat{n}$

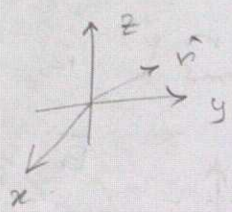
$$V^m_{emf} = \int_2^1 (\vec{u}\times\vec{B})\cdot d\vec{l} + \int_4^3 (\vec{u}\times\vec{B})\cdot d\vec{l}$$

$$= \int_{-l/2}^{l/2}\left(\hat{n}\omega\frac{W}{2}\times\hat{z}B_0\right)\cdot\hat{x}\,dx + \int_{-l/2}^{l/2}\left(-\hat{n}\omega\frac{W}{2}\times\hat{z}B_0\right)\cdot -\hat{x}\,dx$$

$$V_{emf}^{m} = \omega W B_0 \int_{-l/2}^{l/2} (\hat{n}\times\hat{z})\cdot\hat{x}\,dz$$

note: angle b/n is $\alpha$ and is rotating @ $\omega$

$\hat{n}\times\hat{z} = \sin\alpha\,\hat{x}$




if $\alpha = 0 \Rightarrow \hat{n}\times\hat{z} = 0$

$\alpha = 90° \Rightarrow \hat{n}\times\hat{z} = \hat{x}$

$$V_{emf}^{m} = B_0 \omega W \int_{-l/2}^{l/2} \sin\alpha\,\hat{x}\cdot\hat{x}\,dz$$

initial condition constant

$$= B_0\,\omega\,\underbrace{Wl}_{\text{Area}}\sin\alpha \quad \text{and} \quad \alpha = \omega t + C_0$$

$$V_{emf}^{m} = B_0\,\omega A \sin(\omega t + C_0)$$

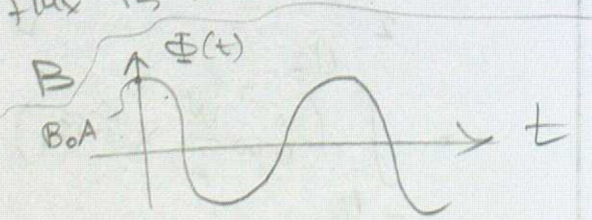
if $\alpha = 0$ @ $t = 0$
$\Rightarrow C_0 = 0$

if $C_0 \Rightarrow$ $\underline{V_{emf}^{m} = B_0\,\omega A \sin(\omega t)}$

## Method 2

Apply Faraday's Law : $V_{emf} = -\dfrac{d\Phi}{dt}$

note: this is the general case and does not care if change in flux is due to change in area or change in B

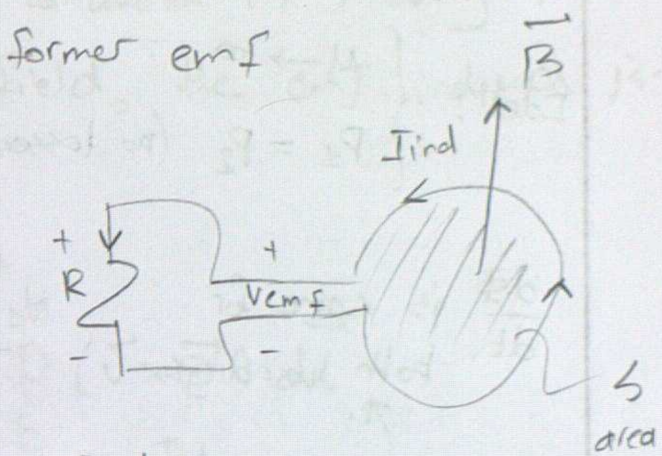



$$V_{emf} = -\frac{d}{dt}\left(\Phi(t)\right)$$

## 6-2: Stationary Loop in a Time Varying Mag Field

* emf is induced when $\vec{B}$ changes
* this emf is called Transformer emf

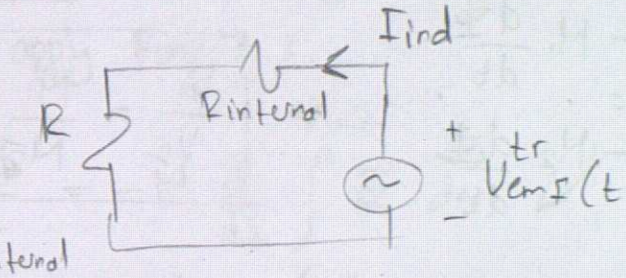
$$V_{emf}^{tr} = -N\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{s}$$




* if no load, still $V_{emf}$ appears

$$I = \frac{V_{emf}^{tr}}{R + R_i}$$

internal to ckt




* Can write $V_{emf}^{tr} = \oint_C \vec{E}\cdot d\vec{l} = -N\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{s}$

* for $N = 1 \Rightarrow \oint_C \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{s}$

* Apply Stoke's:

can equate

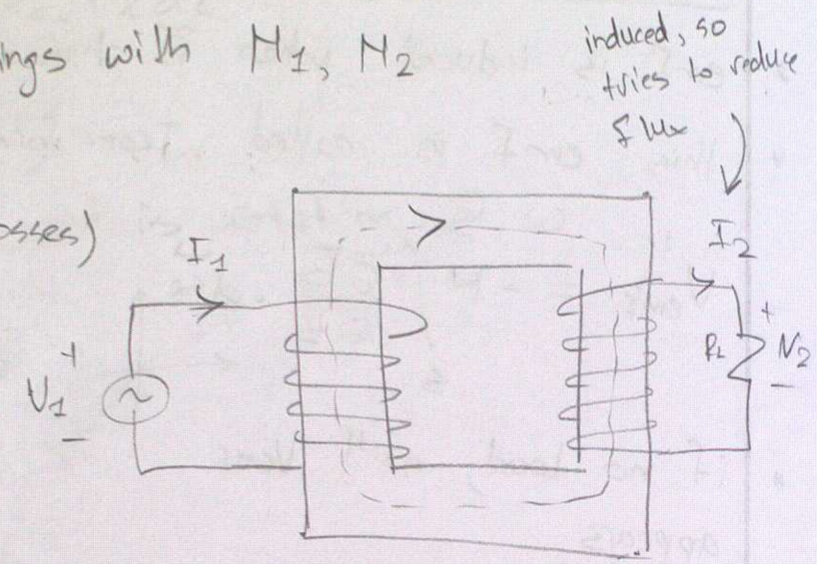
$$\oint_{C=\partial S} \vec{E}\cdot d\vec{l} = \int_S (\nabla\times\vec{E})\cdot d\vec{s} \Rightarrow \int_S (\nabla\times\vec{E})\cdot d\vec{s} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{s}$$

$$\Rightarrow \boxed{\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}} \quad \text{Faraday's Law (differential form)}$$

## 6.3: Ideal Transformer

* has 2 coils of windings with $N_1, N_2$

* Ideal: $\begin{cases} \mu \to \infty \\ P_1 = P_2 \quad \text{(no losses)} \end{cases}$




$\frac{d\Phi}{dt}$ is same for both windings

$$\begin{cases} V_1 = -N_1 \frac{d\Phi}{dt} \\ V_2 = -N_2 \frac{d\Phi}{dt} \end{cases} \longrightarrow \boxed{\frac{V_1}{V_2} = \frac{N_1}{N_2}} \quad \boxed{\frac{I_1}{I_2} = \frac{V_2}{V_1} = \frac{N_2}{N_1}}$$

* want to find relation for resistance

$$R_{in} = \frac{V_1}{I_1} = V_2 \frac{N_1}{N_2} \cdot \frac{1}{I_1} = V_2 \frac{N_1}{N_2} \cdot \frac{1}{I_2 \frac{N_2}{N_1}} = \underbrace{\frac{V_2}{I_2}}_{R_L} \cdot \frac{N_1^2}{N_2^2}$$

$$\Rightarrow \boxed{\frac{R_{in}}{R_L} = \left(\frac{N_1}{N_2}\right)^2}$$

if impedence

$$\boxed{\frac{Z_{in}}{Z_L} = \left(\frac{N_1}{N_2}\right)^2}$$

## 6-6: Moving Conductor in a time-varying mag field

* in a general case where a conductor is moving in a time-varying magnetic field, the emf induced is:

$$V_{emf} = V_{emf}^{tr} + V_{emf}^{m}$$

$$= -N \int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds} + \oint_C (\vec{u} \times \vec{B}) \cdot \vec{dl}$$

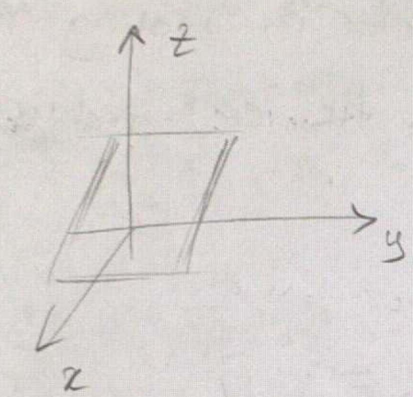
* or can directly apply Faraday's

$$V_{emf} = -N \frac{d\Phi}{dt} = -N \frac{d}{dt} \int_S \vec{B} \cdot \vec{ds}$$

choose which based on case..

Week 11 Tutorial 10 — Mar 24, 2025

6.3 $\boxed{V_{emf} = -N \frac{d\Phi}{dt}}$ $\quad \Phi = \int d\Phi = \int \vec{B} \cdot d\vec{s}$




$d\vec{s} = \hat{z}\, dx\, dy$

a) $\Phi = \int \vec{B}\cdot d\vec{s} = \int \hat{z}\, 20e^{-3t} \cdot \hat{z}\, dx\, dy$

$= 20e^{-3t} \int_{-0.125}^{0.125} dx \int_{-0.125}^{0.125} dy = 20e^{-3t}(0.25)(0.25)$

$= 1.25e^{-3t}$

$V_{emf} = -N\frac{d\Phi}{dt} = -100\left[(-3)(1.25)e^{-3t}\right] = \underline{375e^{-3t}\ [V]}$

b) $\Phi = \int \vec{B}\cdot d\vec{s} = 20\int_{-0.125}^{0.125} \cos x\, dx \int_{-0.125}^{0.125} dy \times \cos 10^3 t$

$= (20)(0.25)(0.25)\cos 10^3 t$

$V_{emf} = -100 \frac{d}{dt}\left[1.25 \cos 10^3 t\right]$

$= \underline{124.6 \sin 10^3 t\ [V]}$

c) $\Phi = \int \vec{B}\cdot d\vec{s} = 20\cos 10^3 t \int_{-0.125}^{0.125} \cos x\, dx \int_{-0.125}^{0.125} \sin 2y\, dy$

$V_{emf} = -100 \frac{d}{dt}\Phi = 0$ $\qquad \underline{\therefore V_{emf} = 0\ [V]}$

6.4 $V_{emf} = 5A \times 0.5\Omega = 2.5V$ $\qquad \underline{\therefore I_{new} = 1A}$

$I = \frac{V_{emf}}{2+0.5} = \frac{2.5V}{2.5\Omega} = 1A$

6.5 $\Phi = \vec{B}\cdot\vec{A} = BA\cos\theta$ ← max when B and A alligned so all the B flows through all the A

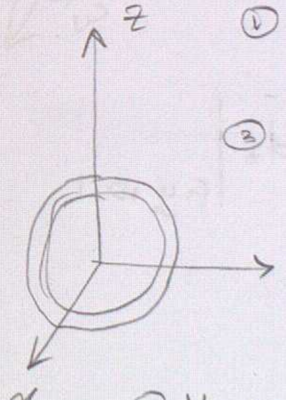
$B = B_0 \cos\left[(6\pi\times10^8)t + \phi_0\right]$ and $V_{emf} = -N\frac{d\Phi}{dt}$

$V_{emf} = -0.02 \frac{d}{dt}\left[B_0\cos((6\pi\times10^8)t+\phi_0)\right]$

$30\times10^{-3} = (0.02)(6\pi\times10^8)B_0 \underbrace{\sin\left[(6\pi\times10^8)t+\phi_0\right]}_{\text{at max, } = 1}$

↳ $\underline{B_0 = 8\times10^{-8}\ [T]}$

6.8 a)




① $V_{emf} = -N\frac{d\Phi}{dt}$, $\Phi = \int_S \vec{B}\cdot d\vec{s}$ ②

$d\vec{s} = \hat{\phi}\, c\, dr$

$\vec{B} = \hat{\phi}\frac{\mu_0 I_0}{2\pi r}\cos\omega t$

③ $\Phi = \int_a^b \frac{\mu_0 I_0}{2\pi r}\cos\omega t\, c\, dr \times \mu_r$

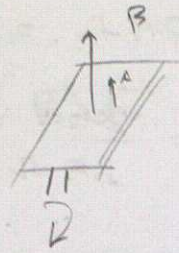
④ $\Phi = \frac{c\mu_0 I_0 \cos\omega t}{2\pi}\ln(b/a)\,\mu_r$

⑤ $V_{emf} = -N\frac{d}{dt}\left[\frac{c\mu_0 I_0 \ln(b/a)}{2\pi}\cos\omega t\right] = +\frac{Nc\mu_0\mu_r\ln(b/a)\,\omega}{2\pi}\sin\omega t$

b) plug ⇒ $\underline{V_{emf} = 5.5\sin 377t\ [V]}$

6.9 $\omega = \frac{2\pi \times 7200}{60} = 290\pi$ rad/s

$A = 50\times10^{-4}\ m^2$



$\Phi = BA$

$= A_o B\cos(\omega t + \Phi_o)$

$(V_{ems})_{peak} = (1) A\omega B_o$

$= (50\times10^{-4})(240\pi)(6\times10^{-6})$

$= \underline{22.6\times10^{-6}\,V}$

6.11 (eq.6.26) $\longrightarrow$ $V_{emf} = \oint_l (\vec{u}\times\vec{B})\cdot d\vec{l}$

$\vec{B} = -\hat{x}\frac{\mu_0 I_1}{2\pi r}$ $\quad \vec{u}\times\vec{B} = -(\hat{y}\times\hat{x})(7.5)\frac{\mu_0 I_1}{2\pi r}$

$= \hat{z}\frac{\mu_0 I_1}{2\pi r}$

$V_{ems} = \int_0^{0.2} \hat{z}\frac{\mu_0 I_1}{2\pi r}\cdot\hat{z}\,dz\Big|_{r=y_0} + \int_{0.2}^{0}\hat{z}\frac{\mu_0 I_1}{2\pi r}\cdot\hat{z}\,dz\Big|_{r=y_0+0.1}$

$= 3\times10^{-6}\left(\frac{1}{y_o} - \frac{1}{y_o+0.1}\right)$

$\underline{I_2 = \frac{V_{emf}}{2R} = 150\times10^{-9}\left(\frac{1}{y_o} - \frac{1}{y_o+0.1}\right)\ [A]}$

6.12 $\Phi = BA\cos(\omega t + \Phi_o)$ and $\omega = \frac{3600\times2\pi}{60} = 120\pi$

$= BA\cos(120\pi t + \Phi_o)$

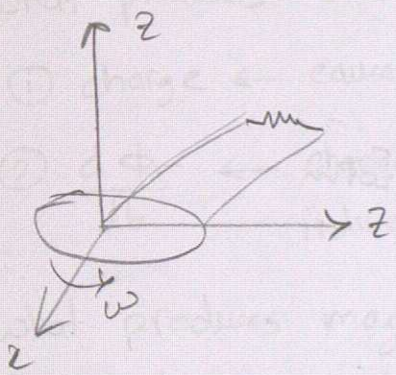
$\underbrace{V_{emf}}_{} = +NBA\omega_{,}\sin(120\pi t + \Phi_o)$

set = 1

$\hookrightarrow (V_{ems})_{pk} = (0.4)(0.1)(120\pi) = 15.08\,V$

$I_{pk} = \frac{(V_{ems})_{pk}}{R} = \frac{15.08\,V}{150\,\Omega} = 0.1\,A$ $\quad \underline{\therefore I_{pk} = 0.1\,A}$

6.13



$\vec{u} = \omega\vec{r} = \hat{\phi}\,\omega r$

$\vec{B} = \hat{z}B_o$ $\quad d\vec{l} = \hat{r}dr$

$V_{emf} = \int(\vec{u}\times\vec{B})\cdot d\vec{l}$

$= \int_0^a (\hat{\phi}\times\hat{z})\cdot\hat{r}\,\omega r B_o dr$

$= \omega B_o\int_0^a r\,dr = \frac{\omega B_o a^2}{2}$

Week 11: Lecture 1

## 6-7: Displacement Current

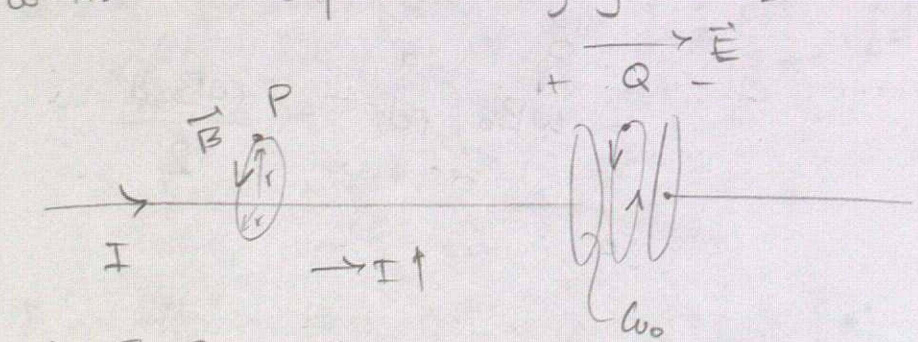
Ampere's Law: $\boxed{\oint \vec{H}\cdot d\vec{l} = I_{enc} = \int \vec{J}\cdot d\vec{s}}$

Maxwell added displacement current

$$\boxed{\oint \vec{H}\cdot d\vec{l} = \int_s (\underbrace{\vec{J}}_{\text{conduction current}} + \underbrace{\frac{\partial}{\partial t}\vec{D}}_{\text{displacement current}})\cdot d\vec{s}} \Big\} \rightarrow \text{integral form}$$

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}\vec{D}} \Big\} \rightarrow \text{point form}$$

* consider a capacitor charging as I increases:




* Find B @ P and Q
* apply Ampere's Law: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$

@P $\Rightarrow B(2\pi r) = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi r}$ [T]

@Q $\Rightarrow I_{enc} = 0 \Rightarrow B_Q = 0$

? how? →

* but experiment gives $B_Q \neq 0$
* in fact, $B_Q|_{experiment} = B_P|_{Ampere's\ Law}$
* Ampere: says current produces B

  But $I = 0$ inside cap $\Rightarrow$ why is there $B_Q$?
* Maxwell: maybe there is something else that is missing in Ampere's Law
* what produces electric field?

  ① charge ← causes electric field

  ② $\frac{d\Phi_m}{dt}$ ← changing magnetic flux ... Faraday's Law induces $V_{emf} \Rightarrow$ electric field, $E = \frac{V}{d}$ ($V = Ed$)
* what produces magnetic field?

  ① current

  ② $\frac{d\Phi_E}{dt}$ ← changing electric field ... Maxwell hypothesized that if: change in $\Phi_m$ produces $\vec{E}$, then change in $\Phi_E$ produces $\vec{H}$
* does this answer the capacitor question and the existence of $B_Q$
* when I increases, then cap charges and E is formed inside cap
* that means electric flux starts increasing

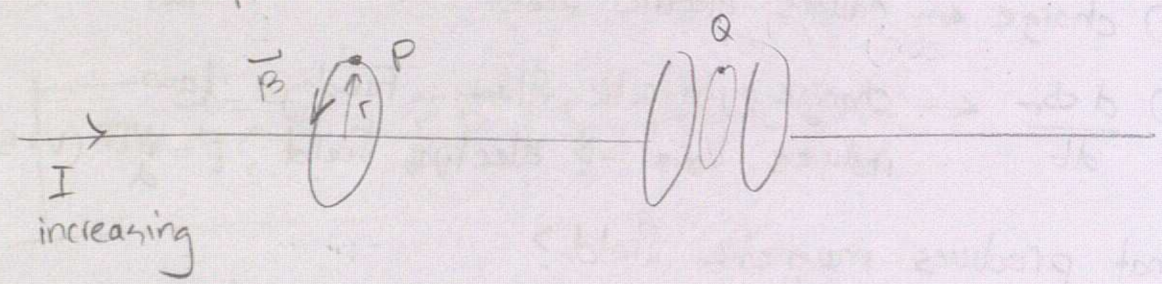
$\Phi_E = \oint \vec{D}\cdot d\vec{s} \Rightarrow \Phi_E = \varepsilon_0 \oint \vec{E}\, ds \rightarrow \frac{d}{dt}\Phi_E$

* Maxwell added the missing term to the Ampere's Law and called it <u>Displacement Current</u>

$$\oint \vec{H}\cdot d\vec{l} = \int_S \Big(\underbrace{\vec{J} + \underbrace{\frac{\partial}{\partial t}\vec{D}}_{}}_{[A/m^2]}\Big)\cdot d\vec{s}$$

* can write $\oint_C \vec{B}\cdot d\vec{l} = \mu_0\left(I_{enc} + \dfrac{d\Phi_E}{dt}\right)$

* back to cap : Find $\vec{B}$ @ P and Q




$$\oint \vec{B}\cdot d\vec{l} = \mu_0\left(I_{enc} + \frac{d\Phi_E}{dt}\right)$$

@ P: $\Phi_E = \epsilon_0 \int \vec{E}\cdot d\vec{s}$ ideal wire $\sigma \to \infty$, $R = 0$, $V = 0 \Rightarrow E = 0$

$\Phi_E = 0$

$\therefore B(2\pi r) = \mu_0(I + 0) \Longrightarrow B_P = \dfrac{\mu_0 I}{2\pi r}$ [T]

from next page : $\boxed{B_P = B_Q}$

@Q : $\Phi_E = \epsilon_0 \int \vec{E}\cdot ds = \epsilon_0 EA = \epsilon_0 E(\pi r^2)$

Recall : parallel plate $\vec{E} = \dfrac{\rho_s}{\epsilon_0}$ or $D = \rho_s$ $[C/m^2]$ $[C/m^2]$

$\Rightarrow \Phi_E = \epsilon_0 \dfrac{\rho_s}{\epsilon_0}\pi r^2 = \underbrace{\rho_s}_{C/Area}\underbrace{\pi r^2}_{A}$

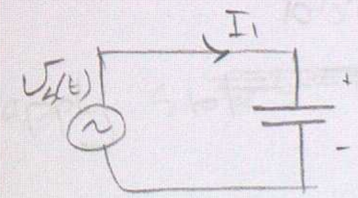
$\therefore \rho_s \pi r^2 = Q$, and $\dfrac{d\Phi_E}{dt} = \dfrac{dQ}{dt} = \underline{I}$

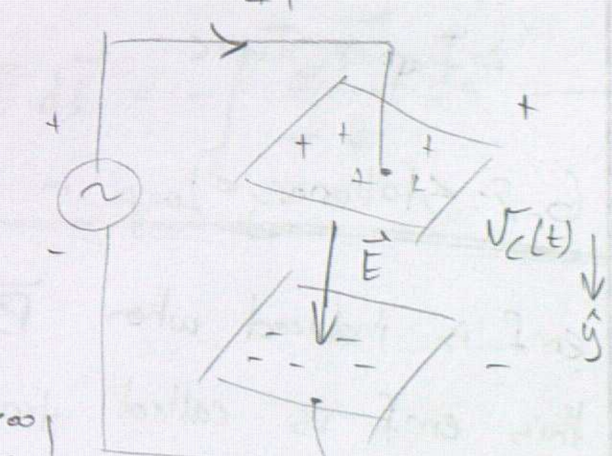
ideal cap $\Rightarrow \sigma = 0,\ R \to \infty \Rightarrow I_{conduction} = 0$

$\Rightarrow B(2\pi r) = \mu_0(0 + I) \Longrightarrow B_Q = \dfrac{\mu_0 I}{2\pi r}$ [T]

<u>Consider Capacitor</u>

$V_s(t) = V_0 \cos\omega t$ [V]




* perfect wire : $\vec{E} = \vec{D} = 0$ $\begin{pmatrix}\sigma \to \infty \\ R \to 0\end{pmatrix}$

$\Rightarrow I_{displ.} = \int_S \dfrac{\partial \vec{D}}{\partial t}\cdot d\vec{s} = 0$ (no displacement current)

* the conduction ($I_{1c}$) is the same as the current charging cap

$\Rightarrow I_{1c} = C\dfrac{dV_c}{dt} = C\dfrac{dV_s}{dt} = C\dfrac{d}{dt}V_0\cos\omega t \Rightarrow \underline{I_{1c} = -CV_0\omega\sin\omega t}$

$$I_{\substack{total\\(wire)}} = I_C + I_{1D} = -CV_0\omega \sin\omega t \ [A]$$

* perfect cap: $\sigma = 0$, $R \to \infty \Rightarrow I_{2C} = 0$

$$\vec{E} = \hat{y}\frac{V_C}{d} = \hat{y}\frac{V_0}{d}\cos\omega t \ [V/m] \Rightarrow \vec{D} = \epsilon_0\vec{E}$$

$$I_{2D} = \int_S \frac{\partial \vec{D}}{\partial t}\cdot d\vec{s} = \int_S \frac{\partial}{\partial t}\left(\hat{y}\frac{\epsilon_0 V_0}{d}\cos\omega t\right)\cdot \hat{y}\,ds$$

$$= -\frac{\epsilon_0 A}{d}V_0\omega\sin\omega t \ [A] \qquad \int ds = A$$

$$\frac{\epsilon_0 A}{d} = C$$

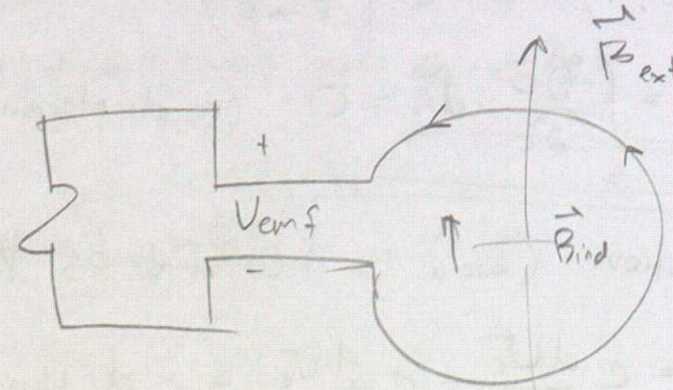
$$I_{\substack{cap\\total}} = -CV_0\omega\sin\omega t \ [A]$$

$\therefore I_{cap} = I_{wire}$

## 6-2: Stationary Loop in t-varying Field

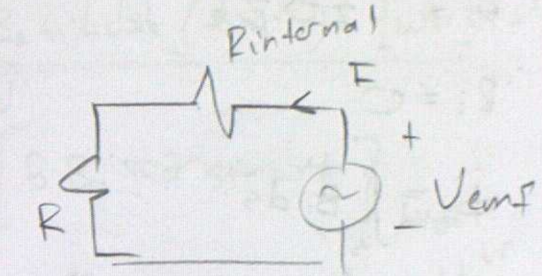
* emf is induced when $\vec{B}$ changes
* this emf is called transformer emf

* $V^{tr}_{emf} = -N\int_S \frac{\partial B}{\partial t}\cdot d\vec{s}$




* Note: If no load, still $V_{emf}$ appears

$$I = \frac{V^{tr}_{emf}}{R + R_i}$$




* can also write

$$V^{tr}_{emf} = \oint_C \vec{E}\cdot d\vec{l} \quad \text{and} \quad V^{tr}_{emf} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

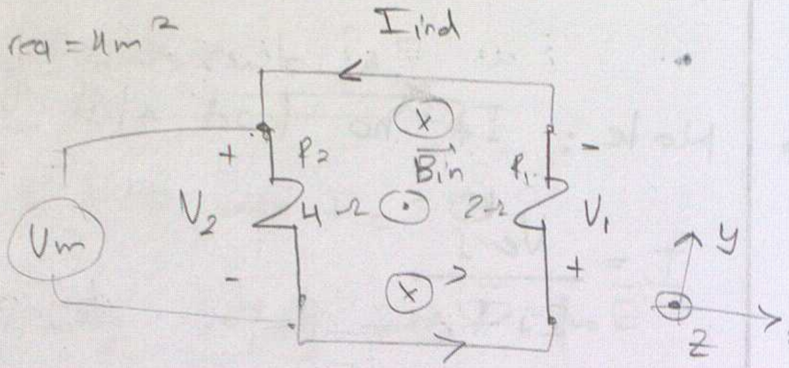
for $N = 1$

$$\Rightarrow \oint_C \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} \quad \text{— (1)}$$

* apply Stoke's Thm:

Week 11 Lecture 3 Mar 28

**Ex:** Find $V_1, V_2$, Area $= 4m^2$




$\vec{B} = -\hat{z}\,0.3t$ [T]

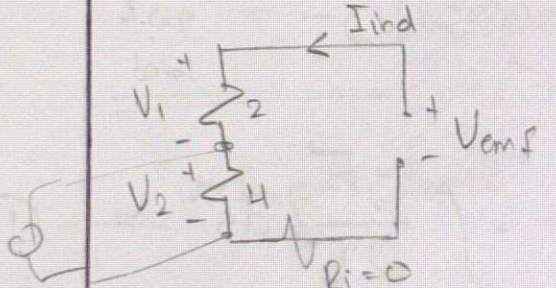
$R_i = 0$

$\Phi_{applied} = \int_s \vec{B}\cdot d\vec{s}$

$= \int (-\hat{z}\,0.3t)\cdot\hat{z}\,ds$

$= -0.3t\,(\text{Area}) = -0.3(4)t = -1.2t$ [wb]

$V_{emf} = -N\dfrac{d\Phi_{app}}{dt} = -(1)\dfrac{d}{dt}(-1.2t) = 1.2$ [V]



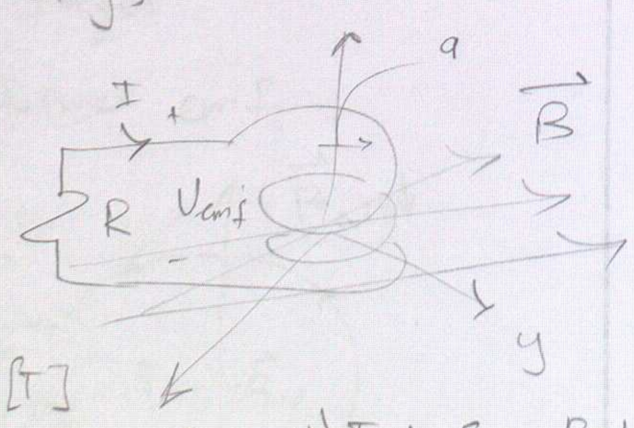

$I = \dfrac{V^{tr}_{emf}}{R_1+R_2} = \dfrac{1.2}{2+4} = 0.2$ [A]

$V_1 = 2\times 0.2 = 0.4$ [V]

$V_2 = 4\times 0.2 = 0.8$ [V]

**Ex:** Given N turns of windings of loop radius a with R

$\vec{B} = B_0(\hat{y}\,2 + \hat{z}\,3)\sin\omega t$ [T]




a) Flux ← find

b) $V_{emf}$ if $N = 10$, $B_0 = 0.2$ [T]

$a = 10$ cm, $\omega = 10^3$ rad/s

c) polarity of $V_{emf}$ at $t = 0$

d) Iind for $R = 1k\Omega$ and $R_i << R$

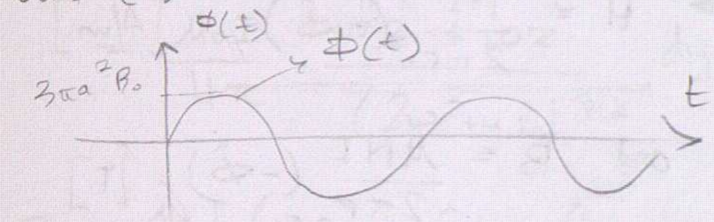
a) $\Phi = \int \vec{B}\cdot d\vec{s} = \int B_0(\hat{y}\,2 + \hat{z}\,3)\sin\omega_0 t \cdot \hat{z}\,ds$

$= \int 3B_0 \sin\omega_0 t\, ds = 3B_0\sin\omega t\,(\pi a^2)$ [wb] flux

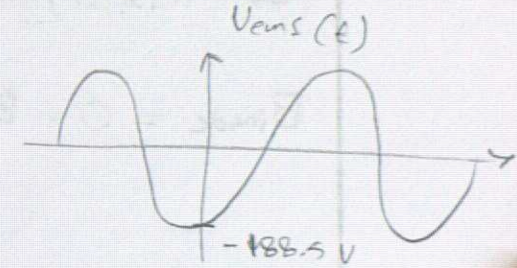
b) $V_{emf} = -N\dfrac{d\Phi}{dt} = -N\dfrac{d}{dt}\left[3B_0\pi a^2 \sin\omega t\right]$

$V_{emf} = -N\,3B_0\pi a^2\,\omega\cos\omega t$ [V] → plug: $V^{tr}_{emf} = -188.5\cos(10^3 t)$ [V]

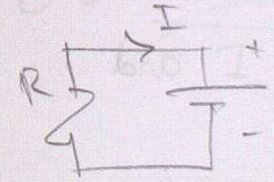
c) $V^{tr}_{emf}(0) = -188.5\cos(0) = -188.5$ [V]




⇒



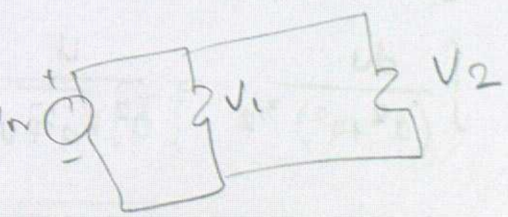

d)




$V_{emf} = -188.5\cos(\omega t)$

$I = \dfrac{-(-188.5\cos\omega t)}{10^3} = 0.19\cos(10^3 t)$ [A]

Eddy Current



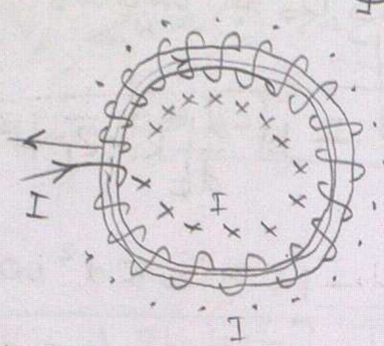
Week 12: Lecture 1 April 1, 2025

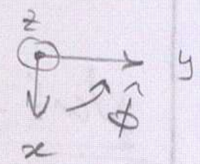
**Ex:** Find magnetic field inside a toroidal coil

turns = N } Find $\vec{H}$ and $\vec{B}$
current = I }

$\Phi_B$ is CW

inner a
outer b




* RHR gives direction of H: $-\hat{\phi}$
* Apply Ampere's Law:

$$\oint \vec{H}\cdot d\vec{l} = I_{enc} \rightarrow I_{enc} = N \times I$$

$$\Rightarrow H(2\pi r) = NI \Rightarrow \vec{H} = \frac{NI}{2\pi r}(-\hat{\phi}) \quad [A/m]$$

$B_{inside} = 0 = B_{outside}$

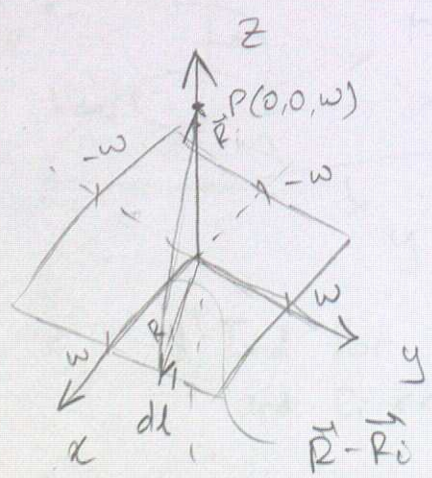
and $$\vec{B} = \frac{\mu N I}{2\pi r}(-\hat{\phi}) \quad [T]$$

**Ex:** The square loop shown has current I and is defined by 4 vertices:

$(w,-w,0)$, $(-w,w,0)$, $(-w,-w,0)$, $(w,w,0)$

Derive expression for B @ $R(0,0,w)$ using Biot Savart Law using

$$\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}}$$




B.S. Law: $$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\, d\vec{l}\times\hat{R}}{R^2}$$

$d\vec{l} = dy\,\hat{y}$, $\vec{R} = w\hat{z}$, $R_1 = w\hat{x} + y\hat{y}$

$$\vec{R} - \vec{R}_1 = -w\hat{x} - y\hat{y} + w\hat{z}$$

$$dB = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l}\times(\vec{R}-\vec{R}_1)}{|\vec{R}-\vec{R}_1|^3}$$

$$= \frac{\mu_0 I}{4\pi}\frac{dy\hat{y}\times(-w\hat{x} - y\hat{y} + w\hat{z})}{(w^2+y^2+w^2)^{3/2}}$$

$$= \frac{\mu_0 I}{4\pi}\frac{w\hat{z} + w\hat{x}}{(2w^2+y^2)^{3/2}}\,dy$$

$$B = \frac{\mu_0 I(w\hat{z} + w\hat{x})}{4\pi}\int_{-w}^{w}\frac{dy}{(2w^2+y^2)^{3/2}}$$

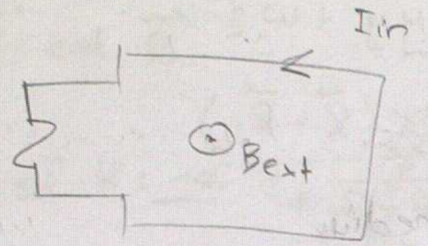
$$B = \frac{\mu_0 I w(\hat{z}+\hat{x})}{4\pi}\left(\frac{y}{2w^2\sqrt{2w^2+y^2}}\right)\Bigg|_{-w}^{w}$$

$$B = \frac{\mu_0 I(\hat{x}+\hat{z})}{4\pi}\frac{1}{\sqrt{3w^2}} \longrightarrow B = \frac{\mu_0 I}{4\pi\sqrt{3}\,w}(\hat{x}+\hat{z}) \quad [T]$$

$$\vec{B}_{tot} = 4\vec{B} \Rightarrow \vec{B}_{tot} = \frac{\mu_0 I}{\sqrt{3}\,\pi w}\hat{z} \quad [T]$$

* note that $\hat{x}$ comp. cancel out with $-\hat{x}$ comp. and likewise for $\hat{y}$.

Ex: $\vec{B}_{ext}$ changing




A) shows $B_{ext}$ (+)ve

B) shows $B_{ext}$ (-)ve

C) both $B_{ext}$ and $B_{ind}$ are (+)ve

D) none of the above

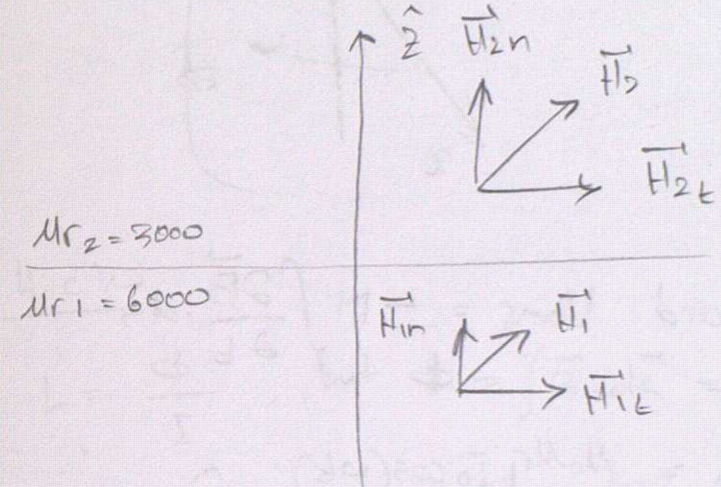
Week 12: Lecture 2 — Apr 2, 2025

Ex: the magnetic field is given by

$\vec{H}_1 = 6\hat{x} + 2\hat{y} + 3\hat{z}$ [A/m] in medium

with a $\mu_{r_1} = 6000$ that exists in $z<0$

Find $\vec{H}_2$ in medium with $\mu_{r_2} = 3000$ for $z>0$




$$\begin{cases} \vec{B}_{1n} = \vec{B}_{2n} \\ \mu_1 \vec{H}_{1n} = \mu_2 \vec{H}_{2n} \\ \vec{H}_{1t} - \vec{H}_{2t} = \vec{J}_s \end{cases}$$

Knowns

$\vec{H}_{1n} = 3\hat{z}$ and $\vec{H}_{1t} = 6\hat{x} + 2\hat{y}$

$\vec{B}_{1n} = \mu_0 \mu_{r_1} \vec{H}_{1n} = \mu_0 (6000)(3\hat{z}) = \mu_0 18000\hat{z}$

Knowing $\vec{B}_{1n} = \vec{B}_{2n}$ $\Rightarrow$ $\vec{B}_{2n} = 18000\mu_0 \hat{z}$

$\mu_1 \vec{H}_{1n} = \mu_2 \vec{H}_{2n} \Rightarrow \vec{H}_{2n} = \frac{6000}{3000}(3\hat{z}) = 6\hat{z}$ [A/m]

$\vec{H}_{1t} - \vec{H}_{2t} = \vec{J}_s \Rightarrow J_s = 0$

$\vec{H}_{1t} = \vec{H}_{2t} = 6\hat{x} + 2\hat{y} \Rightarrow \underline{\vec{H}_2 = 6\hat{x} + 2\hat{y} + 6\hat{z}}$ [A/m]

Ex: Consider a toroid and $I = I_0 \cos(\omega t)$ [A]

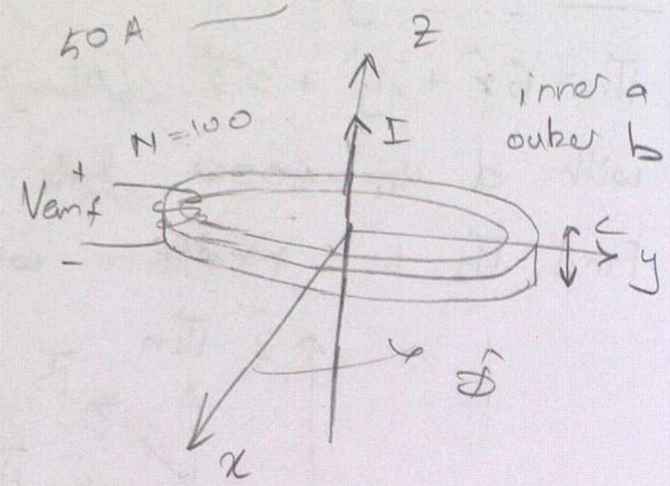
$I_0$ = 50 A

a) develop expression for emf

b) Find emf for

$f = 60\,Hz$, $\mu_r = 4000$

$a = 5$ cm, $b = 6$ cm, $c = 2$ cm




$$V_{emf} = -N\frac{d\Phi}{dt} \quad \text{and} \quad V_{emf} = -N\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

$$\vec{B} = \frac{\mu I}{2\pi r} \Rightarrow \vec{B}(t) = \frac{N\mu_0\mu_r I_0\cos(\omega t)}{2\pi r}\hat{\phi}$$

$$V_{emf} = -N\int_a^b\int_0^c \frac{\partial B}{\partial t}\,dz\,dr$$

$$= -N\int_a^b\int_0^c \frac{N\mu_0\mu_r I_0}{2\pi r}(-\omega\sin\omega t)\,dz\,dr$$

$$V_{emf} = \frac{N\mu_0\mu_r I_0 c\,\omega\sin\omega t}{2\pi}\ln(b/a) \quad [V]$$

plug : $V_{emf} = 5.5$

Week 12 : Lec 3 Apr 4, 2025

Past Exam

$$B = \begin{cases} 0 \\ \dfrac{\mu_0 J_0 (r^2 - a^2)}{2r} \\ \end{cases}$$

$q = CV$

$C = \frac{V}{q}$

$L = \frac{\Phi}{I}$

4. c)

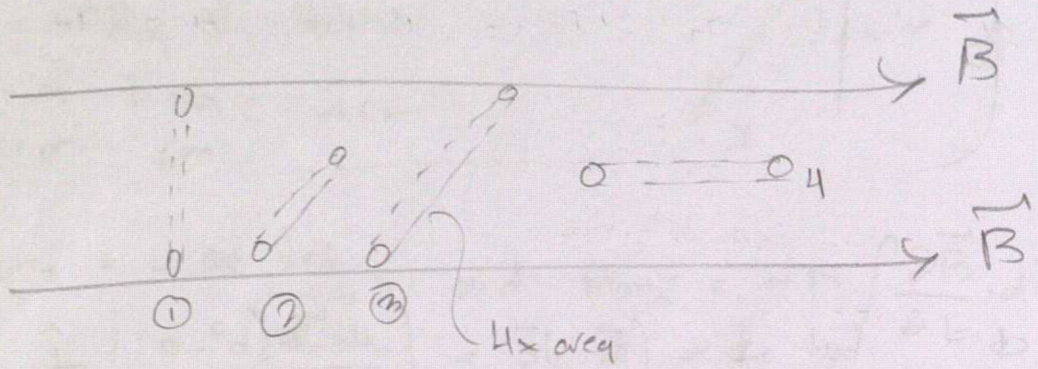
$$L = \frac{\Phi}{I}, \quad \text{but } \Phi = \int \vec{B}\cdot d\vec{s} = \int \frac{\mu_0 J_0 (b^2 - a^2)}{2r}\hat{\phi}\cdot\hat{\phi}\,dr\,dz$$

$$\Phi = \frac{\mu_0 J_0 (b^2 - a^2)}{2}\int_0^d dz\int_h^{h+d}\frac{1}{r}dr = \frac{\mu_0 J_0 (b^2 - a^2)}{2}(d)\ln\left(\frac{h+d}{h}\right)$$

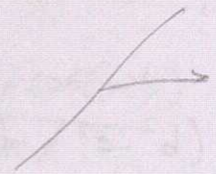
$$L = \frac{\Phi}{I} = \frac{\dfrac{\mu_0 \cancel{J_0}\cancel{(b^2 - a^2)}}{2}\times d\ln\left(\dfrac{h+d}{h}\right)}{\cancel{J_0}\pi\cancel{(b^2 - a^2)}}$$

$$\Rightarrow L = \frac{\mu_0 d}{2\pi}\ln\left(\frac{h+d}{h}\right) \quad [H]$$

Ex: Consider 4 loops perpendicular to the page. radius of loops 3,4 are twice loops 1 and 2. B is same. Rank in order from largest to smallest fluxes




$\pi r^2 \rightarrow \boxed{4\pi r^2}$



$\boxed{3, 1, 2, 4}$ 4

$\Phi = BA\cos\theta = B(\pi r^2)\cos\theta$

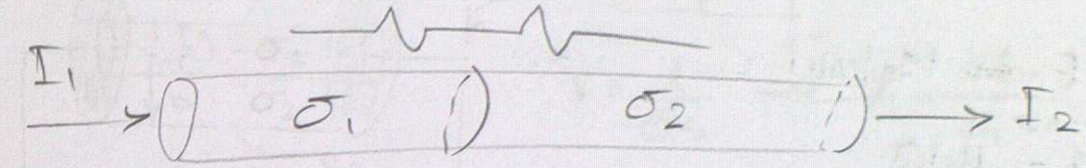
$\Phi_1 = B\pi(1)^2\cos 0 = B\pi$ c

$\Phi_2 = B\pi(1)^2\cos 45^\circ = 0.707 B\pi$ b

$\Phi_3 = B\pi(2)^2\cos 45^\circ = 0.707(4)B\pi$ a

$\Phi_4 = B\pi(2)^2\cos 90^\circ = 0$ d

Ex: A wire consists of two equal-diameter with $\sigma_2 > \sigma_1$. The current in segment 1 is $I_1$




a) compare values of currents in the two segments

Is $I_2$ >, <, or = to $I_1$?

Due to KCL (conservation of current)

$I_1 = I_2$

b) Compare strengths of $J_1$ and $J_2$

$I = JA \rightarrow J_1 = \frac{I_1}{A_1}$, $J_2 = \frac{I_2}{A_2} = \frac{I_1}{A_1}$

$\boxed{\therefore J_1 = J_2}$

c) Compare strength of electric fields $E_1$ and $E_2$ in 2 segments

ohm's Law

$\boxed{E_2 < E_1}$ $J = \sigma E$

$E_1 = \frac{J_1}{\sigma_1}$, $E_2 = \frac{J_2}{\sigma_2}$

$\sigma_2 > \sigma_1$

# Final Exam Practice

## Midterm Material

$\vec{D} = \epsilon_0 \epsilon_r \vec{E} = \epsilon \vec{E}$

$\vec{B} = \mu_0 \mu_r \vec{H} = \mu \vec{H}$

- Ch. 4.1 – 4.10

## Final Exam Material

- Ch. 4.1 – 4.10
- Ch. 5.1 – 5.8 (–5.4 –5.5.1) } Focus
- Ch. 6.1 – 6.9 } Focus

## Tutorials

$\underbrace{1, 2, 3, 4, 5}_{\text{midterm}}, \underbrace{6, 7, 8, 9, 10, 11}_{\text{final exam}}$

## Calc 3 Review

### Divergence Theorem

$$\oint\!\!\oint_{S=\partial R} \vec{F}\cdot\hat{n}\,dS = \iiint_R \nabla\cdot\vec{F}\,dV$$

⇓ Gauss Law

$$\oint\!\!\oint_S \vec{D}\cdot\vec{dS} = Q_{enc}$$

$$\nabla\cdot\vec{D} = \rho_V$$

$$\nabla\times\vec{D} = 0$$

$$\iiint \nabla\cdot\vec{D}\,dV = \iiint \rho_V\,dV = Q_{enc}$$

### Stoke's Theorem

$$\oint_{C=\partial S} \vec{F}\cdot\hat{T}\,ds = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$$

⇓ Ampere's Law

$$\oint_C \vec{B}\cdot\vec{dl} = \mu I_{enc}$$

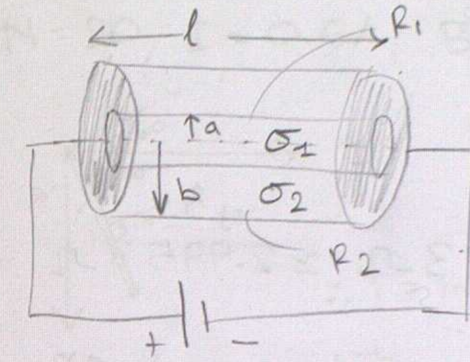
$$\nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{B} = \mu\vec{J}_s$$

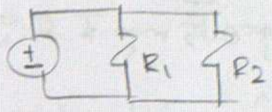
$$\iint_S (\nabla\times\vec{B})\cdot\vec{dS} = \mu\iint_S \vec{J}_s\cdot\vec{dS} = \mu I_{enc}$$

# ECE221 Tutorial 6

## ✓ 4.44 (from Tut 5)




ckt




$R_T = \dfrac{R_1 R_2}{R_1 + R_2}$

$R = \dfrac{l}{\sigma S} \rightarrow R_1 = \dfrac{l}{\sigma_1 \pi a^2}, \quad R_2 = \dfrac{l}{\sigma_2 \pi (b^2 - a^2)}$

$R_1 R_2 = \dfrac{l^2}{\sigma_1 \sigma_2 \pi^2 a^2 (b^2 - a^2)}$

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{\sigma_1 \pi a^2}{l} + \frac{\sigma_2 \pi (b^2 - a^2)}{l} = \frac{\pi(\sigma_1 a^2 + \sigma_2 b^2 - \sigma_2 a^2)}{l}$$

$$R_T = \frac{l}{\pi a^2(\sigma_1 + \sigma_2 b^2 - \sigma_2)} = \frac{l}{\pi(\sigma_1 a^2 + \sigma_2(b^2 - a^2))}\ [\Omega] \ ✓$$

## ✓ 4.45

$R_T = \dfrac{l}{\pi(\sigma_1 a^2 + \sigma_2(b^2 - a^2))}, \quad \sigma_1 = 0,\ a = 0.02\,m,\ b = 0.03\,m$

$\sigma_2 = 3\times10^4,\ l = 0.2\,m$

$$R_T = \frac{0.2}{\pi(3\times10^4)(0.03^2 - 0.02^2)} = \underline{4.244\ m\Omega} \ ✓$$

## ✓ 4.46

$\sigma_{Al} = 3.5\times10^7,$

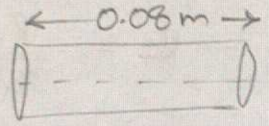
$\} \ 2\times10^{-3}\times10^{-2} = 2\times10^{-5}\ m$

sides $0.05 \times 0.05\ m$

a) $R = \dfrac{l}{\sigma S} = \dfrac{0.05}{(3.5\times10^7)(2\times10^{-5})(0.05)} = \underline{1.43\ m\Omega}$ ✓

b) $R = \dfrac{l}{\sigma S} = \dfrac{2\times10^{-5}}{(3.5\times10^7)(0.05)^2} = \underline{2.3\times10^{-10}\ \Omega}$ ✓

4.47

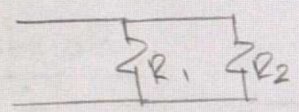
a) 

$\uparrow 0.5\,mm = 0.5\times10^{-3}$

$\sigma_{cr} = 3.4\times10^4 \qquad \sigma_{cp} = 5.8\times10^7$

$$R = \frac{l}{\sigma S} = \frac{0.08}{(3.4\times10^4)\pi(0.5\times10^{-3})^2} = 3\,\Omega \simeq 2.9959\,\Omega$$

b) $R_{new} = 0.60\times 2.9959 = 1.7975$ $\qquad t = b-a$, $a = 0.5\times10^{-3}$



$$R_{new} = \frac{l}{\pi(\sigma_1 a^2 + \sigma_2(b^2-a^2))}$$

$$\pi R_{new}(\sigma_1 a^2 + \sigma_2 b^2 - \sigma_2 a^2) = l$$

$$b = \left(\left[\left(\frac{l}{\pi R_{new}}\right) - \sigma_1 a^2 + \sigma_2 a^2\right]\frac{1}{\sigma_2}\right)^{1/2}$$

$$b = \left(\left[\left(\frac{0.08}{\pi(1.7975)}\right) + (0.5\times10^{-3})^2(5.8\times10^7 - 3.4\times10^4)\right]\frac{1}{5.8\times10^7}\right)^{1/2}$$

$b = a+t = 5.000976936\times10^{-4}$

$t = b - a = \underline{0.0977\,\mu m} = t$

x
z y

☆ 4.62, 5.2

✓ 5.1 $\vec{u} = 8(10)^6\,\hat{x}$, $B = \hat{x}4 - \hat{z}3$ $\qquad F = ma$, $a = \frac{F}{m}$

$e = 1.6\times10^{-19}$, $m_e = 9.1\times10^{-31}$

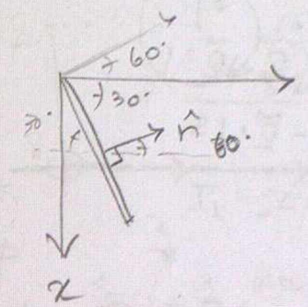
$$F = q(\vec{u}\times B) = -e\left(8(10)^6\,\overbrace{\hat{x}\times 4\hat{x}}^{=0} - 3\hat{z}\right) \quad \text{(}-\hat{y}\text{)}$$

$$a = -\frac{(1.6\times10^{-19})(8\times10^6)(-3)(-\hat{y})}{9.1\times10^{-31}} = \underline{-4.22\times10^{18}\,\hat{y}\ [m/s^2]}\ ✓$$

ECE221 Tutorial 7

✓ 5.4

$N=20$, $I=0.5A$, $B=\hat{y}\,2.4T$, $\vec{T} = \vec{m}\times\vec{B}$, $T = mB\sin\theta$



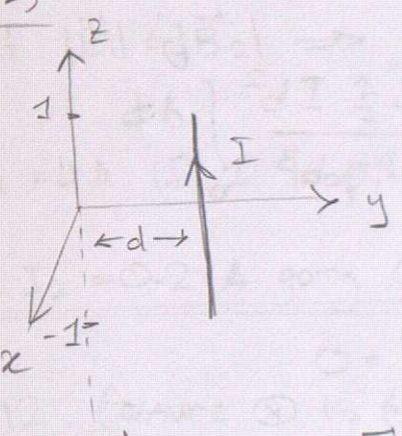
$\theta$ is angle b/n $\hat{n}$ and $\hat{y} \rightarrow \theta = 60°$

$\vec{m} = NAI\hat{n}$

$\therefore T = mB\sin\theta = (20)(0.4\times0.2)(0.5)(2.4)\sin 60°$

$\underline{|\vec{T}| = 1.66\ N\cdot m}$ ✓, $\underline{\text{direction CW}}$ ✓

✓ 5.5

a) 

$\vec{F_m} = I(\vec{L}\times\vec{B})$, $\vec{L} = 2\hat{z}$, $\vec{B} = \hat{r}\,0.2\cos\phi$

$= 5(2\hat{z}\times\hat{r}\,0.2\cos\phi)$, $\phi = \frac{\pi}{2}$

$\underline{\vec{F_m} = 0}$ ✓

b) $w = \vec{F}\cdot\vec{d} \Rightarrow dw = \vec{F}\cdot d\vec{l} = \vec{F}\cdot\hat{\phi}\,r\,d\phi$, $r = 0.04$

$\vec{F} = 5(2\hat{z}\times\hat{r}\,0.2\cos\phi) = 2\cos\phi\,\hat{\phi}$

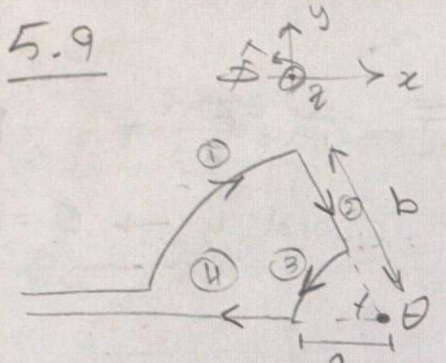
$$\begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ 0 & 0 & 2 \\ Br & 0 & 0 \end{vmatrix} = -\hat{\phi}(-2Br) = 2Br = 0.4\cos\phi\,\hat{\phi}$$

$$w = \int \vec{F}\cdot d\vec{l} = \int_{\phi=0}^{\phi=2\pi} 2\cos\phi\,\hat{\phi}\cdot 0.04\,\hat{\phi}\,d\hat{\phi} = 2(0.04)\int_0^{2\pi}\cos\phi\,d\phi = 0$$

$\underline{\therefore W_{total} = 0\ J}$ ✓

c) $\vec{F} = 2\cos\phi\,\hat{\phi} \leftarrow$ max when $\cos\phi = 1 \Rightarrow \underline{\phi = 0, \pm\pi, \pm2\pi\cdots}$ ✓

5.9



$B = \mu H = \frac{\mu I}{4\pi} \oint_c \frac{d\vec{l}' \times (\vec{R}-\vec{R}')}{|\vec{R}-\vec{R}'|^3}$

$\therefore d\vec{H} = \frac{I}{4\pi} \frac{d\vec{l}' \times (\vec{R}-\vec{R}')}{|\vec{R}-\vec{R}'|^3}$

① : $d\vec{l}' = -\hat{\phi} r d\phi = -\hat{\phi} b d\phi$

$\vec{R} = 0, \vec{R}' = \hat{r} b \Rightarrow \vec{R}-\vec{R}' = -\hat{r} b$

$|\vec{R}-\vec{R}'|^3 = (b^2)^{3/2} = b^3$

$\hat{r}\ \hat{z}\ \hat{\phi}$ $\quad \hat{\phi} \times \hat{r} = -\hat{z}$

$\vec{H}_1 = \frac{I}{4\pi} \int_0^\theta \frac{-\hat{\phi} b d\phi \times -\hat{r} b}{b^3} = \frac{-\hat{z}\, I b^2}{4\pi b^3} \int_0^\theta d\phi$

$\vec{H}_1 = -\hat{z} \frac{I\theta}{4\pi b}$

② : $\vec{H}_2 = 0$ since $d\vec{l}' = \hat{r}$ and $\hat{r} \times \hat{r} = 0$

③ : $d\vec{l}' = \hat{\phi} a d\phi$

$\vec{R} = 0, \vec{R}' = a\hat{r} \Rightarrow \vec{R}-\vec{R}' = -\hat{r} a$

$|\vec{R}-\vec{R}'|^3 = (a^2)^{3/2} = a^3$

$\hat{\phi} \times \hat{r} = -\hat{z}$

$\hat{\phi} \times -\hat{r} = \hat{z}$

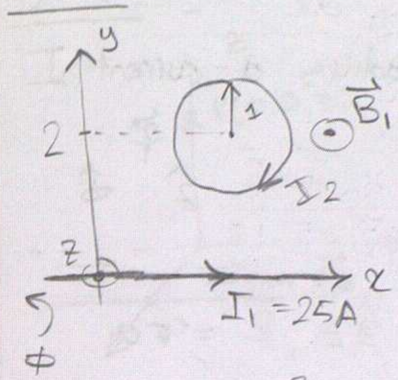
$\vec{H}_3 = \frac{I}{4\pi} \int_0^\theta \frac{\hat{\phi} a d\phi \times -\hat{r} a}{b^3} = \hat{z} \frac{I\theta}{4\pi a}$

④ : $\vec{H}_4 = 0$ for same reason as ②

$\therefore \vec{H} = \vec{H}_1 + \vec{H}_2 + \vec{H}_3 + \vec{H}_4 = -\hat{z} \frac{I\theta}{4\pi b} + 0 + \hat{z} \frac{I\theta}{4\pi a} + 0$

$\therefore \vec{H} = \frac{I\theta}{4\pi} \left( \frac{1}{a} - \frac{1}{b} \right) \hat{z}$ ✓

5.11



Using R.H.R, $\vec{B}_1$ is in $\hat{z}$. For $\vec{B}$ to be zero at centre of loop, $\vec{B}_2$ must be $-\hat{z}$ and $\therefore$ by RHR, direction of loop current $-\hat{\phi}$ ✓

$\vec{B}_1 = \frac{\mu I_1}{4\pi} \oint_c \frac{dl' \times (\vec{R}-\vec{R}')}{|\vec{R}-\vec{R}'|^3} = \frac{\mu I_1}{2\pi(2)}$

$\hat{r}\ \hat{z}\ \hat{\phi}$

$\vec{B}_2 = \frac{\mu I_2}{4\pi} \int_0^{2\pi} \frac{-d\phi \hat{\phi} \times -\hat{r}}{1^3} = -\hat{z} \frac{2\pi \mu I_2}{4\pi} = -\hat{z} \frac{\mu I_2}{2}$ $\quad -\hat{\phi} \times -\hat{r} = -\hat{z}$

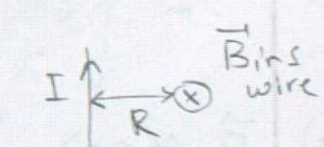
set $|\vec{B}_1| = |\vec{B}_2| \Rightarrow \frac{\mu I_1}{2\,4\pi} = \frac{\mu I_2}{2} \Rightarrow I_2 = \frac{I_1}{2\pi} = \frac{25}{2\pi} \approx 4\text{ A}$

$I_2 = 4\text{A}$ if 1 loop, but since 20 loops, $I_2 = \frac{I_{total}}{20} = \frac{4}{20} = 0.2$

$\therefore I_2 = 0.2$ A going CW ✓

5.12 (assume ⊗ is (-)ve, ⊙ is (+)ve)

From E&M theory, we know $\vec{B}_{\substack{ins\\line}} = \frac{\mu I}{2\pi R}$ :

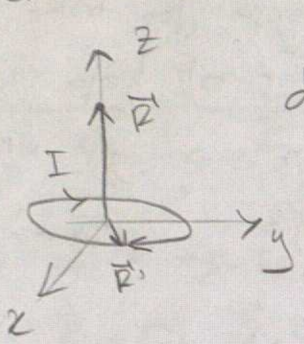


$\therefore \vec{B}_P = \vec{B}_{P_1} + \vec{B}_{P_2} = -\frac{\mu I_1}{2\pi 0.5} + -\frac{\mu I_2}{2\pi 1.5} = \frac{-\mu}{2\pi} \left( \frac{I_1}{0.5} + \frac{I_2}{1.5} \right)$

$\vec{B}_P = \frac{4\pi \times 10^{-7}}{2\pi} \left( \frac{6}{0.5} + \frac{6}{1.5} \right) = 3.2\,\mu T$ [into page] $= \vec{B}_P$ ✓

✓ 5.14

Derive the $\vec{B}$ at P(0,0,z) for loop radius a current I




$$d\vec{B} = \frac{\mu I}{4\pi}\frac{d\vec{l}' \times (\vec{R}-\vec{R}')}{|\vec{R}-\vec{R}'|} \qquad \hat{z} \quad \hat{r} \quad \hat{\phi}$$

$$d\vec{l}' = -\hat{\phi}\, a\, d\phi, \quad \vec{R} = \hat{z}z, \quad \vec{R}' = \hat{r}a$$

$$\vec{R}-\vec{R}' = \hat{z}z - \hat{r}a, \quad |\vec{R}-\vec{R}'| = (z^2+a^2)^{3/2}$$

$$d\vec{l}' \times (\vec{R}-\vec{R}') = -\hat{\phi}\, a\, d\phi \times (\hat{z}z - \hat{r}a) = -\hat{r}az\,d\phi - \hat{z}a^2 d\phi$$

by inspection, $\hat{r}$ component of $\vec{B}$ cancels out

$$\therefore d\vec{B} = \frac{\mu I}{4\pi}\frac{-\hat{z}a^2 d\phi}{(z^2+a^2)^{3/2}} \Rightarrow \vec{B} = -\hat{z}\frac{\mu I a^2}{4\pi(z^2+a^2)^{3/2}}\int_0^{2\pi} d\phi$$

$$\vec{B}_m(0,0,z) = -\hat{z}\frac{\mu I a^2}{2(z^2+a^2)^{3/2}} \quad [T]$$

given $a = 3$, $I = 40$

a) $\vec{B}_T(0,0,0) = \vec{B}_m(0,0,0) + \vec{B}_m(0,0,-2)$

$$= \frac{-\hat{z}\mu a^2 I}{2}\left(\frac{1}{a^3} + \frac{1}{(2^2+a^2)^{3/2}}\right) = -\hat{z}\,1.32\times10^{-5}\ [T]$$

b) $\vec{B}_T(0,0,1) = \vec{B}_m(0,0,1) + \vec{B}_m(0,0,-1) = -\hat{z}\,1.43\times10^{-5}\ [T]$

c) $\vec{B}_T(0,0,2) = \vec{B}_m(0,0,2) + \vec{B}_m(0,0,0) = -\hat{z}\,1.32\times10^{-5}\ [T]$

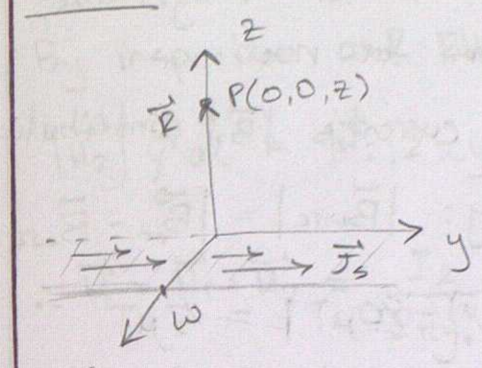
$\therefore$ $\vec{B}$ @ 0 $= -\hat{z}$ 13.2 μT ✓

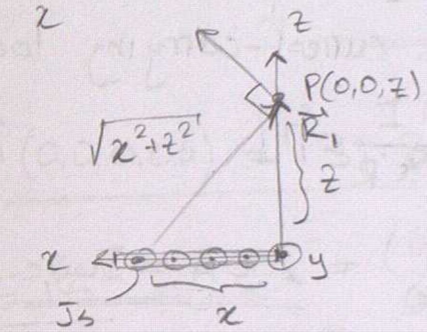
@ 1 $= -\hat{z}$ 14.3 μT ✓

@ 2 $= -\hat{z}$ 13.2 μT ✓

ECE221 Tutorial 8

✓ 5.10




By inspection and RHR, $\vec{B}$ @ P should have no $\hat{y}$ component and positive $\hat{z}$ and $\hat{x}$ components.

We know $I = \int J_s$, and considering a thin infinity-in-the-y wire, the current in each thin wire is: $\frac{I}{w} = \frac{J_s w}{w} = J_s$

Recall that the $\vec{B}$ at a radial distance R from an infinite line current is $\vec{B} = \frac{\mu I}{2\pi R}\hat{\phi}$

$\therefore |d\vec{B}| = \frac{\mu}{2\pi}\frac{J_s\,dx}{\sqrt{x^2+z^2}}$ and we can integrate over x

direction: $\hat{y} \times \left(\frac{x}{\sqrt{x^2+z^2}}(-\hat{x}) + \frac{z}{\sqrt{x^2+z^2}}\hat{z}\right) \Rightarrow \ldots\hat{z} + \hat{x}$

$$\Rightarrow \vec{B} = \frac{\mu|J_s|}{2\pi}\left[\hat{z}\int_0^w \frac{x}{(\sqrt{x^2+z^2})^2}dx + \hat{x}\int_0^w \frac{z}{(\sqrt{x^2+z^2})^2}dx\right]$$

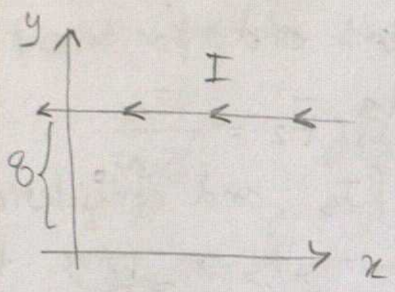
$$= \frac{5\mu}{2\pi}\left[\hat{z}\int_0^w \frac{x}{x^2+z^2}dx + \hat{x}z\int_0^w \frac{1}{x^2+z^2}dx\right]$$

$$= \frac{5\mu}{2\pi}\left[\hat{z}\left(\frac{1}{2}\ln(x^2+z^2)\right)\Big|_0^w + \hat{x}z\left(\frac{1}{z}\arctan\left(\frac{x}{z}\right)\right)\Big|_0^w\right]$$

$$\vec{B} = \frac{5\mu}{2\pi}\left[\hat{z}\frac{1}{2}\ln\left(\frac{w^2+z^2}{z^2}\right) + \hat{x}\arctan\left(\frac{w}{z}\right)\right]\ [T]$$ ✓

## 5.13

Since question asks for magnitude, dirsegard direction for now.




Observe that the current's $|\vec{B}|$ contribution can be found by: $|\vec{B}_{wire}| = |\vec{B}_{on} - \vec{B}_{off}|$

$\therefore |\vec{B}_{wire}| = |15\mu T - 20\mu T| = 5\mu T$

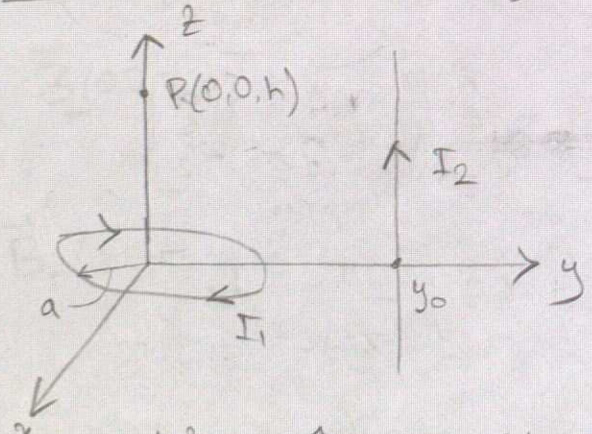
Recall that the $\vec{B}$ due to an infinite current-carrying loop at a radial distance $R$ is $|\vec{B}| = \frac{\mu I}{2\pi R}$ [T]

$$\therefore I_{wire} = \frac{2\pi R|\vec{B}_{wire}|}{\mu} = \frac{2\pi(8)(5\mu)}{4\pi\times10^{-7}} = \underline{\underline{200A = I_{wire}}}$$

## 5.15

$B = \mu H$

a)




We apply superposition:

$\vec{H}_1$ due to $I_1$, $\vec{H}_2$ " " $I_2$ $\Big\}$ $\vec{H} = \vec{H}_1 + \vec{H}_2$

$$d\vec{H} = \frac{I}{4\pi}\frac{d\vec{l}'\times(\vec{R}-\vec{R}')}{|\vec{R}-\vec{R}'|^3}$$

$d\vec{l}' = -\hat{\phi}\, a\, d\phi$, $\vec{R} = \hat{z}h$, $\vec{R}' = \hat{r}a$ $\qquad \hat{z}\ \hat{r}\ \hat{\phi}$

$\vec{R}-\vec{R}' = -a\hat{r} + h\hat{z} \rightarrow d\vec{l}'\times(\vec{R}-\vec{R}') = -a^2 d\phi\,\hat{z} - ah\,d\phi\,\hat{r}$

by inspection and RHR, $\vec{H}_1$ has only $\hat{z}$-comp ($\hat{r}$ comp cancels out)

$$\therefore d\vec{H}_1 = \frac{I_1}{4\pi}\frac{-a^2 d\phi\,\hat{z}}{(a^2+h^2)^{3/2}} \Rightarrow \vec{H}_1 = -\hat{z}\frac{\mu I_1 a^2}{4\pi(a^2+h^2)^{3/2}}\int_0^{2\pi} d\phi$$

$$\therefore \vec{H}_1 = -\hat{z}\frac{I_1 a^2}{2(a^2+h^2)^{3/2}} \text{ [H/m]}$$

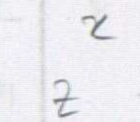
For $I_2$, use Ampere's Law due to symmetry: $\oint_c \vec{H}\cdot d\vec{l} = I_{enc}$

By inspection and RHR, direction of $\vec{H}_2$ is $\hat{x}$.

$$|\vec{H}_2|\oint_c dl = |\vec{H}_2|2\pi y_0 = I_{enc} = I_2 \Rightarrow \vec{H}_2 = \frac{I_2}{2\pi y_0}\hat{x}$$

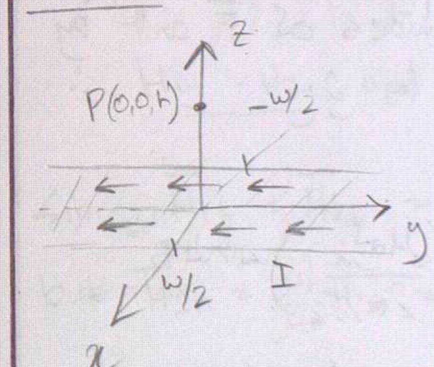
$$\therefore \vec{H} = \vec{H}_1 + \vec{H}_2 = \frac{I_2}{2\pi y_0}\hat{x} - \frac{I_1 a^2}{2(a^2+z^2)^{3/2}}\hat{z} \text{ [H/m]} = \vec{H}(0,0,h)$$

b) $\vec{H}(0,0,0.04) = 31.83\,\hat{x} - 36\,\hat{z}$ [H/m]

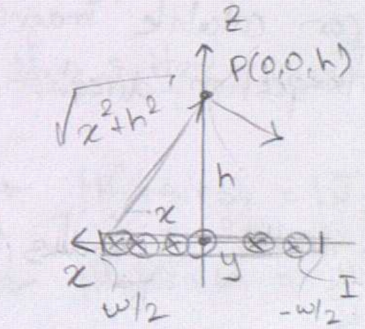



## 5.18

a)




could cancel out




Due to RHR and symmetry:
* direction of $\vec{B}$ is in (+)ve $\hat{z}$ and (-)ve $\hat{z}$
* can find contribution of 0 to $w/2$ and multiply result by 2

$$|\vec{B}| = \frac{\mu I}{2\pi R} = \frac{\mu I/w}{2\pi\sqrt{x^2+h^2}}, \text{ direction } -\hat{y}\times\left(\frac{-x}{\sqrt{x^2+h^2}}\hat{x} + \frac{h}{\sqrt{x^2+h^2}}\hat{z}\right)$$

due to symmetry, $\hat{z}$ comp. gone $\rightarrow$ $= \frac{-x}{\sqrt{x^2+h^2}}\hat{z} - \frac{h}{\sqrt{x^2+h^2}}\hat{x}$

$$\therefore \vec{B} = 2\times\vec{B}_{\text{one side}} = \frac{2\mu I}{2\pi w\sqrt{x^2+h^2}}\left(-\hat{x}\int_0^{w/2}\frac{h}{\sqrt{x^2+h^2}}dx\right)$$

goes inside integral

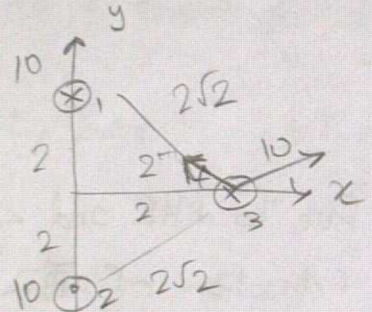
$$= \frac{-\hat{x}\,\mu I h}{\pi w}\int_0^{w}\frac{1}{x^2+h^2}dx \Rightarrow \vec{B} = \frac{-\hat{x}\,\mu I}{w\pi}\arctan\left(\frac{w}{2h}\right) \text{ [T]}$$

b) $F = I(\vec{L} \times \vec{B})$ $\quad \hat{z}\ \hat{x}\ \hat{y}$

$\vec{L} = \hat{y}$, $\vec{B} = \dfrac{\mu I \arctan(w/2h)}{\pi w}(-\hat{x})$ $\quad \hat{y} \times -\hat{x} = \hat{z}$

$\therefore \vec{F}_{\substack{\text{wire} \\ \text{due to} \\ \text{sheet}}} = \hat{z}\,\dfrac{\mu I^2 \arctan(w/2h)}{\pi w}$ [N]

## 5.19




$\vec{F}_3 = \vec{F}_{3,1} + \vec{F}_{3,2}$

Due to symmetry and equal currents, can calculate magnitudes of $\vec{F}$ and by inspection, direction is $\hat{y}$

$\vec{F}_3 = \hat{y}\, I_3 \left(\dfrac{\mu_0 I_1}{2\pi 2\sqrt{2}}\right) \sin 45^\circ + \hat{y}\, I_3 \left(\dfrac{\mu_0 I_2}{2\pi 2\sqrt{2}}\right) \sin 45^\circ$

$= \hat{y}\, \dfrac{2I^2 \mu_0}{4\pi\sqrt{2}} \sin 45^\circ = \hat{y}\, 1\times10^{-5}$ [N] $= \vec{F}$

## 5.25

$\vec{H} = \hat{\phi}\, \dfrac{2}{r}\left[1-(4r+1)e^{-4r}\right]$ [A/m] for $r \le 0.05$

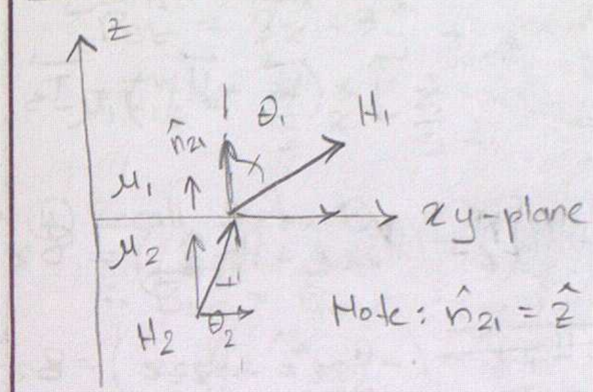
$\oint_c \vec{H}\cdot d\vec{l} = I_{enc} = \int_0^{2\pi} \hat{\phi}\, \dfrac{2}{r}\left[1-(4r+1)e^{-4r}\right] \cdot \hat{\phi}\, 0.05\, d\phi$

$I_{enc}(0.05) = \dfrac{4\pi(0.05)}{0.05}\left[1-(4\times0.05+1)e^{-4\times0.05}\right] = \underline{0.2202\ \text{A}}$

ECE221 Tutorial 9

## 5.32

a)




Note: $\hat{n}_{21} = \hat{z}$

Let $\vec{H}_1 = \vec{H}_{1n} + \vec{H}_{1t} = \vec{H}_{1z} + \vec{H}_{1x} + \vec{H}_{1y}$

$\vec{H}_2 = \vec{H}_{2n} + \vec{H}_{2t} = \vec{H}_{2z} + \vec{H}_{2x} + \vec{H}_{2y}$

Recall $B_{1n} = B_{2n}$ $\rightarrow H_{2n} = \dfrac{\mu_1}{\mu_2} H_{1n}$

$\mu_1 H_{1n} = \mu_2 H_{2n}$

$\therefore H_{2z} = \dfrac{\mu_1}{\mu_2} H_{1z}$

Since $J_s = 0$, then $\hat{z} \times (\vec{H}_1 - \vec{H}_2) = 0$ $\quad \hat{x}\ \hat{z}\ \hat{y}$

$\hat{z} \times (\vec{H}_{1x} + \vec{H}_{1y} - (\vec{H}_{2x} + \vec{H}_{2y})) = 0$

$\vec{H}_{1y} - \vec{H}_{1x} - (\vec{H}_{2y} - \vec{H}_{2x}) = 0 \Rightarrow \vec{H}_{1y} - \vec{H}_{1x} = \vec{H}_{2y} - \vec{H}_{2x}$

$\therefore$ $H_{1y} = H_{2y}$ and $H_{2x} = H_{2x}$ $\quad \therefore \vec{H}_2 = H_{1x}\hat{x} + H_{1y}\hat{y} + \dfrac{\mu_1}{\mu_2} H_{1z}\hat{z}$

b) Since $\vec{H}_{1t} = \vec{H}_{2t} \rightarrow |\vec{H}_1|\sin\theta_1 = |\vec{H}_2|\sin\theta_2$

but $\vec{H}_{2n} = \dfrac{\mu_1}{\mu_2}\vec{H}_{1n} \rightarrow |\vec{H}_1|\cos\theta_1 = \dfrac{\mu_1}{\mu_2}|\vec{H}_2|\cos\theta_2$

divide since $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$

$\tan\theta_1 = \dfrac{\mu_1}{\mu_2}\tan\theta_2$

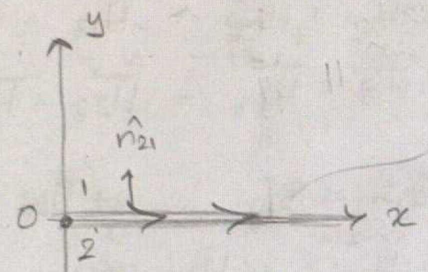
c) $\vec{H}_1 = 2\hat{x} + 0\hat{y} + 4u\hat{z}$ and $\mu_1 = \mu_0$, $\mu_2 = 4\mu_0$

$\vec{H}_2 = 2\hat{x} + 0\hat{y} + \dfrac{\mu_0}{4\mu_0}\cdot 4u\,\hat{z} = 2\hat{x} + 0\hat{y} + (10)^{-6}\hat{z}$ [A/m] $= \vec{H}_2$

$\theta_1 = \arctan\left(\dfrac{H_{1t}}{H_{1n}}\right) = \arctan\left(\dfrac{2}{4u}\right) = 89.9999^\circ$

$\theta_2 = \arctan\left(\dfrac{\mu_2}{\mu_1}\tan(\theta_1)\right) = 89.9999^\circ$

## 5.33




$\hat{z}\ \hat{x}\ \hat{y}$

Note that $\hat{n}_{21} = \hat{y}$

$\vec{J}_s = 8\hat{x}$ [A/m]

$\therefore \hat{y} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s$

$\vec{H}_{1n} = 0 = \vec{H}_{2n}$

$\vec{H}_{1t} = 11\hat{z}$

$\vec{H}_{2t} =$

$\hat{y} \times (H_{1x}\hat{x} + H_{1z}\hat{z} - (H_{2x}\hat{x} + H_{2z}\hat{z})) = 8\hat{x}$

$-H_{1x}\hat{z} + H_{1z}\hat{x} - (-H_{2x}\hat{z} + H_{2z}\hat{x}) = 8\hat{x}$

0

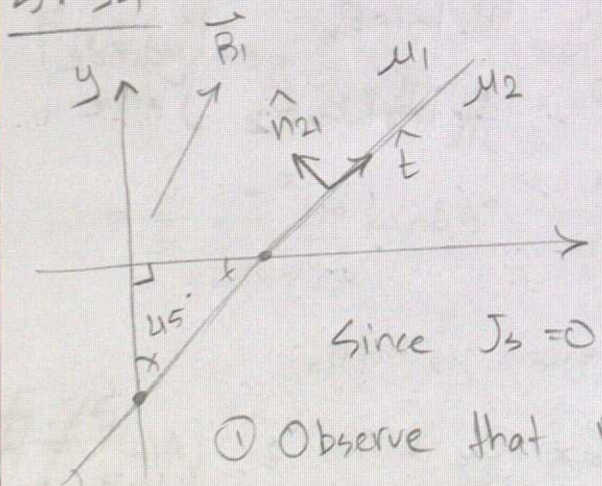
$-H_{1x}\hat{z} + H_{1z}\hat{x} - H_{2z}\hat{x} = 8\hat{x}$

0

$\hat{x}: H_{1z} - H_{2z} = 8 \rightarrow H_{1z} = 11 - 8 = 3$

$\therefore \vec{H}_2 = 0\hat{x} + 0\hat{y} + 3\hat{z}$ [A/m]

## 5.34




$\hat{n}_{21} = -\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y}$

$\vec{B}_1 = 2\hat{x} + 3\hat{y}$

Since $J_s = 0 \rightarrow B_{1n} = B_{2n}$, $H_{1t} = H_{2t}$

① Observe that $B_{1n} = \hat{n}_{21} \cdot \vec{B}_1$

$B_{1n} = -\frac{2}{\sqrt{2}} + \frac{3}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

② But $\hat{n}_{21} = \frac{\vec{B}_{1n}}{|\vec{B}_{1n}|} = \frac{\vec{B}_{1n}}{B_{1n}} \rightarrow \vec{B}_{1n} = \hat{n}_{21}|\vec{B}_{1n}| = \hat{n}_{21} B_{1n}$

$\vec{B}_{1n} = \frac{\sqrt{2}}{2}\left(-\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y}\right) = -\frac{1}{2}\hat{x} + \frac{1}{2}\hat{y}$ //

③ Observe that $\vec{B}_1 = \vec{B}_{1n} + \vec{B}_{1t} \rightarrow \vec{B}_{1t} = \vec{B}_1 - \vec{B}_{1n}$

$\vec{B}_{1t} = 2\hat{x} + 3\hat{y} + \frac{1}{2}\hat{x} - \frac{1}{2}\hat{y} = \frac{5}{2}\hat{x} + \frac{5}{2}\hat{y}$

$\mu_1 \vec{H}_{1t} = \frac{5}{2}\hat{x} + \frac{5}{2}\hat{y}$

④ Recall that $B_{1n} = B_{2n}$ and $H_{1t} = H_{2t}$

$\therefore \vec{B}_{2n} = \vec{B}_{1n} = -\frac{1}{2}\hat{x} + \frac{1}{2}\hat{y}$ //

$\because \mu_1 \vec{H}_{2t} = \frac{5}{2}\hat{x} + \frac{5}{2}\hat{y}$, but $\vec{B}_{2t} = \mu_2 \vec{H}_{2t} \rightarrow \vec{H}_{2t}\ \frac{\vec{B}_{2t}}{\mu_2}$

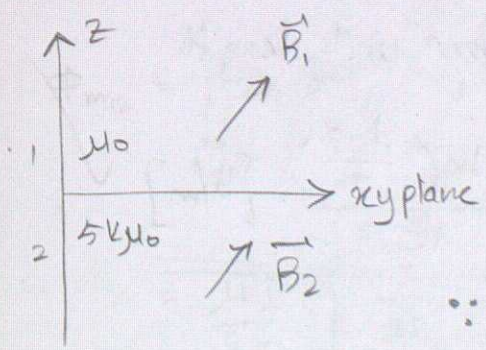
$\frac{\mu_1}{\mu_2}\vec{B}_{2t} = \frac{5}{2}\hat{x} + \frac{5}{2}\hat{y} = \frac{5\mu_2}{\mu_2}\vec{B}_{2t} = 5\vec{B}_{2t}$

$\therefore \vec{B}_{2t} = \frac{1}{2}\hat{x} + \frac{1}{2}\hat{y}$

⑤ Since $\vec{B}_2 = \vec{B}_{2n} + \vec{B}_{2t} = -\frac{1}{2}\hat{x} + \frac{1}{2}\hat{y} + \frac{1}{2}\hat{x} + \frac{1}{2}\hat{y} = 1\hat{y}$

$\therefore \vec{B}_2 = 1\hat{y}$

## 5.35




Since no $J_s$ mention: $B_{1n} = B_{2n}$, $H_{1t} = H_{2t}$

$\vec{B}_1 = \underbrace{4\hat{x} - 6\hat{y}}_{\vec{B}_{1t}} \underbrace{+ 8\hat{z}}_{\vec{B}_{1n}}$

$\because B_{1n} = B_{2n}$, $\therefore \vec{B}_{2n} = 8\hat{z}$ //

$\because \vec{B}_{1t} = \mu_1 \vec{H}_{1t} = 4\hat{x} - 6\hat{y} = \mu_1 \vec{H}_{2t}$

and $\vec{H}_{2t} = \frac{\vec{B}_{2t}}{\mu_2}$, $\therefore \frac{\mu_1}{\mu_2}\vec{B}_{2t} = 4\hat{x} - 6\hat{y}$

$\frac{\mu_0}{5k\mu_0}\vec{B}_{2t} = 4\hat{x} - 6\hat{y}$

$\vec{B}_{2t} = 20k\hat{x} - 30k\hat{y}$ //

$\therefore \vec{B}_2 = 20000\hat{x} - 30000\hat{y} + 8\hat{z}$

## 5.37

Recall that self inductance $L = \frac{N\Phi_m}{I}$ and observing the symmetry, we can compute contribution due to one wire and multiply by 2:

① $\vec{B} = \frac{\mu I}{2\pi x}\hat{y}$ (recall $\vec{B}$ due to infinite line current, can derive using Ampere's Law)

② $\Phi_m = \iint_S \vec{B}\cdot d\vec{A} = \int_a^{d-a}\int_0^{l} \frac{\mu I}{2\pi x}\hat{y}\cdot dz\,dx\,\hat{y}$

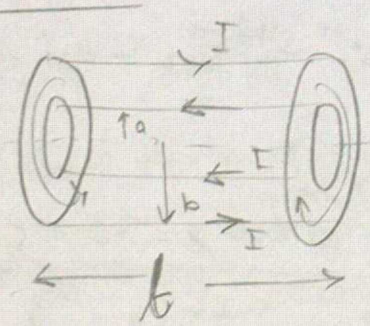
$$= \frac{\mu I}{2\pi}\int_0^l dz\int_a^{d-a}\frac{dx}{x} = \frac{\mu I l}{2\pi}\ln\left(\frac{d-a}{a}\right)$$

③ $L = \frac{N\Phi_m}{I} = \frac{1\times\frac{\mu \cancel{I} l}{2\pi}\ln\left(\frac{d-a}{a}\right)}{\cancel{I}} = \frac{\mu l}{2\pi}\ln\left(\frac{d-a}{a}\right)$

④ multiply $L$ by 2 to account for second wire and divide by $l$ to get inductance per unit length

$$\frac{2L}{l} = \frac{\frac{2\mu\cancel{l}}{2\pi}\ln\left(\frac{d-a}{a}\right)}{\cancel{l}} = \frac{\mu\ln\left(\frac{d-a}{a}\right)}{\pi}\ [H/m]$$

## 5.39




$W = \frac{1}{2}LI^2$

$L = \frac{1\times\Phi_m}{I} = l(b-a)$

$$W = \frac{1}{2}\iiint_V \vec{B}\cdot\vec{H}\,dV = \frac{1}{2}\iiint_V \mu H^2\,dV = \frac{1}{2}\int_0^l\int_a^b\int_0^{2\pi}\mu H^2 r\,d\phi\,dr\,dz$$

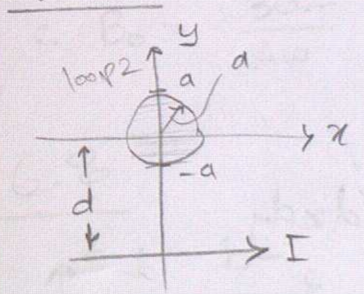
Ampere's Law: $\oint_C B\cdot dl = \mu I \Rightarrow H(2\pi r) = I$

$|\vec{H}| = \frac{I}{2\pi r} \Rightarrow W = \frac{1}{2}\mu\int_0^l\int_a^b\int_0^{2\pi}\frac{I^2}{4\pi^2 r^2}r\,d\phi\,dr\,dz$

$W = \frac{\mu I^2 l\,2\pi}{8\pi^2}\int_a^b\frac{1}{r}dr = \frac{\mu I^2 l}{4\pi}\ln(b/a)$

$\therefore W = \frac{\mu I^2(3)}{4\pi}\ln(2) = 2(10)^{-7}I^2\ [J]$

## ☆ 5.40




$L_{12} = \frac{N_2\Phi_{m12}}{I_1} = \frac{1\times\iint_S B_1\cdot dS_2}{I_1}$

$|\vec{B}_1| = \frac{\mu I_1}{2\pi d}$, $dS_2 = \hat{z}\,rd$

$x^2+y^2 = a^2$

$x = \pm\sqrt{a^2-y^2}$

$$\Phi_{m12} = \iint_S B_1\cdot dS_2 = \int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{+\sqrt{a^2-y^2}}\frac{\mu I}{2\pi(d+y)}\,dx\,dy$$

$$\Phi_{12} = \frac{\mu I}{2\pi}\int_{-a}^{a}\frac{1}{d+y}2\sqrt{a^2-y^2}\,dy$$

ECE221 Tutorial 10

## 6.3

$$V_{emf} = -N\frac{d}{dt}\Phi_B$$

$$= -N\frac{d}{dt}\iint_S B\cdot ds$$

a) If $\vec{B} = \hat{z}\,20e^{-3t}$ [T]

$$V_{emf} = -100\frac{d}{dt}\int_0^{0.25}\int_0^{0.25} 20e^{-3t}\,\hat{z}\cdot\hat{z}\,dx\,dy$$

$$= -2000(0.25)^2(-3)e^{-3t}$$

$$\underline{V_{emf} = 375\,e^{-3t}\ [V]}\ \checkmark$$

b) If $\vec{B} = \hat{z}\,20\cos x\cos 10^3 t$ [T]

$$V_{emf} = -100\frac{d}{dt}\int_0^{0.25}\int_0^{0.25} \cancel{\hat{z}}\,20\cos x\cos 10^3 t\,\cancel{\hat{z}}\,dx\,dy$$

$$= -2000(0.25)\int_0^{0.25}\cos x\,dx\,\frac{d}{dt}\cos 10^{3t}$$

$$= -2000(0.25)\sin(0.25)(10)^3(-\sin 10^3 t)$$

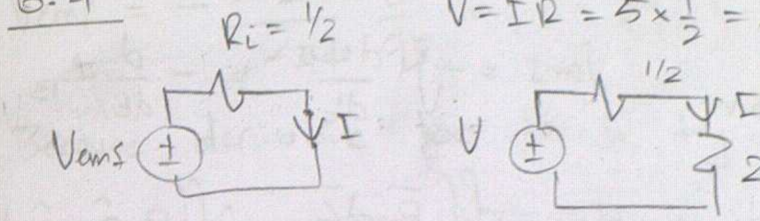
$$\underline{V_{emf} = -123.7\sin 10^3 t\ [kV]}\ \checkmark$$

c) $$V_{emf} = -100\frac{d}{dt}\int_0^{0.25}\int_0^{0.25} 20\cos x\sin 2y\cos 10^3 t$$

$$= -2000\sin(0.25)\left(-\tfrac{1}{2}\cos(0.5)+\tfrac{1}{2}\cos(0)\right)(-\sin 10^3 t)$$

$$\underline{V_{emf} = 30.29\sin 10^3 t\ [V]}$$ ← wrong since bounds have to be $\int_{-0.125}^{0.125} \rightarrow \underline{\underline{=0}}$

## 6.4

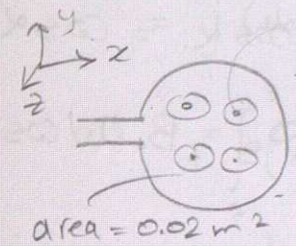



$V = IR = 5\times\frac{1}{2} = 2.5\ V$

$$I = \frac{V}{R_t} = \frac{2.5}{2.5} = 1\ A$$

$$\therefore \underline{I = 1\ [A]}$$

## 6.5




$\vec{B} = \hat{z}\,B_0\cos(\omega t) \rightarrow \omega = 2\pi f = 600\pi$ Mrad/s

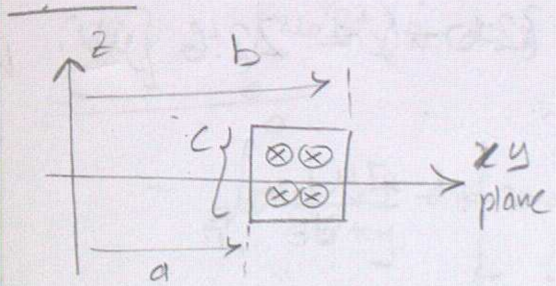
$$V_{emf} = -N\frac{d}{dt}\Phi_B = -\frac{d}{dt}B\cdot A\cdot\cos\alpha$$ ← angle b/n loop an B

Since oriented for max response, $\alpha = 0° \rightarrow \cos\alpha = 1$

$$\therefore 30m = -A\frac{d}{dt}B_0\cos(\omega t) = A\omega B_0\sin(\omega t)$$ → set $\sin(\omega t) = 1$ for max $B_0$

$$\therefore B_0 = \frac{30m}{A\omega} = \frac{30m}{(0.02)(600\pi M)} = \underline{795.8\ [pT]}\ \checkmark$$

## 6.8




Recall that $\vec{B}_{inc,line} = \frac{\mu I(t)}{2\pi R}\hat{\phi}$

$$\therefore |\vec{B}(t)| = \frac{\mu I_0\cos\omega t}{2\pi x}$$

$$\Phi_B = \iint_S B\cdot ds = \int_{-c/2}^{c/2}\int_a^b \frac{\mu I_0\cos\omega t}{2\pi x}\,dx\,dy$$

$$\therefore \Phi_B = \frac{c\mu I_0\cos\omega t}{2\pi}\ln(b/a) \rightarrow V_{emf} = -N\frac{d}{dt}\Phi_B$$

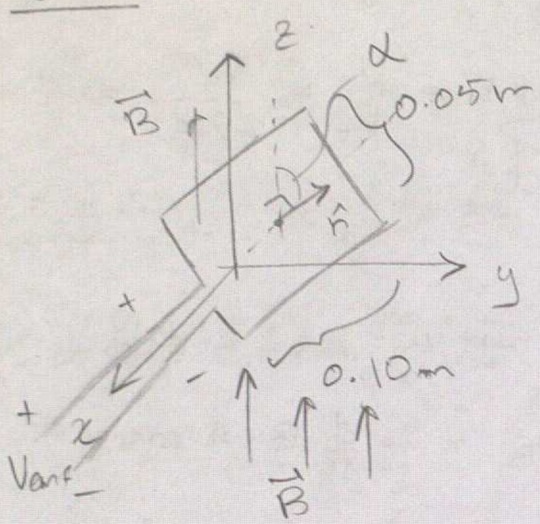
4000 00

a) $$\underline{V_{emf} = +\frac{Nc\mu I_0\ln(b/a)}{2\pi}\omega\sin\omega t\ [V]}$$ since $\mu r = 4000$

b) $$V_{emf} = \frac{400k(0.02)(4\pi\times10^{-7})(50)\ln(6/5)}{2\pi}2\pi(60)\sin(60 2\pi t)\ [V]$$ $\times 4000$ 377

$$\underline{V_{emf} = 5.5\sin(377t)\ [V]}\ \checkmark$$

## 6.9




$$V_{emf} = -N\frac{d\Phi_B}{dt} = -\frac{d}{dt}\Phi_B$$

$$\Phi_B = \iint_S \vec B\cdot d\vec S = \iint_S B_0\hat z\cdot\hat n\,dz\,dy$$

$$\hat z\cdot\hat n = |\hat z||\hat n|\cos\alpha = \cos\alpha$$

? $\boxed{\alpha = \omega t} \Rightarrow \Phi_B = B_0 l W\cos\omega t$

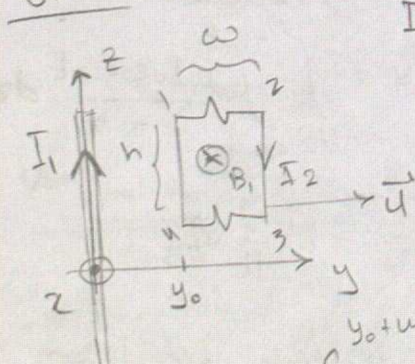
$$V_{emf} = -\frac{d}{dt}\left[B_0 l w\cos\omega t\right] = B_0 l W\omega\sin\omega t$$

peak voltage implies $\sin\omega t = 1$

$$\omega = \frac{7200\text{ rev}}{1\text{ min}}\times\frac{1\text{ min}}{60\text{ s}}\times\frac{2\pi\text{ rad}}{1\text{ rev}} = 240\pi\text{ rad/s}$$

$$\therefore V_{emf\,pk} = (6\mu)(0.05)(0.10)(240\pi) = 22.6\ [\mu V]$$

## 6.11




$I_2 = \frac{V_{emf}}{2R}$, $V_{emf} = -1\times\frac{d}{dt}\Phi_B$

$$\Phi_B = \iint_S \vec B\cdot d\vec S$$

$\vec B = -\hat x\frac{\mu I_1}{2\pi y}$ $\quad d\vec S = -\hat x\,dz\,dy$

$$\Phi_B = \int_{y_0}^{y_0+w}\int_0^h \frac{\mu I_1}{2\pi y}\,dz\,dy = \frac{\mu I_1 h}{2\pi}\ln\left(\frac{y_0+w}{y_0}\right)$$

$$\frac{d}{dt}\Phi_B = \frac{\mu I_1 h}{2\pi}\frac{d}{dt}\ln\left(\frac{y_0(t)+w}{y_0(t)}\right)$$

$$V_{emf} = -\frac{\mu I_1 h}{2\pi}\frac{d}{dt}\ln\left(1+\frac{w}{y_0(t)}\right)$$

Because derivative too trash, try a different method

$V_{emf} = \oint_C(\vec u\times\vec B)\cdot d\vec l$ $\quad \vec u = 7.5\hat y = u_0\hat y$, $\vec B = -\hat x\frac{\mu I_1}{2\pi y} = B(y)(-\hat x)$

$\hat x$, $\hat z \circlearrowleft \hat y$

$\hat y\times\hat x = -\hat z$

$\hat y\times -\hat x = \hat z$

$$= \int_1^2(u_0\hat y\times -\hat x B(y))\cdot dy\,\hat y + \int_2^3(u_0\hat y\times -\hat x B(y))\cdot dz\,\hat z + \int_3^4(u_0\hat y\times -\hat x B(y))\cdot dy\,\hat y + \int_4^1(u_0\hat y\times -\hat x B(y))\cdot dz\,\hat z$$

$$= \int_{y_0}^{y_0+w} u_0B(y)\cancelto{0}{\hat z\cdot\hat y}\,dy + \int_h^0 u_0B(y)\hat z\cdot dz\,\hat z + \int_{y_0+w}^{y_0} u_0B(y)\cancelto{0}{\hat z\cdot dy\,\hat y} + \int_0^h u_0B(y)\hat z\cdot dz\,\hat z$$

$$= \int_h^0 u_0\frac{\mu I_1}{2\pi y}dz\Big|_{y=y_0} + \int_0^h u_0\frac{\mu_1 I_1}{2\pi y}dz\Big|_{y=y_0+w}$$

$h = 0.2$
$w = 0.1$
$u_0 = 7.5$
$I_1 = 10$
$R = 10$

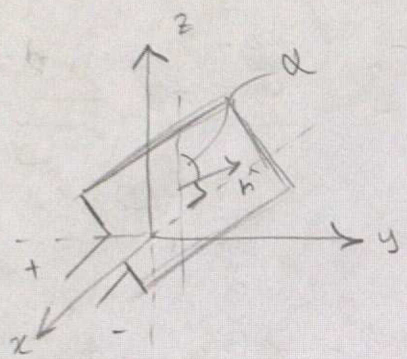
$$= \frac{-hu_0\mu I_1}{2\pi y_0} + \frac{hu_0\mu_1 I_1}{2\pi(y_0+w)} = \frac{hu_0\mu I_1}{2\pi}\left(\frac{1}{y_0+w}-\frac{1}{y_0}\right)$$

$$\therefore I_2 = \frac{V_{emf}}{2R} = \frac{hu_0\mu I_1}{4\pi R}\left(\frac{1}{y_0+w}-\frac{1}{y_0}\right) = 150\left(\frac{1}{y_0+0.1}-\frac{1}{y_0}\right)\ [nA]$$

## 6.12 ✓

$$I = \frac{V_{emf}}{R}, \quad \mathcal{E}_{emf} = -1\frac{d}{dt}\iint \vec{B}\cdot d\vec{s} = -\frac{d}{dt}\left[B_0\hat{z}\cdot A\hat{n}\right]$$

assum



$$= -\frac{d}{dt}\left[B_0 A\cos\alpha\right] \text{ where } \alpha = \omega t$$

$$= -B_0 A \frac{d}{dt}\cos\omega t$$

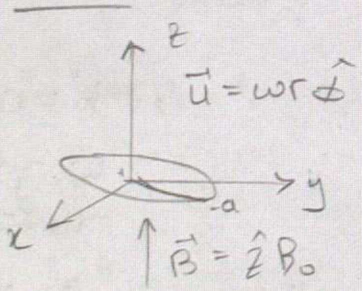
set $\sin\omega t = 1$

$$= +B_0 A\omega\sin\omega t \Rightarrow |V_{emf}| = B_0 A\omega$$

Note $\omega = \frac{3600\,\cancel{rev}}{1\,\cancel{min}} \times \frac{1\,\cancel{min}}{60\,s} \times \frac{2\pi\,rad}{1\,\cancel{rev}} = 120\pi$ rad/s

$$\therefore I = \frac{V_{emf}}{R} = \frac{B_0 A\omega}{R} = \frac{(0.4)(0.1)(120\pi)}{150} = \underline{0.1\,[A]}$$ ✓

## 6.13 ✓



~~Consider thin "bar" and later integrate~~

$$V_{emf} = \int_C (\vec{u}\times\vec{B})\cdot dl = \int_0^a B_0\omega r\,\hat{r}\cdot dr\hat{r}$$

$$= B_0\omega\int_0^a r\,dr = \frac{B_0\omega}{2}a^2$$ ← for one thin bar

$\hat{r}$ $\hat{z}$ $\hat{\phi}$

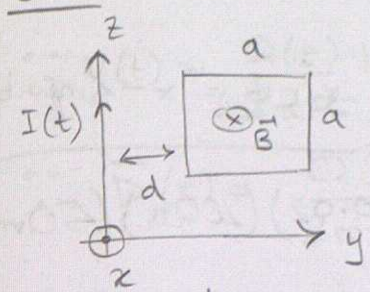
$\hat{\phi}\times\hat{z} = \hat{r}$

$$\therefore V_{emf} = \frac{B_0\omega a^2}{2}\,[V]$$ ✓

# ECE221 Tutorial 11

## 6.6 ✓

a)



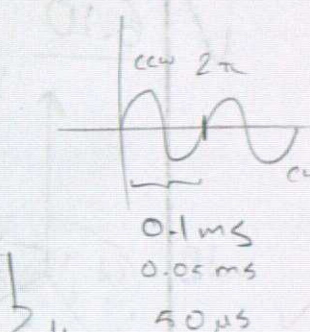
$$V = -\frac{d}{dt}\Phi_B, \quad \Phi_B = \iint_S \vec{B}\cdot d\vec{s}$$

$$|\vec{B}| = \frac{\mu I(t)}{2\pi y}, \quad |d\vec{s}| = dy\,dz$$

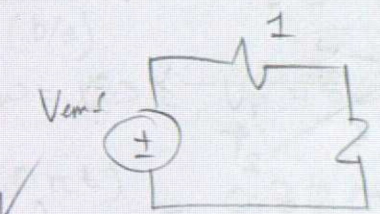
$a = 0.1\,m$
$d = 0.05\,m$
$\omega = 2\pi\times10^4$

$$\Phi_B = \int_d^{d+a}\int_0^a \frac{\mu I(t)}{2\pi y}\,dy\,dz = \frac{\mu I(t)a}{2\pi}\ln\left(\frac{d+a}{d}\right)$$

$$V_{emf} = -\frac{d}{dt}\Phi_B = -\frac{\mu a\ln\left(\frac{d+a}{d}\right)}{2\pi}\frac{d}{dt}\left[5\cos\omega t\right]$$



$$= +\frac{\mu a\ln\left(\frac{d+a}{d}\right)(5)\omega}{2\pi}\sin\omega t$$

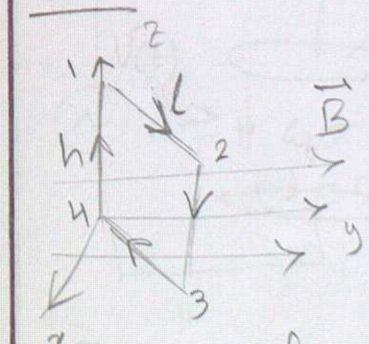


$$V_{emf} = 6.9\sin(2\pi\times10^4 t)\,[mV]$$ ✓

b)

$$|I| = \frac{V_{emf}}{5} = 1.38\,mA \begin{cases} ccw, & 50\mu s(2k+1),\ k\in 0,\pm1,\pm2\ldots \\ cw, & \end{cases}$$

## 6.7 ✓



$$I = \frac{V_{emf}}{R}, \quad V_{emf} = \oint_C(\vec{u}\times\vec{B})\cdot dl$$

$$\vec{B} = B_0\hat{y}, \quad \vec{u}_{41} = 0, \quad \vec{u}_{12} = \vec{u}_{34} = \omega r\hat{\phi}, \quad \vec{u}_{23} = l\omega\hat{\phi}$$

$$\hat{\phi}\times\hat{y} = (-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})\times\hat{y} = -\sin\phi\,\hat{z}$$

$$\therefore\oint_C(\vec{u}\times\vec{B})\cdot dl = -\int_1^2 u_{12}B_0\cancelto{0}{\hat{z}\cdot dr\hat{r}} + \int_2^3 u_{23}B_0\hat{z}\cdot dz\hat{z}$$

$$-\int_3^4 u_{34}B_0\cancelto{0}{\hat{z}\cdot dr\hat{r}} - \int_4^1 \cancelto{0}{u_{41}}B_0\hat{z}\cdot dz\hat{z}$$

$\hat{r}$ $\hat{z}$ $\hat{\phi}$

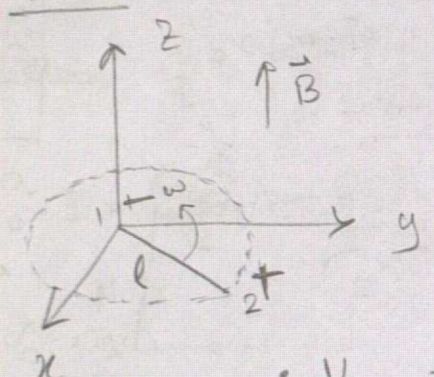
$\hat{z}\cdot\hat{r} = 0$

$\therefore \oint_c (\vec{u}\times\vec{B})\cdot d\vec{l} = -\int_h^0 u_{23} B_0 \, dz = h u_{23} B_0$

$\therefore I = \frac{\varepsilon_{enc}}{R}$, $\therefore I = \frac{h u_{23} B_0}{1/2} = -2\underbrace{(0.03)(0.02)(200\pi)(50m)}_{}$   $x - \sin\phi = x - \sin\omega t$ ✓

$\omega = \frac{6k\,rev}{min} \times \frac{1}{60} \times 2\pi = 200\pi \Rightarrow \underline{I = 37.7\sin(200\pi t)\ [mA]}$

✓ **6.10**



$V_{12} = \int_1^2 (\vec{u}\times\vec{B})\cdot d\vec{l}$   $\hat{r}$   $\hat{z}$   $\hat{\phi}$

$\vec{u} = r\omega\hat{\phi}$, $\vec{B} = B_0\hat{z} \Rightarrow \hat{\phi}\times\hat{z} = \hat{r}$

$d\vec{l} = \hat{r}dr$

$\omega = 90^3 \times \frac{2\pi}{60} = 3\pi$

$\therefore V_{12} = -\int_0^l r\omega B_0 \hat{r}\cdot\hat{r}dr = \frac{-\omega B_0 l^2}{2}$

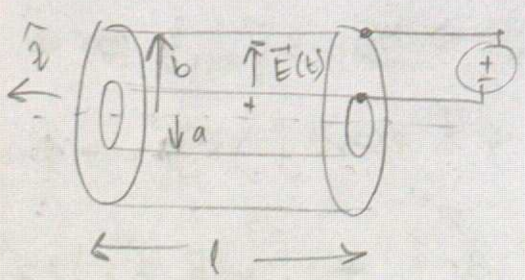
$B_0 = 0.2m$

$l = 0.5$

$\therefore V_{12} = \frac{-(3\pi)(0.2m)(0.5)^2}{2} = \underline{-0.236[V]}$ ✓

P ✓ used EM instead of ckt method

**6.15**



$V(t) = 50\sin(120\pi t) = V_0\sin\omega t$

Apply $\nabla\cdot\vec{D} = \rho_s$

$\iint_S \vec{D}\cdot d\vec{s} = \iiint_V \rho_s dV = Q_{enc} = +Q$

$2\pi r l D = Q$   $Q = CV$, $V = \int \frac{D}{\varepsilon}\cdot dl = \frac{Q}{\varepsilon 2\pi l}\ln(b/a)$

$\vec{D} = \frac{Q}{2\pi r l}\hat{r}$

Since $V(t)$, then $Q(t)$ and $\therefore \vec{D}(t)$

$\therefore V(t) = \frac{Q(t)}{\varepsilon 2\pi l}\ln(b/a) \Rightarrow Q(t) = \frac{V(t)2\pi l\varepsilon}{\ln(b/a)}$

$\therefore \vec{D}(t) = \frac{Q(t)}{2\pi r l}\hat{r} \Rightarrow \frac{\varepsilon V(t)\cancel{2\pi l}}{\cancel{2\pi}r\cancel{l}\ln(b/a)} = \frac{\varepsilon V(t)}{\ln(b/a)r} = \frac{\varepsilon V_0\sin\omega t}{\ln(b/a)r}\hat{r}$

From Ampere-Maxwell Law: $\oint_c \vec{H}\cdot d\vec{l} = \int_s \vec{J}\cdot d\vec{s} + \frac{d}{dt}\int_s \vec{D}\cdot d\vec{S} = I$

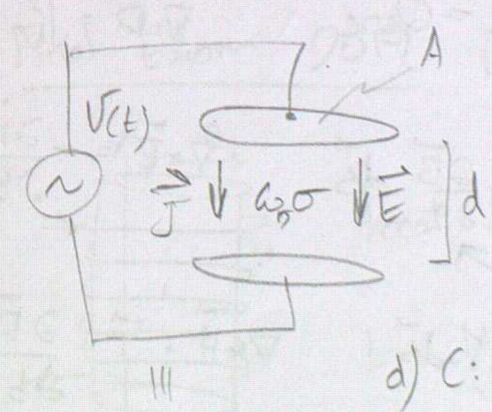
$I = \oint_c \vec{H}\cdot d\vec{l} = 0 + \frac{d}{dt}\int_0^l\int_0^{2\pi} \frac{\varepsilon V_0\sin\omega t}{\ln(b/a)\cancel{r}}\hat{r}\cdot\hat{r}\,\cancel{r}d\phi dz$

$= \frac{d}{dt}\left[\frac{V_0\sin\omega t\, 2\pi l}{\ln(b/a)}\right] = \frac{\varepsilon 2\pi l\omega V_0\sin\omega t}{\ln(b/a)}$ [A]

$l = 0.06$, $b = 0.01$, $a = 0.005$, $\omega = 120\pi$, $V_0 = 50$, $\varepsilon = 9\varepsilon_0$ ✓

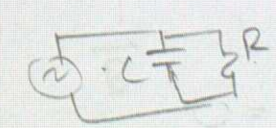
$I = \frac{9\varepsilon_0 2\pi(0.06)(120\pi)(50)\sin(120\pi t)}{\ln(0.01/0.005)} = \underline{0.82\cos(120\pi t)\ [\mu A]}$

✓ **6.16** (Recall $V = Ed \Rightarrow E = \frac{V}{d}$) and assume all surfaces ⊥ and uniform



$I_c = \iint_S \vec{J}\cdot d\vec{s} = JA = \sigma EA = \frac{\sigma VA}{d}$ //

$I_d = \frac{d}{dt}\iint_s \vec{D}\cdot d\vec{s} = \frac{d}{dt}\varepsilon EA = \frac{d}{dt}\varepsilon\frac{V}{d}A = \frac{\varepsilon V'(t)A}{d}$



d) C: $I_d = C\frac{V(t)}{dt} \Rightarrow C = \frac{I_d}{V'(t)} = \frac{\varepsilon A}{d} = \frac{4\varepsilon_0 0.4m}{0.005}$ ✓

R: $V = IR \Rightarrow R = \frac{V(t)}{I_c} = \frac{d}{\sigma A} = \frac{0.005}{2.5(0.4m)}$ ✓

$A = 4cm^2 \times \left(\frac{1m}{100cm}\right)^2 = 0.4m$

$d = 0.005\,m$, $\varepsilon_r = 4$, $\sigma = 2.5$

$\therefore \underline{C = 2.8\ [pF]}$ ✓

$\underline{R = 5\,\Omega}$ ✓

$1+sRC=0$

$s = -\frac{1}{RC}$

## 6.18

$\vec{E} = \hat{z} E_0 \cos\omega t$, $\epsilon = 81\epsilon_0$, $\sigma = 4$ → assume scalar integration

$I_c = \iint_S \vec{J}\cdot\vec{ds} = \sigma E(t) A \rightarrow J_c = \sigma E(t)$

$I_d = \frac{d}{dt}\iint \vec{D}\cdot\vec{ds} = \frac{d}{dt} A\epsilon E(t) \Rightarrow J_d = \epsilon E'(t)$

$\left.\begin{array}{l} J_c = \sigma E_0\cos\omega t \rightarrow |J_c| = \sigma E_0 \\ J_d = -\epsilon\omega E_0\sin\omega t \rightarrow |J_d| = \epsilon\omega E_0 \end{array}\right\} \frac{|J_c|}{|J_d|} = \frac{\sigma \cancel{E_0}}{\epsilon\omega \cancel{E_0}} = \frac{\sigma}{\epsilon\omega}$

a) $\frac{\sigma}{\epsilon 2\pi f} = \frac{4}{81(8.85\times10^{-12})2\pi(1k)} = 888079$ ✓

b) 888 ✓

c) 0.888 ✓

d) 0.00888 ✓

} as ω increases contribution from displacement current increases

---

1) $\oiint_S \vec{D}\cdot\vec{ds} = \iiint_V (\nabla\cdot\vec{D})\,dV = \iiint_V \rho_v\, dV = Q_{enc}$ $\qquad \nabla\cdot\vec{D} = \rho_V$

2) $\mathcal{E}_{ems} = \oint_C \vec{E}\cdot\vec{dl} = \iint_S (\nabla\times\vec{E})\cdot\vec{ds} = \iint_S -\frac{\partial\vec{B}}{\partial t}\cdot\vec{ds}$ $\qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

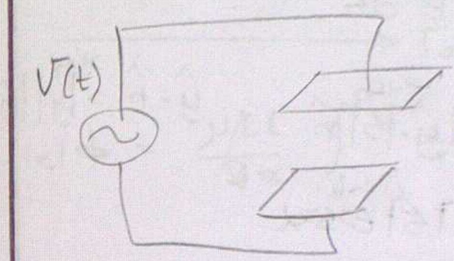
3) $\oint_C \vec{H}\cdot\vec{dl} = \iint_S (\nabla\times\vec{H})\cdot\vec{ds} = \iint_S \left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)\cdot\vec{ds}$ $\qquad \nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$

4) $\oiint_S \vec{B}\cdot\vec{ds} = \iiint_V (\nabla\cdot\vec{B})\,dV = 0$ $\qquad \nabla\cdot\vec{B} = 0$

ECE221 Stickel Lec 37

## Final Exam 2012

$\nabla\times\vec{H} = \underbrace{\vec{J}}_{I_c} + \underbrace{\frac{\partial}{\partial t}\vec{D}}_{I_d}$ (Maxwell equ. 4)




$I_c = \iint_S \vec{J}\cdot\vec{ds}$, $I_d = \iint_S \frac{\partial}{\partial t}\vec{D}\cdot\vec{ds}$

Given $C$, $V(t)$, $\omega$, $\sigma$, $A$
Knowing $C = \frac{Q}{V}$, but $V = Ed \rightarrow E = \frac{V}{d}$ } Assuming scalar integrals ...

$\frac{|I_c|}{|I_d|} = \frac{|\int_S \vec{J}\cdot\vec{ds}|}{|\int_S \frac{\partial}{\partial t}\vec{D}\cdot\vec{ds}|} = \frac{|JA|}{|\frac{\partial}{\partial t}\epsilon E| \times A} = \frac{|\sigma E A|}{|\frac{\partial}{\partial t}\epsilon\frac{V(t)}{d}| \times A} = \frac{|\sigma\frac{V(t)}{d}A|}{|\frac{\epsilon}{d}V'(t)| \times A} = \frac{|\sigma V(t) A|}{|d|}\cdot\frac{|d|}{|\epsilon V'(t) A|}$

$= \frac{|\sigma V_0\cos\omega t A|}{|-\epsilon V_0\omega\sin\omega t A|} = \frac{|\sigma V_0 A|}{|-\epsilon V_0\omega|} = \frac{\sigma A}{\epsilon\omega} = \frac{(2.12\times10^{-8})(3.2\times10^{-5})}{(4.22)(8.85\times10^{-12})(290)}$

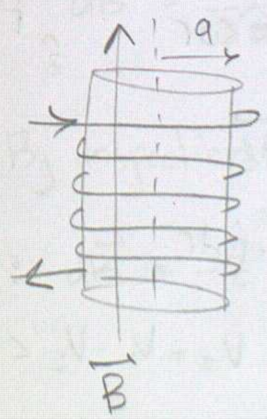
$\therefore \frac{|I_c|}{|I_d|} = \underline{1.96}$ ✓

## Final Exam 2019

$\vec{B} = \hat{z}\frac{\mu I N}{l}$, $a = 0.005$, $l = 2\pi^2\times10^{-2}$, $N = 20$

$\mu = 5k\mu_0$

a)




Recall that inductance $L = \frac{N\Phi}{I} = \frac{N|B|A}{I}$

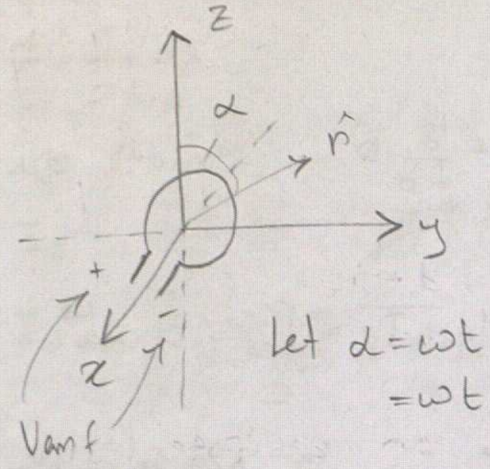
$L = \frac{N|\vec{B}|\pi a^2}{I} = \frac{N\frac{\mu I N}{l}\pi a^2}{I} = \frac{\mu N^2\pi a^2}{l}$

$= \frac{(5k)\mu_0(20)^2\pi(0.005)^2}{(2\pi^2\times10^{-2})}$

$L = 1\,[mH]$

Review Examples 2025

√ 1)




Recall that $V_{emf} = -N\frac{\partial}{\partial t}\Phi_B$

where $\Phi_B = \iint_S \vec{B}\cdot d\vec{s}$

$= |\vec{B}|\hat{y}\cdot|s|\hat{n}$ $\quad \hat{y}\cdot\hat{n} = |\hat{y}||\hat{n}|\cos\alpha = 1\times1\times\cos\alpha$

Let $\alpha = \omega t + 0 = \omega t$

$= |\vec{B}||s|\cos\alpha$

$= |\vec{B}|(\pi a^2)\cos\omega t$

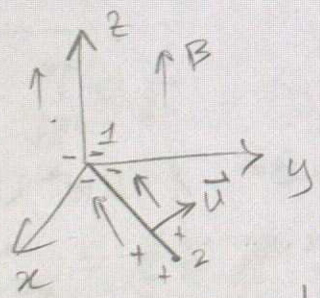
$\therefore V_{emf} = -1\frac{d}{dt}[B\pi a^2\cos\omega t] = B\pi a^2\omega\underbrace{\sin\omega t}_{set\,=1}$

$RPS \triangleq \frac{1\text{ rot}}{1s} \implies \omega = \frac{RPS\text{ rot}}{s}\times\frac{2\pi\text{ rad}}{\text{rot}}$ ✓

$\therefore V_{emf} = B\pi a^2\,RPS\cdot 2\pi$, $\therefore RPS = \frac{V_{emf}}{B\pi a^2 2\pi}$

$\therefore RPS = \frac{120}{(0.5\times10^{-4})(\pi^2)(2)(0.05)^2} = 48.6\ \frac{\text{million rot}}{s}$ ✓

√ 2)




$l = 0.5$, $\omega = \frac{90\text{ rev}}{\text{min}}\times\frac{1\text{min}}{60s}\times\frac{2\pi\text{ rad}}{\text{rev}} = 3\pi$

$\vec{B} = \hat{z}(0.2)[mT]$, $\vec{u} = \hat{\phi}\omega r$ $\quad \hat{r}\ \hat{z}\ \hat{\phi}$

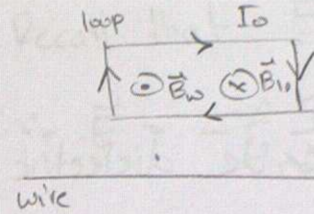
$|V_{12}| = \int^2 \vec{E}\cdot d\vec{l} = \int^2 (\vec{u}\times\vec{B})\cdot d\vec{l}$

$|V_{12}| = \int_0^l (\hat{\phi}\omega r\times B_0\hat{z})\cdot\hat{r}dr \rightarrow = \int_0^l \omega r B_0\hat{r}\cdot\hat{r}dr$

$= \omega B_0\int_0^l r\,dr = \frac{\omega B_0 l^2}{2}$, and since $V_1 < V_2$, $V_{12} = V_1 - V_2 < 0$

$\therefore V_{12} = -\frac{(3\pi)(0.2m)(0.5)^2}{2} = -235[\mu V]$ ✓

3)




option 1 ✓

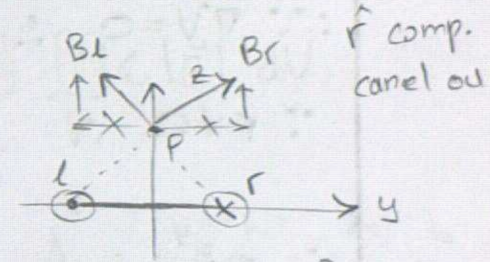
$\Phi = \iint B\cdot dA = \int_0^l\int_{d-w/2}^{d+w/2}\frac{\mu I}{2\pi r}\,dr\,dz$

$\Phi = \frac{\mu I l}{2\pi}\int_{d-w/2}^{d+w/2}\frac{1}{r}dr = \frac{\mu I l}{2\pi}\ln\left(\frac{d+w/2}{d-w/2}\right)$

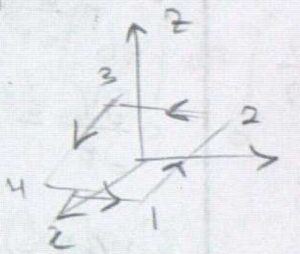
4) Recall that a perfect conductor is an equipotential surface. ∴ $\vec{E}$ alway normal to surface ⟶ $\boxed{E}$

5) $\boxed{f}$

6) Recall the: $d\vec{B} = \frac{\mu I}{4\pi}\frac{d\vec{l}'\times(\vec{R}-\vec{R}')}{|\vec{R}-\vec{R}'|^3}$




☆ Consider seg. 1-2




$d\vec{l}' = \hat{x}dx$, $\vec{R} = \hat{z}w$, $\vec{R}' = \hat{x}x + \hat{y}y$ $\quad \hat{x}\ \hat{z}\ \hat{y}$

$\vec{R}-\vec{R}' = \hat{z}w - \hat{x}x - \hat{y}y = -x\hat{x} - y\hat{y} + w\hat{z}$

$|\vec{R}-\vec{R}'|^3 = (x^2+y^2+z^2)^{3/2}$

$\therefore d\vec{B} = \frac{\mu I}{4\pi}\frac{\hat{x}dx\times(-x\hat{x}-y\hat{y}+w\hat{z})}{(x^2+y^2+z^2)^{3/2}} = \frac{\mu I}{4\pi}\frac{-\hat{z}ydx - \hat{y}wdx}{(x^2+y^2+z^2)^{3/2}}$

By inspection and RHR, $\hat{y}$ comp. cancel out

$\therefore d\vec{B} = -\hat{z}\frac{\mu I y}{4\pi}\int_w^{-w}\frac{1}{(y^2+z^2+x^2)^{3/2}}dx = -\frac{\hat{z}\mu I y}{2\pi}\left[\frac{x}{(y^2+z^2)\sqrt{(y^2+z^2)^2+z^2}}\right]_w^{-w}$

$\therefore \vec{B}(0,0,$

ECE221 Final 2023

Question 1

a) Because we assume zero free charge inside dielectric, we recall the Laplacian $\nabla^2 V = 0$

Knowing Boundary conditions $V(a) = V_0$ and $V(b) = 0$:

$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + 0 + 0$ since we assume $V$ is function of $r$ only

$\because \nabla^2 V = 0 \quad \therefore \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$

$\because r \neq 0 \quad \therefore \frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] = 0$

$\therefore \frac{\partial}{\partial t}[A] = 0$

$\therefore$ A must be a constant

$\therefore r\frac{\partial V}{\partial r} = A$ for A a constant

$\frac{\partial V}{\partial r} = \frac{A}{r}$ for $r \neq 0$

$V(r) = \int \frac{A}{r}\,dr = A\ln|r| + C$

Applying boundary cond.

$V(a) = V_0 = A\ln(a) + C$ ①

$V(b) = 0 = A\ln(b) + C$ ②

from ②: $C = -A\ln(b) \Rightarrow$ ①: $V_0 = A\ln(a) - A\ln(b) = A\ln(a/b)$

from ①: $A = \frac{V_0}{\ln(a/b)}$ // and $\therefore C = -\frac{V_0\ln(b)}{\ln(a/b)}$ //

Combing our constants, we see that

$V(r) = A\ln|r| + C = \frac{V_0\ln(r)}{\ln(a/b)} - \frac{V_0\ln(b)}{\ln(a/b)}$

$V(r) = \frac{V_0}{\ln(a/b)}\left[\ln(r) - \ln(b)\right] \Rightarrow V(r) = \frac{V_0}{\ln(a/b)}\ln\left(\frac{r}{b}\right)$ [V]

b) Recall that $\vec{E} = -\nabla V = -\left(\frac{\partial}{\partial r}V\hat{r} + 0 + 0\right)$ since $V = V(r)$

$\frac{1}{r/b}\cdot\frac{1}{b} = \frac{b}{r}\cdot\frac{1}{b}$

$\therefore \vec{E} = -\hat{r}\frac{\partial}{\partial r}\left[\frac{V_0}{\ln(a/b)}\ln(r/b)\right]$

$= -\frac{\hat{r}V_0}{\ln(a/b)}\left[\frac{1}{r/b}\cdot\frac{1}{b}\right] = -\hat{r}\frac{V_0}{\ln(a/b)}\cdot\frac{1}{r}$

$\therefore \vec{E} = -\frac{V_0}{\ln(a/b)}\frac{1}{r}\hat{r}$ [V/m]

d) $C = \frac{Q}{V}$, but $W_e = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{1}{2}\iiint_V \varepsilon|\vec{E}|^2\,dV$

Cap 1 $\varepsilon_1$

$W_e = \frac{1}{2}\int_0^{L/2}\int_0^{2\pi}\int_a^b \varepsilon_1 - \frac{V_0^2}{\ln^2(a/b)}\frac{1}{r^{\cancel{2}}}\cancel{r}\,dr\,d\phi\,dz = \frac{-\varepsilon_1 V_0^2}{2\ln^2(a/b)}\int_0^{L/2}dz\int_0^{2\pi}d\phi\int_a^b\frac{1}{r}dr$

$= -\frac{\varepsilon_1 V_0^2 L\, 2\pi}{4\ln^2(a/b)}\ln(b/a) = -\frac{\varepsilon_1 V_0^2 L\pi}{2\ln^2(a/b)}\ln(b/a)$ [J]

Cap 2 $\varepsilon_2$: $W_e = -\frac{\varepsilon_2 V_0^2 L\pi}{2\ln^2(a/b)}\ln(b/a)$ [J]

$\therefore$ whole cable energy $= W_1 + W_2$

$W_e = -\frac{\varepsilon_0 V_0^2 L\pi\ln(b/a)}{2\ln^2(a/b)}\left(\varepsilon_1 + \varepsilon_2\right)$ [J]

c) Recall $C = \frac{Q}{V} = \frac{\oiint_s \vec{D}\cdot d\vec{s}}{\left|-\int_l \vec{E}\cdot d\vec{l}\right|} = \frac{\epsilon \oiint_s \vec{E}\cdot d\vec{s}}{\int_l \vec{E}\cdot d\vec{l}}$ —— I —— II

Cap 1 ($\epsilon_1$)

I: $\epsilon_1 \oiint_s \vec{E}\cdot d\vec{s} = \epsilon_1 \int_0^{L/2}\int_0^{2\pi} -\frac{V_0}{\ln(a/b)}\frac{1}{r}\, a\, d\phi\, dz = -\epsilon_1 V_0 a 2\pi$

Question 2

a) $I_o = \int_s \vec{J}\cdot d\vec{s} = |\vec{J}|\int_s ds = J\frac{4\pi a^2}{2} = J2\pi(R)^2$

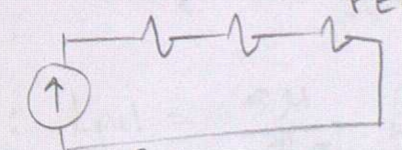
$\therefore \vec{J} = \frac{I_o}{2\pi R^2}\hat{R}$ [A/m²]

b) Recall that $\vec{E} = \frac{\vec{J}}{\sigma}$

$\sigma_1: \vec{E} = \frac{\vec{J}}{\sigma_1}$

$\sigma_2: \vec{E} = \frac{\vec{J}}{\sigma_2}$

$$\therefore \vec{E} = \begin{cases} 0, & 0<R<R_1 \\ \frac{I_0}{\sigma_1 2\pi R^2}\hat{R}, & R_1<R<t_1 \\ \frac{I_0}{\sigma_2 2\pi (R)^2}\hat{R}, & t_1<R<t_2 \\ 0, & R>0 \end{cases} \quad [V/m]$$

c) 


$R_1 = \frac{l}{\sigma s} = \frac{V}{I} = \left|\frac{\int_l \vec{E}\cdot d\vec{l}}{\iint_s \vec{J}\cdot d\vec{s}}\right| = \frac{\int_l \vec{E}\, d\vec{l}}{\sigma_1 \iint \vec{E}\cdot d\vec{s}}$

$R_1 = \frac{\int_{R_1}^{R_1+t_1}\frac{I_0}{\sigma_1 2\pi R^2}dR}{I_0} = \frac{\left[-\frac{1}{R}\right]_{R_1}^{t_1}}{\sigma_1 2\pi}$

$R_1 = \left|\frac{-\frac{1}{R_1+t_1}+\frac{1}{R_1}}{\sigma_1 2\pi}\right|, \quad R_2 = \left|\frac{-\frac{1}{R_1+t_1+R_2}+\frac{1}{R_1+t_1}}{\sigma_2 2\pi}\right|$

$R = \left|\frac{\frac{1}{R_1}-\frac{1}{R_1+t_1}}{\sigma_1 2\pi}\right| + \left|\frac{\frac{1}{R_1+t_1}-\frac{1}{R_1+t_1+R_2}}{\sigma_2 2\pi}\right|$ [Ω]

d) $P = I^2 R$
$= I_o^2 R$

Question 3

a) $\oint_C \vec{H}\cdot d\vec{l} = I_{enc} \Rightarrow \oint_C \vec{B}\cdot d\vec{l} = \mu I_{enc}$

inner J (volume current density) $\vec{J} = \frac{I}{\pi a^2}\hat{z}$ [A/m²]

outer $J_s$ (surface current density) $\vec{J_s} = \frac{I}{2\pi b}(-\hat{z})$ [A/m]

for $0<r<a$

$2\cancel{\pi}\cancel{r}B = \mu J \cancel{\pi} r^{\cancel{2}} \Rightarrow \vec{B} = \frac{\mu J}{2} r\hat{\phi} = \frac{\mu I}{2\pi a^2} r\hat{\phi}$

$\therefore \vec{B} = \frac{\mu I r}{2\pi a^2}(-\sin\phi\hat{x} + \cos\phi\hat{y})$ [T]   on xz plane, $\phi = 0$

$\therefore \vec{B} = \frac{\mu I r}{2\pi a^2}\hat{y}$ [T]

b) for $a<r<b$

$2\pi r B = \mu I \longrightarrow \vec{B} = \frac{\mu I}{2\pi r}\hat{y}$ [T]

c) for $r>b$

$2\pi r B = \mu I_{enc} = \mu(I - I) = 0 \longrightarrow \vec{B} = 0$ [T]

d) $\Phi_B = \iint_S \vec{B}\cdot d\vec{s} = \int_a^b \int_0^l \frac{\mu I}{2\pi r}\hat{y}\cdot\hat{y}\, dz\, dr = \frac{\mu I l}{2\pi}\ln(b/a)$

$\Phi_{Bpul} = \frac{\Phi_B}{l} = \frac{\mu I}{2\pi}\ln(b/a)$ [wb/m] $= \Phi_{Bpul}$

e) $W_m = \frac{1}{2}\iiint_V B\cdot H\, dV = \frac{1}{2}\int_V \mu|\vec{H}|^2 dV = \frac{\mu}{2}\int_0^{2\pi}\int_0^l\int_0^a \frac{I^2 r^2}{4\pi^2 a^4} r\, dr\, dz\, d\phi$

$= \frac{\mu I^2}{8\pi^2 a^4} 2\pi l \int_0^a r^3 dr = \frac{\mu I^2 l}{4\pi a^4}\left[\frac{1}{4}r^4\right]_0^a = \frac{\mu I^2 l a^4}{16\pi a^4}$

$\therefore W_m = \frac{\mu I^2 l}{16\pi}$   $\therefore W_{mpul} = \frac{W_m}{l} = \frac{\mu I^2}{16\pi}$ [J/m] $= W_{mpul}$

f) $L = \frac{2W_m}{I^2}$ and we already know $W_{m_1} = \frac{\mu I^2 l}{16\pi}$

$W_{m_2} = \frac{\mu}{2}\int_0^{2\pi}\int_0^l\int_a^b \frac{I^2}{4\pi^2 r^{\cancel{2}}}\cancel{r}\, dr\, dz\, d\phi = \frac{\mu I^2 \cancel{2}\pi l}{\cancel{2}\times 4\pi^{\cancel{2}}}\ln(b/a)$

$W_m = W_1 + W_2 = \frac{\mu I^2 l}{16\pi} + \frac{\mu I^2 l}{4\pi}$ ln(b/a)   forgot

$L_{pul} = \frac{L}{l} = \frac{2W_m}{lI^2} = \frac{\cancel{2}\left(\frac{\mu \cancel{I^2}\cancel{l}}{\cancel{16}8\pi} + \frac{\mu \cancel{I^2}\cancel{l}}{\cancel{4}2\pi}\right)}{\cancel{l}\cancel{I^2}} = \frac{\mu}{8\pi} + \frac{\mu}{2\pi}$

$\therefore L_{pul} = \frac{5\mu}{8\pi}$ [H/m]

Question 4

a) Recall that $\oint_C \vec{H}\cdot d\vec{l} = I_{enc}$ , by RHR: $\vec{H} = -\hat{x}|\vec{H}|$

$$2\pi r|\vec{H}| = I_{enc} \Rightarrow \boxed{\vec{B} = -\hat{x}\,\frac{\mu i(t)}{2\pi y}\ [T]}$$

b) $$\Phi_B = \iint_S \vec{B}\cdot d\vec{s} = \int_{z_0-a/2}^{z_0+a/2}\int_{y_0-b}^{y_0+f(t)+b} \cancel{-\hat{x}}\,\frac{\mu i(t)}{2\pi y}\cdot \cancel{-\hat{x}}\,dy\,dz \qquad \hat{y}\times -\hat{x} = +\hat{z}$$

$$= \frac{\mu i(t)}{2\pi}\int_{z_0-a/2}^{z_0+a/2} dz \int_{y_0-b}^{y_0+f(t)+b}\frac{1}{y}\,dy \qquad z_0+\frac{a}{2}-z_0+\frac{a}{2} = a$$

$$\Phi_B = \frac{\mu i_1(t)a}{2\pi}\ln\left(\frac{y_0+f(t)+b}{y_0-b}\right)\ [Wb]$$

$$V^m_{emf\,AB} = \int_A^B (\vec{u}\times\vec{B})\cdot d\vec{l} = \int_{A'}^{B'}\left(f'(t)\hat{y}\times -\hat{x}\,\frac{\mu i(t)}{2\pi y}\right)\cdot\hat{z}\,dy$$

$$= \int_{A'}^{B'}\frac{f'(t)\mu i(t)}{2\pi y}dy = \frac{f'(t)\mu i(t)}{2\pi}\int_{z_0+a/2 = \frac{2z_0+a}{2}}^{z_0-a/2 = \frac{2z_0-a}{2}}\frac{1}{y}\,dy$$

$$= \frac{f'(t)\mu i(t)}{2\pi}\ln\left(\frac{\frac{2z_0-a}{2}}{\frac{2z_0+a}{2}}\right)$$

$$V^m_{emf\,AB} = \frac{f'(t)\mu i_1(t)}{2\pi}\ln\left(\frac{2z_0-a}{2z_0+a}\right)\ [V]$$

c) $$V^m_{emf\,BC} = \int_B^C\left(f'(t)\hat{y}\times -\hat{x}\frac{\mu i(t)}{2\pi y}\right)\cdot\hat{y}\,dy \Rightarrow \hat{z}\cdot\hat{y} = 0$$

$$\therefore V^m_{emf\,BC} = 0\ [V]$$

d) $$i_2(t) = \frac{V_{emf}(t)}{R_T}, \quad V_{emf}(t) = \iint_S -\frac{\partial}{\partial t}\vec{B}(t)\cdot d\vec{s} = -\frac{\partial}{\partial t}\Phi_B(t)$$

$$-\frac{\partial}{\partial t}\Phi_B(t) = -\frac{\partial}{\partial t}\left[\frac{\mu i_1(t)a}{2\pi}\underbrace{\ln\left(\frac{y_0+f(t)+b}{y_0-b}\right)}_{g(t)}\right]$$

$$= -\frac{\mu a}{2\pi}\frac{\partial}{\partial t}\left[i_1(t)g(t)\right] = -\frac{\mu a}{2\pi}\left(i_1'(t)g(t) + i_1(t)g'(t)\right)$$

$$g'(t) = \frac{1}{\frac{y_0+f(t)+b}{y_0-b}}\times\frac{f'(t)}{y_0-b} = \frac{\cancel{y_0-b}}{y_0+f(t)+b}\times\frac{f'(t)}{\cancel{y_0-b}} = \frac{f'(t)}{y_0+f(t)+b}$$

$$V_{emf}(t) = -\frac{\mu a}{2\pi}\left[i'(t)\ln\left(\frac{y_0+f(t)+b}{y_0-b}\right) + \frac{i(t)f'(t)}{y_0+f(t)+b}\right]$$

$$R_T = R\times(\text{total length } A'B'C'D') = R(4b+2a)$$

$$\therefore i_2(t) = -\frac{\mu a}{2\pi R(4b+2a)}\left[i'(t)\ln\left(\frac{y_0+f(t)+b}{y_0-b}\right) + \frac{i(t)f'(t)}{y_0+f(t)+b}\right]\ [A]$$

made error in orig. b) calculation $\int_{A'}^{B}\ldots\hat{z}\,dz$ ← dz!!! not <u>dy</u>