ECE216 Course Summary

Basics

* given $x(t)$ and you want to graph $y(t) = x(\alpha t + \beta)$
  - ↳ first apply time shift β
  - ↳ then scale t-axis by α → $\begin{cases} |\alpha| > 1 : \text{horiz. "wider"} \\ |\alpha| < 1 : \text{horiz. "squished"} \end{cases}$
    (divide all t's by α)

* is $y(t) = x_1(t) + x_2(t)$ periodic and with what T_0?
  - ↳ yes if $\left.\frac{T_1}{T_2} = \frac{l}{k}\right\}$ and l/k is rational
  - ↳ then $T_0 = lT_2 = kT_1 \Rightarrow T_0 = LCM(T_1, T_2)$

* DT comp. expo. periodic iff ω is a rational multiple of 2π
  - ↳ $\omega_0 = k\frac{2\pi}{N_0}$ ← ω_0 must satisfy for smallest integer k

3. Fourier Series

* trigonometric FS: $x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(\omega_0 n t) + b_n \sin(\omega_0 n t)\right]$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt \quad a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(\omega_0 n t)\,dt \quad b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(\omega_0 n t)\,dt$$

* CTFS v.s. DTFS (CTFS generally aperiodic, but DTFS is N_0-periodic)

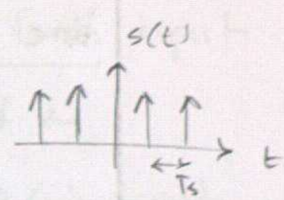
$x(t)$ periodic $\{$
  - → CTFS represents $x(t)$ as infinite discrete sum of CT comp. exp.
  - → DTFS 〃 $x[n]$ 〃 finite discrete sum of DT 〃 〃

* FS repr. of $x(t)$ is $x(t)$ repr. by sum of comp. exp. and lets us analyze frequency content of $x(t)$ by examining α_k

4. Fourier Transform

- FT is extension of FS to aperiodic signals
- idea: aperiodic signal is periodic signal with infinite period
- $x(t) \xleftrightarrow{F} X(j\omega)$ ← called "FT" or "spectrum" of x
- $u(t+a) - u(t-a) \xleftrightarrow{FT} \frac{2\sin(a\omega)}{\omega}$
- $\frac{\sin(at)}{\pi t} \xleftrightarrow{FT} u(\omega+a) - u(\omega-a)$
- if $x(t)$ very concentrated, $X(j\omega)$ very spread out
- $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega_0 t} \xleftrightarrow{FT} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \alpha_k \delta(\omega - k\omega_0)$
- CTFT of a periodic signal is a sum of impulse functions located at multiples of the fundamental frequency ω_0
- CTFT v.s. DTFT
  - → CTFT represents aperiodic $x(t)$ as continuous sum of CT comp. exp. and in general is not periodic (spectrum)
  - → DTFT " " $x[n]$ " continuous sum of DT comp. exp. " " " is 2π-periodic (spectrum)
  - → both $X(j\omega)$ and $X(e^{j\omega})$ capture the "amount" of frequency ω contained in x

5. Sampling, Aliasing, Interpolation



* sampling function : $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$ ← sampling period T_s

* weighted sum of impulses : $\underbrace{s(t)\, x(t)}_{x_s(t)} = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)\, x(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)\, x(nT_s)$

sampled function

* CTFS of sampling func. : $\alpha_k = \frac{1}{T_s} \Rightarrow s(t) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$

* CTFT of sampling func. : $s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t} \overset{FT}{\longleftrightarrow} S(j\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$

* The CTFT of the sampling function is a sampling function!
(impulse in t-dom spaced by T_s, in $j\omega$-dom spaced by ω_s)

* The spectrum of the sampled signal is a <u>periodized</u> version of spectrum of the original signal, with period ω_s in $j\omega$-dom

* $\omega_s \geq 2\omega_{max}$ or else will have aliasing (Nyquist Rate)

6. CT Systems

* Linearity : it holds that $T\{\alpha x_1(t) + \beta x_2(t)\} = \alpha T\{x_1(t)\} + \beta T\{x_2(t)\}$

→ can apply superposition

→ input of 0 always results in 0

→ can apply to sums and integrals : $T\{\int \alpha x_1(t)\, dt\} = \alpha \int T\{x_1(t)\}\, dt$

* time invariance: if it holds that $y(t-a) = T\{x(t-a)\}$
  - → to check, compute $y(t-a)$ and $T\{x(t-a)\}$ and check if they are equal → if yes, then T.I.
  - → deals with only inputs and outputs (if you have say $y(t) = x(t) \cos t$, inputting $x(t-a)$ doesn't change $\cos t$ ... but shifting output does → ∴ not TI)
  - → for integral systems, can apply change of variables

* Causality: if it holds that $y(t)$ only depends on past and present values of t (if future, then not causal)
  - → if two inputs agree up to some time t_0, then the corresponding outputs must also agree up to that time t_0.

* memoryless: only the current time matters (not future or past)
  - → derivative and RC ckt systems not memoryless

* invertibility: you can "undo" the operation of T
  - → delay system invertible, but squaring system not

* BIBO stability: bounded inputs produce bounded outputs
  - → to show system is not BIBO stable, find example bounded input that leads to unbounded output

- impulse response is system's response to a delta func. input
  - → $h(t) = T\{\delta(t)\}$
  - → once $h(t)$ known, can compute $y(t)$ for any $x(t)$ input
- output of any LTI system given by:

  $$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$$

- a CT LTI system with $h(t)$ is
  1) Causal iff $h(t) = 0$ for $\forall t < 0$
  2) Memoryless iff $h(t) = \alpha\,\delta(t)$ for $\alpha \in \mathbb{C}$ ☆
  3) BIBO stable if h has finite action : $\int_{-\infty}^{\infty} |h(t)|\,dt < +\infty$
- changing order of series LTI systems still gives same result
- the inverse of an LTI sys. (if $\exists$ one) is also LTI sys. ($\delta * y = y$)
- the impulse response of parallel LTI's is $h_1(t) + h_2(t)$
- linear, inhomogeneous, constant-coefficient ODE's can define <u>causal</u> <u>LTI</u> systems.
  - → if we restrict ourselves to right-sided inputs $x(t)$
  - → " " " " " " " solutions $y(t)$

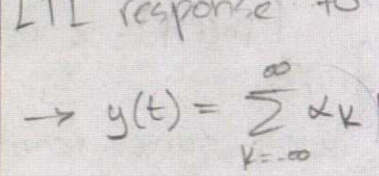
    then
  - → syst. is LTI and causal
  - → syst. is BIBO stable iff order(input) $\leq$ order(LHS)

    RHS ⋯ $x(t)$; $y''(t) + y'(t)$ ⋯

    <u>and</u> poles of transfer function (s-dom) have (-)ve real part

7. CT System Analysis with FT

- FT of impulse response $h(t)$ is $H(j\omega)$ frequency response
- LTI system with comp. exp. input : $x(t) = e^{j\omega_0 t}$
  - $\rightarrow y(t) = H(j\omega)|_{\omega_0} \times e^{j\omega_0 t} = H(j\omega_0) e^{j\omega_0 t}$
  - → due to eigenfunction property, LTI system amplitude-scales and phase-shifts any comp. exp. input
  - → meaning if input is comp. exp. then output is also comp. exp. with same frequency but scaled by $H(j\omega)$
- LTI response to periodic input : $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega_0 t}$
  - $\rightarrow y(t) = \sum_{k=-\infty}^{\infty} \alpha_k H(jk\omega_0) e^{jk_0\omega_0 t}$
  - → if we know α_k of $x(t)$, we can get $X(j\omega)$ and we know $Y(j\omega) = X(j\omega) H(j\omega)$ and tak IFT of that
  - → output $y(t)$ is $T_0 = \frac{2\pi}{\omega_0}$ periodic just like input
- convolution in t-dom means multiplication in $j\omega$-dom
- An LTI system is invertible iff $H(j\omega) \neq 0$ for $\forall\, \omega \in \mathbb{R}$
- The above lets you find $H(j\omega)$ for feedback systems:

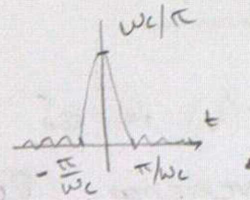


$$y = (x - y * h_2) * h_1$$
$$y = x * h_1 - y * h_2 * h_1$$
$$y + y * h_2 * h_1 = x * h_1$$
$$(\delta + h_2 * h_1)\, y = x * h_1$$
$$y = (\delta + h_2 * h_1)^{-1} x * h_1$$

iff $(1 + H_2(j\omega) H_1(j\omega)) \neq 0$ then

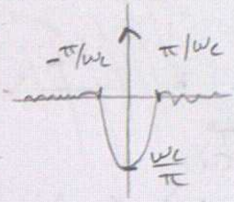
$$H(j\omega) = \frac{H_1(j\omega)}{1 + H_1(j\omega) H_2(j\omega)}$$

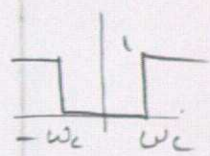
* The ideal LPF

$h(t) = \frac{\sin(\omega_c t)}{\pi t}$ 

$\xleftrightarrow{FT}$ $H(j\omega) = \begin{cases} 1, & -\omega_c < \omega < \omega_c \\ 0, & \text{oth.w.} \end{cases}$ 

* The ideal HPF

$h(t) = \delta(t) - \frac{\sin(\omega_c t)}{\pi t}$ 

$\xleftrightarrow{FT}$ $H(j\omega) = \begin{cases} 1, & |\omega| > \omega_c \\ 0, & \text{oth.w.} \end{cases}$ 

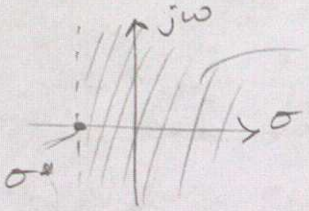
* Ideal non-causal filters V.S. non-ideal causal filters
  - → ideal filter $h(t) \neq 0$ for $\forall\, t < 0$, so not causal and cannot be used for real-time application
  - → ideal filters anticipate the input change (since not causal) and perfectly removes/rejects frequencies
  - → non-ideal filters react only after input changes (since causal) and can only attenuate (not completely remove for eg.) signal
  - → non-ideal: LPF, output @ C and HPF, output @ L → for RC ckts

* FT method V.S. LT methods
  - → FT: * 2-sided CTFT-able inputs
    * causality not required
    * used in signal processing ...
  - → LT: * LT-able inputs which are right-sided from time 0
    * causal systems only
    * used in control, energy, robotics, aerospace...

8. CT System Analysis with LT

- Laplace Transform defined if $x(t)$ is of exponential order/class and on a region of convergence (ROC)

$$X(s) = \int_{0^-}^{\infty} x(t) e^{-st} \, dt$$



$s = \sigma + j\omega$

- if LT ROC contains imaginary ($j\omega$) axis, the FT exists and can be found by subbing $s = j\omega$ into LT
- for causal BIBO systems, freq. response $H(j\omega)$ quatifies the steady-state

9. Fundamentals of DT Systems

- properties from CT systems generally hold
- impulse response of summer : $h[n] = \sum_{k=-\infty}^{n} \delta[k] \overset{n \geq 0}{=} u[n]$

Basics :

- continuous time signal (CT) : $x(t)$, $t \in \mathbb{R}$
- discrete time signal (DT) : $x[n]$, $n \in \mathbb{Z}$
- sampling : converting CT to DT
- interpolating : converting DT to CT
  - zero-order hold (ZoH)
  - first-order (FoH)

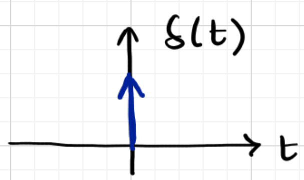
Operations :

- time shift : delays / advances in time $\rightarrow x(t \pm t_0)$ and $x[n+n_0]$
  - advance : shift signal to left t_0 or n_0
  - delay : shift signal to right t_0 or n_0
- time reversal : mirror signal about its vertical zero axis $\rightarrow x(0)$ or $x[0]$
- time scaling : horizontal (time scale) stretch or compression $\rightarrow x(\alpha t)$
  - horizontal "wide" : $|\alpha| < 1 \longrightarrow$ stretch
  - horizontal "squish" : $|\alpha| > 1 \longrightarrow$ compression
- periodic signal : if $x(t) = x(t+T) \longrightarrow$ signal repeats itself after period of time
  - if I can find T s.t. $x(t) = x(t+T)$ for $\forall t$, then periodic, else aperiodic
- constant signal
  - DT case : periodic with $T_0 = 1$
  - CT case : periodic with undefined period
- sum / product of periodic signals periodic ? $\rightarrow \underset{\uparrow \atop T_0}{y(t)} = \underset{\uparrow \atop T_1}{x(t)} + \underset{\uparrow \atop T_2}{h(t)}$
  - yes if $\left.\frac{T_1}{T_2} = \frac{\ell}{k}\right\}$ a rational number
  - then $T_0 = LCM(T_1, T_2)$ or : $T_0 = kT_1 = \ell T_2$

- signal support : where signal is not zero (active)
- right-sided signal : if for all $t < t_0$, signal is zero
- finite duration signal : non-zero over finite (bounded) interval
  - can turn finite duration signal into periodic signal
- even signal : $x(t) = x(-t)$
- odd signal : $x(-t) = -x(t)$
- signal decomposition : given any $x(t)$:

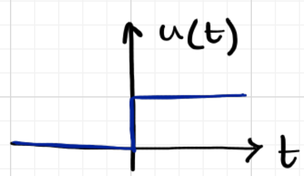
$$x_{even}(t) = \frac{x(t) + x(-t)}{2} \qquad x_{odd}(t) = \frac{x(t) - x(-t)}{2}$$

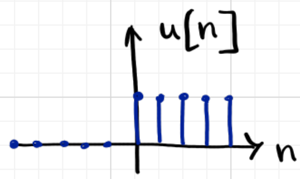
- signal amplitude, $\|x\|_\infty$: peak value (absolute max) of signal
- signal action, $\|x\|_1$: the absolute area under the curve
- signal energy, $\|x\|_2^2$: the strength of the signal
- impulse : dirac delta function



- $\delta(t) = 0,\ t \neq 0$
- $\int_{-\infty}^{\infty} \delta(t) = 1$
- $\int_{0^-}^{0^+} \delta(t) = 1$
- $\int_{-\infty}^{\infty} x(t)\,\delta(t-A) = x(A)$

- unit step : heavyside function





Periodicity

- Euler's :

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$e^{-j\theta} = \cos\theta - j\sin\theta \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

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- DT complex exponentials only periodic iff ω is a rational multiple of 2π

$$\omega_0 = k\frac{2\pi}{N_0} \leftarrow \omega_0 \text{ must satisfy for smallest integer } k$$

- DT complex exponentials are periodic functions of ω

$$e^{j(\omega+2\pi)n} = e^{j\omega n} e^{j2\pi n} = e^{j\omega} \leftarrow \text{period } 2\pi$$

- Aliasing: when CT signal sampled $< 2f_{max}$, called Nyquist frequency
  - $\omega: 0 \rightarrow \pi$, increasing ω increases rate of oscillation
  - $\omega: \pi \rightarrow$ fastest a signal can oscillate in DT
  - $\omega: \pi \rightarrow 2\pi$, increasing ω lowers (slows) rate of oscillation

Fourier Series

- trigonometric FS: $x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(\omega_0 n t) + b_n \sin(\omega_0 n t)\right]$

average (DC)

Fourier coefficients

harmonics

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt \qquad a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(\omega_0 n t)\,dt \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(\omega_0 n t)\,dt$$

- complex exponential FS: $x(t) = \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$

harmonic number

Fourier coefficients (complex numbers) $\rightarrow$ $\alpha_k = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t}\,dt$

- orthogonality relations (if $m = n$)
  - $\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t) = T/2$
  - $\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t) = 0$
  - $\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t) = T/2$

- **Dirichlet Conditions**
  1) must be periodic
  2) finite number of max and mins
  3) finite number of discontinuities
  4) absolutely integrable

- **Convergence of FS**
  1) uniform convergence: FS converges to $x(t)$ everywhere consistently
  2) point-wise convergence: FS converges to $x(t)$ at every point except possibly at discontinuities (FS converges to midpoint for finite dis.)

- **Parseval's Relation** : energy of signal per unit time (aka average pwr) is equal to energy of signal in frequency domain (sum of F.Coef. energy)

- **Tutorial 3 Notes**
  - odd function : $\alpha_{-k} = -\alpha_k \longleftarrow x(-t) = -x(t)$
  - imaginary function : $\alpha_{-k} = -\alpha^*_k \longleftarrow x(-t) = -x^*(t)$
  - $1 - 2\sin^2\theta = \cos 2\theta$
  - $2\cos^2\theta - 1 = \cos 2\theta$
  - Frequency $= \frac{k}{T}$
  - given $z(t) = \alpha \overset{a_k}{x(t)} + \beta \overset{b_k}{y(t)} \longrightarrow c_k = \alpha a_k + \beta b_k$
  - given $\underset{\alpha_k}{x(t)}$ and $\underset{\beta_k}{y(t)} = x(t-\tau) \longrightarrow \beta_k = e^{-j\omega_0 \tau k} \cdot \alpha_k$