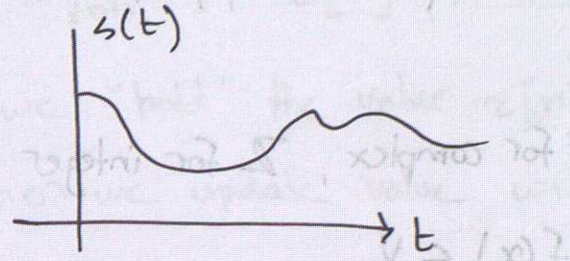
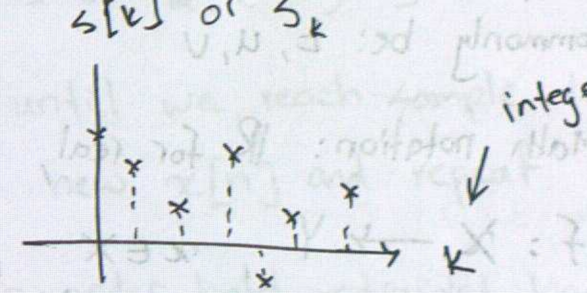
Week 1: Lecture 1 Jan 7, 2024

What is a Signal?

* a signal is any phenomenon carrying information
* pressure, radio signal, counting photons
* changes over time





* "signal" is entire function/plot
* s(t) or s[k] is value of signal at "t" or "k"
* spectrum & frequency domain
  ↳ sharper peaks/troughs in signal imply higher frequencies

Systems

* takes input(s) signal(s) → creates output signal(s)
* some transformation "T" $T: x \longmapsto T\{x\}$

$$x \longrightarrow \boxed{T} \longrightarrow y = T\{x\}$$

value of output signal at time t is: $y(t) = T\{x\}(t)$

* linear time-invariant systems
* some "bloks" we can design (e.g. radio reciever) and others already exist (e.g. atmosphere)

Week 1: Lecture 2 — Jan 9, 2025

Signal Types

$s(t)$, $s(x,y)$ (image), $s(x,y,t)$ (video) — $s[k]$, s_k

- continuous time signal (CT)
- independent variable can commonly be: t, u, v

- discrete time signal (DT)
- indep. var.: k, l, m, n

- Math notation: $\mathbb{R}$ for real, $\mathbb{C}$ for complex, $\mathbb{Z}$ for integer
- $f: X \to Y$, $x \in X$ (domain), $f(x) \in Y$ (codomain)

Continuous Time Signal

- has a value for every value of t
- a "smooth" function

Discrete Time Signal

- values at discrete values of "n" or "k" integers
- no value of signal for "in-betweens"
- bitstream and text-stream inherently discrete

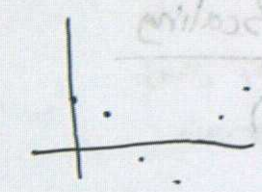
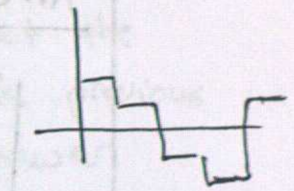
Sampling

- turns CT signal to DT signal

$$x_s[n] = x(t)\Big|_{t=nT_s} = x(nT_s)$$

T_s is sampling period/interval

Interpolation

- turns DT signal to CT signal
- we use "zero-order hold"

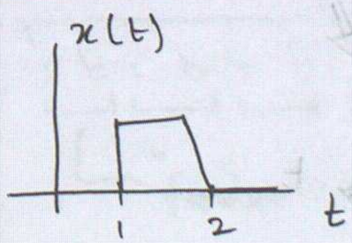
$$x_{zoh}(t) = x[n] \quad \text{for } nT_s \le t < (n+1)T_s$$

- we "hold" the value $x[n]$ until we reach sample time $n+1$ then we update value with new $x[n]$ and repeat
- Reconstruction filters: zero-order hold, straight line interpolate, ideal filter

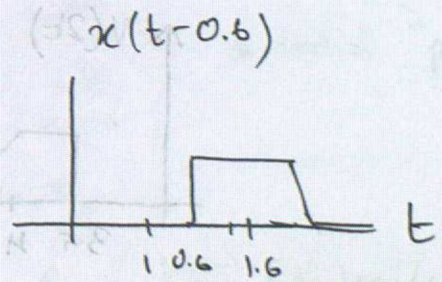
Pointwise Operations

- pointwise addition: $h(t) = f(t) + g(t)$ $\quad h[n] = f[n] + g[n]$
- pointwise scaling: $h(t) = \alpha f(t)$ $\quad h[n] = \alpha f[n]$
- pointwise multiplication: $h(t) = f(t) \cdot g(t)$ $\quad h[n] = f[n] \cdot g[n]$

Time Shifting



$\rightarrow$

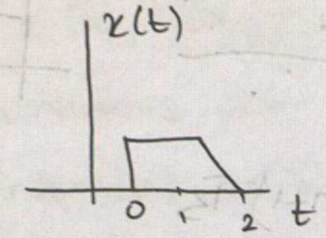


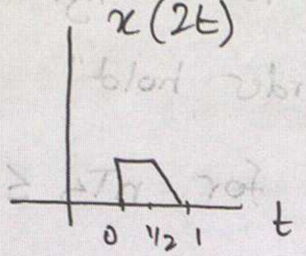
- $t_0 = 0.6$ shifts (delays) new signal to the right by 0.6

CT: $x(t) \to x(t - t_0)$
DT: $x[n] \to x[n - n_0]$

$t_0, n_0 > 0$ means delay
$t_0, n_0 < 0$ means advance

Time Scaling



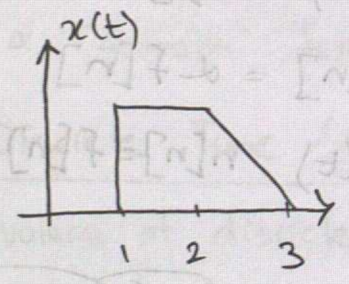


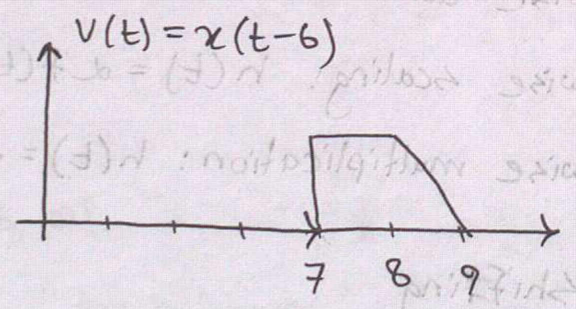
* if $\alpha < 0$, then signal flips

* scale CT signal x by $\alpha \in \mathbb{R}$ to get $x(\alpha t)$
* $|\alpha| > 1$ compresses t (move thru t faster)
* $|\alpha| < 1$ expands t (move thru t slower)
* same idea for DT signals, but α must be integer

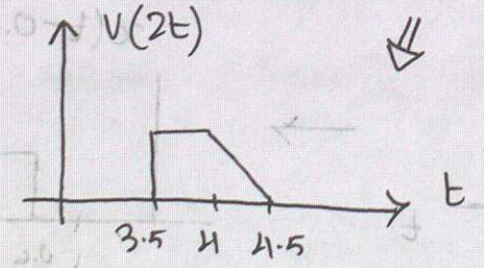
eg. if ~~=~~ $\alpha = 2$, we only sample every other sample

Combining Shifting and Scaling



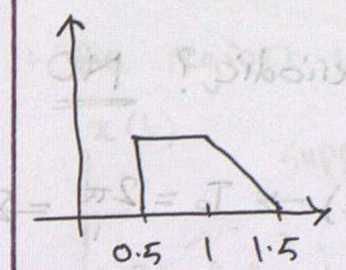


$x(2t+6)$

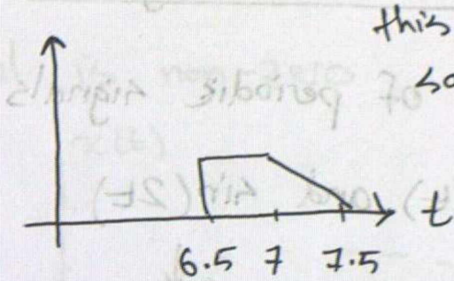


here, we shifted and then scaled

* What if we time scale first?



shift by 6 $\Longrightarrow$



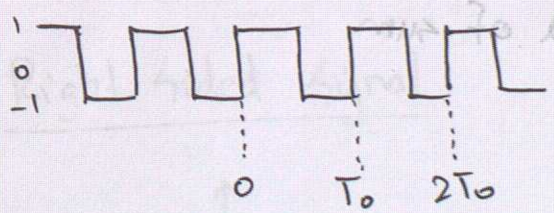
this is not the same as previous answer

$\Downarrow$

Takeaway: Always do <u>time shift</u> first, then time scale

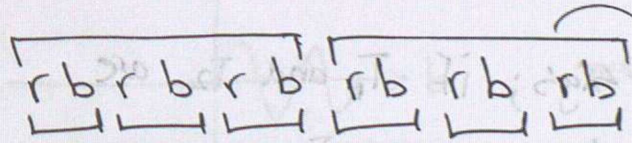
Periodic CT Signals

* periodic if $x(t) = x(t+T)$ for some value of $T > 0$ and all $t \in \mathbb{R}$
* smallest value of this "T" is fundamental period T_0 of x



eg: $\sin(t)$ has $T_0 = 2\pi$

$\sin\left(\frac{2\pi t}{\tau}\right)$ has $T_0 = \tau$



a valid "period" T

fundamental period T_0

* fundamental frequency : $f_0 = \frac{1}{T_0}$ [cycles/s]
* angular frequency : $\omega_0 = 2\pi f_0$ [radians/s] (fundamental)

these and T_0 for DT signals too, but all are integers

Periodicity of Sum Signals

* Is sum of periodic signals always periodic? NO

* eg: $\sin(t)$ and $\sin(2t)$

$$T_0 = \frac{2\pi}{1} = 2\pi \qquad T_0 = \frac{2\pi}{2} = \pi$$

ratio of periods: $\frac{2\pi}{\pi} = 2$ ✓

$\sin(t) \rightarrow T_0 = \frac{2\pi}{1} = 2\pi$

$\sin(\pi t) \rightarrow T_0 = \frac{2\pi}{\pi} = 2$

$\frac{2\pi}{2} = \pi$ ← irrational, not periodic

* Will eventually line up (i.e. periodic) if $\boxed{\ell T_1 = k T_2}$ where $\ell, k \in \mathbb{Z}$

* Period of sum signal is the ratio of the periods

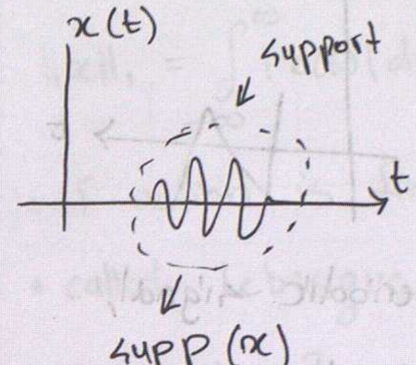
$\boxed{\ell T_1 = k T_2 = T}$ ← period of sum

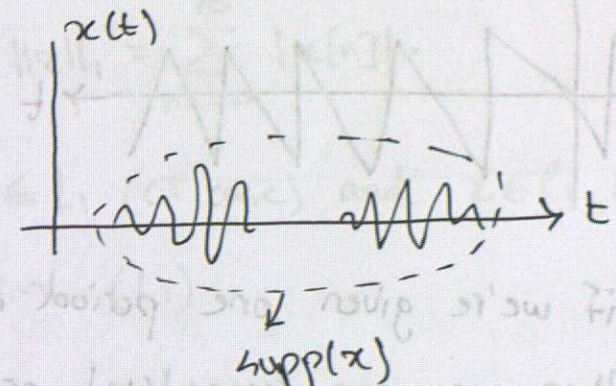
Product of Signals

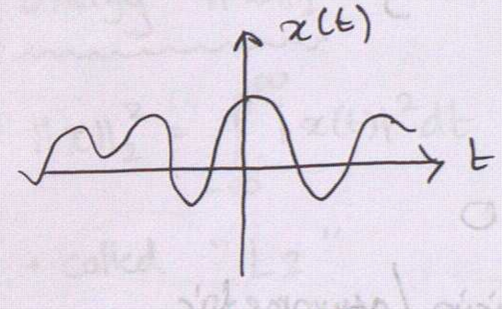
* same idea as sums of sigs; if T_1 and T_2 are fundamental periods of x_1 and x_2 and $\frac{T_1}{T_2}$ is rational, then period of $x_1 \cdot x_2$ is $\boxed{T = \ell T_1 = k T_2}$

Signal Support

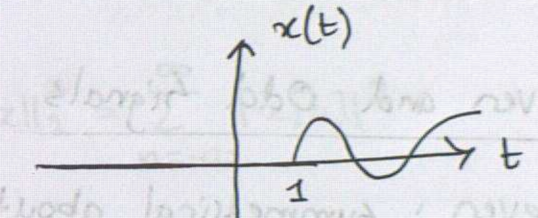
* the region where signal is non-zero





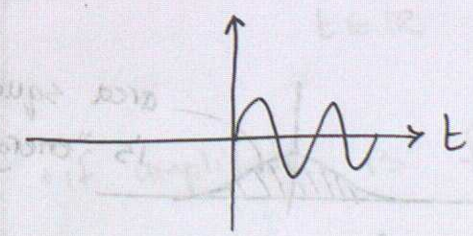


$supp(x) = \mathbb{R}$



$supp(x) = \{t \in \mathbb{R} \mid t \geq 1\}$

Right Sided Signal



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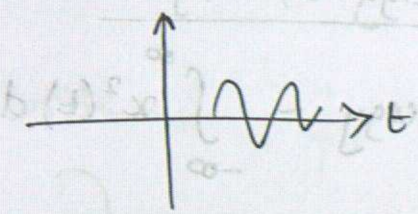

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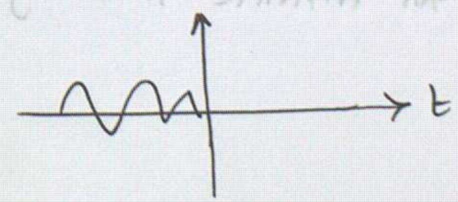
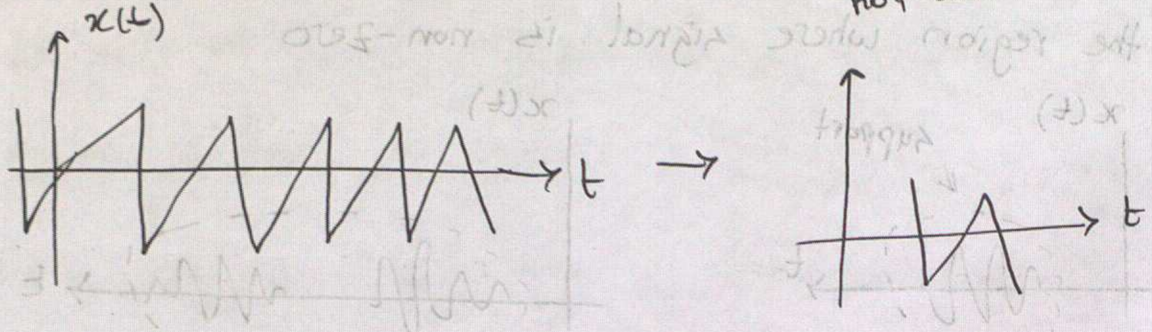


these are all right-handed or right sided signals

Finite Duration Signal

$\ast$ Trash diagram, not accurate



$\ast$ if we're given one period of a periodic signal, then we can reconstruct original signal

Even and Odd Signals

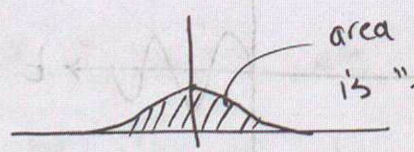
$\ast$ even : symmetrical about $t=0$

$\ast$ odd : point symmetry about origin / asymmetric

$$e(t) = \frac{x(t) + x(-t)}{2} \qquad o(t) = \frac{x(t) - x(-t)}{2}$$

Energy of Signal

$$\text{energy} = \int_{-\infty}^{\infty} x^2(t)\,dt$$



area squared is "energy"

$\ast$ energy is finite for finite signal

$\ast$ power is a better description for infinite time signal

Week 2: Lecture 1 Jan 14, 2025

Size of a Signal

Action $\|x\|_1$

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)|\,dt \qquad \|x\|_1 = \sum_{n=-\infty}^{\infty} |x[n]|$$

$\ast$ if action is finite, $x \in L_1$ (CT case) and $x \in \ell_1$ (DT case)

$\ast$ called Lebesgue 1 norm (L_1)

Energy $\|x\|_2^2$

$$\|x\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2\,dt \qquad \|x\|_2^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$\ast$ called "L_2"

Amplitude $\|x\|_\infty$

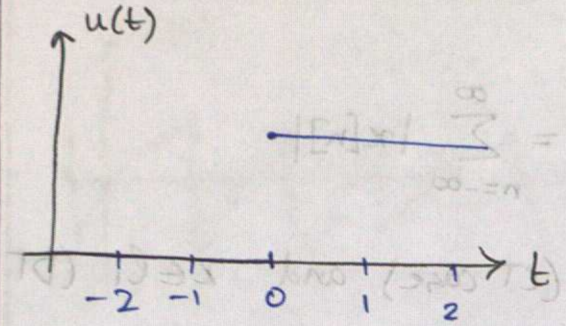
$$\|x\|_\infty = \max_{t \in \mathbb{R}} |x(t)| \qquad \|x\|_\infty = \max_{n \in \mathbb{Z}} |x[n]|$$

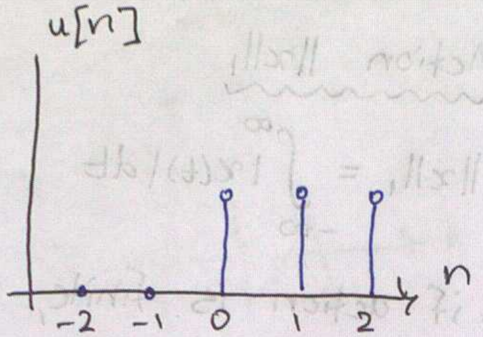
$\ast$ if amplitude is finite, we write

$x \in L_\infty$ (CT) and $x \in \ell_\infty$ (DT)

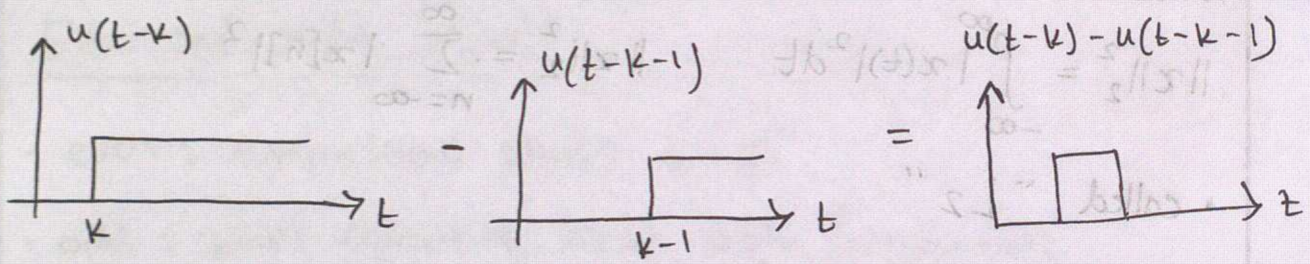
finite for **BOUNDED** sigs

Unit Step Function

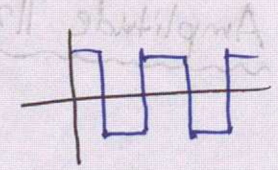




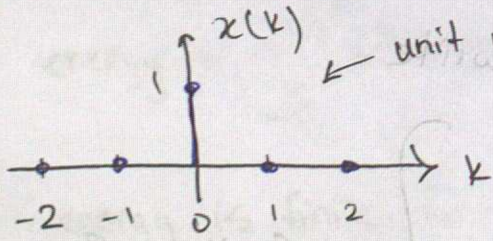
+ multiply signal by unit step function creates a right-sided signal

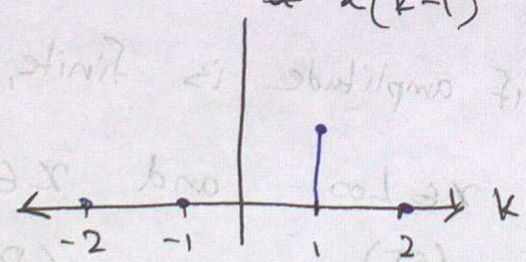


$$x(t) = \sum_{k=-\infty}^{\infty} (-1)^k \left(u(t-k) - u(t-k-1)\right) \Rightarrow$$



Unit Impulse $\delta[n]$





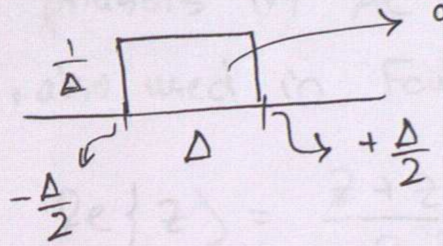
Sifting Formula

$$\sum_{n=-\infty}^{\infty} x[n]\,\delta[n-n_0]$$

this is $=1$ for $n=n_0$ and zero otherwise

this "sifts" out the value of the signal at $n=n_0$

The CT Unit Pulse

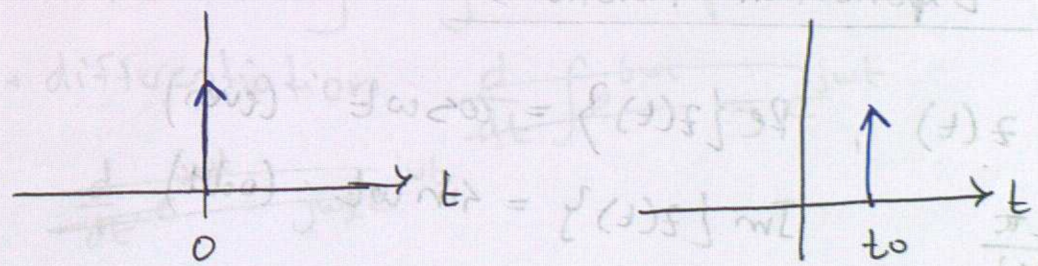


area $= \frac{1}{\Delta} \cdot \Delta = 1$

$$P_\Delta(t) = \begin{cases} \frac{1}{\Delta} & \text{if } -\frac{\Delta}{2} \le t \le \frac{\Delta}{2} \\ 0 & \text{otherwise} \end{cases}$$

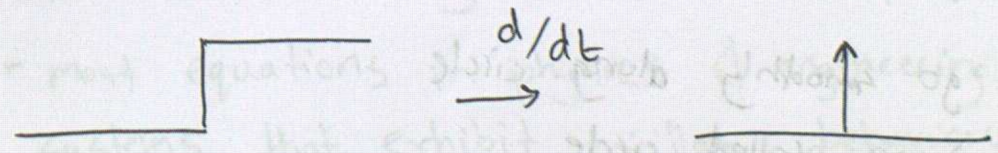
Sifting formula: $\displaystyle\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$

Drawing it:



Notes:

- derivative of unit step is unit impulse



Sinusoids

$\theta(t) = \omega_0 t$, $x(t) = \cos(\omega_0 t)$, $y(t) = \sin(\omega_0 t)$

* time shift is a phase shift $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$

Complex Numbers

$\cos\theta + j\sin\theta = e^{j\theta}$ ← Euler's Formula

* cartesian makes add/sub ez
* polar makes multip/divis. ez
* complex conjugate z^*

$\Rightarrow zz^* = |z|^2$

Complex Exponential Functions

$e^{j\omega t} = z(t)$, $Re\{z(t)\} = \cos\omega t$ (even)

$T_0 = \frac{2\pi}{\omega}$ $Im\{z(t)\} = \sin\omega t$ (odd)

DT Case

$x[n] = e^{j\omega(n\cdot 1)} = \cos(\omega n) + j\sin(\omega n)$ $T_s = 1$

* CT case, go smoothly along circle
* DT case, jump around circle

Week 2: lecture 1 Jan 15, 2025

Complex Numbers

* 3 forms $\Rightarrow$ exponential $z = re^{j\theta}$

$\Rightarrow$ polar $z = r\angle\theta$

$\Rightarrow$ rectangular $z = x + jy$

* conjugate is usefol for calulating amplitude of phasors in AC ckt
* also used in Fourier transform

$Re\{z\} = \frac{z + z^*}{2}$ $Im\{z\} = \frac{z - z^*}{2j}$

Complex Exponential Funct^n s / Signals

* mathematically easy to manipulate and solve
* differentiation: $\frac{d}{dt} \int \cancel{e^{j\omega t}} = e^{j\omega t}$

$\cancel{\frac{d}{dt} e^{j\omega t} = j\omega k} e^{j\omega t}$

Sinusoidal Signals

* are embedded in natural world due to their inherent oscillatory and wave-like property
* most equations in physics & engineering describe systems that exhibit oscillatory behaviour

eg: sound wave, light wave, mechanical vibration, Fourier analysis

Euler's Formula

* allows to express sinusoidal natural phenomena as complex exponentials that are easier to manipulate and solve mathematically

Relations: $e^{j\theta} = \cos\theta + j\sin\theta$

$e = 2.71828$

$e^{-j\theta} = \cos\theta - j\sin\theta$

Add $\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

Sub $\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

DT exponential case

* must take care that my "jumps" land me where I started, or else will not be periodic
* ω must be a rational multiple of 2π in order to have a periodic signal : $x[n] = e^{j\omega n}$
* $\boxed{\omega = k\dfrac{2\pi}{N}} \iff 2\pi k = \omega N$ ← period
* $e^{j\omega n}$ is periodic with period N if the frequency ω is an integer multiple of $\dfrac{2\pi}{N}$

Ex: $x[n] = e^{j\frac{\pi}{2}n}$, find fundamental period to ensure $x[n]$ is periodic.

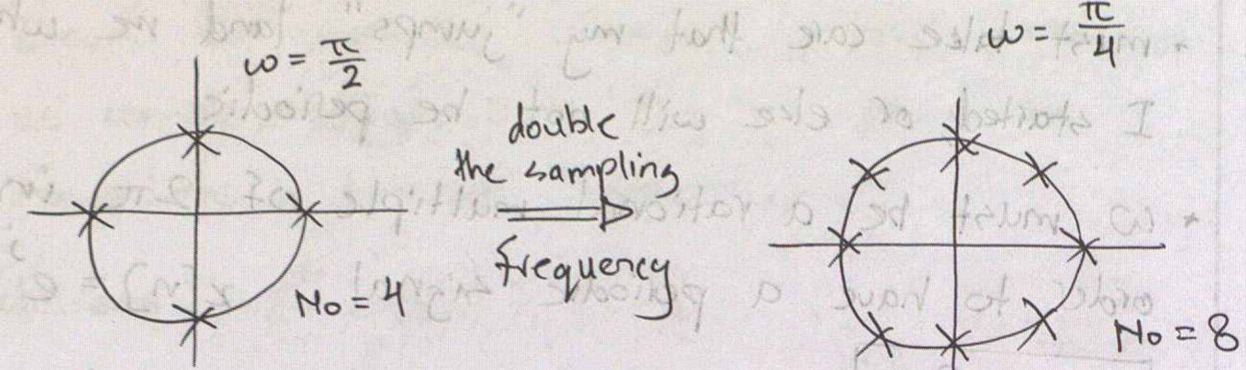
$\omega = \dfrac{\pi}{2}$, condition is $\omega = k\dfrac{2\pi}{N}$

$k \Rightarrow k\dfrac{2\cancel{\pi}}{N} = \dfrac{\cancel{\pi}}{2} \Rightarrow N = 4k$

Find fundamental period ⟹ smallest positive integer N that satisfies the condition is when $k=1 \Longrightarrow \boxed{N_0 = 4}$

Week 2: Lecture 2 Jan 16, 2025

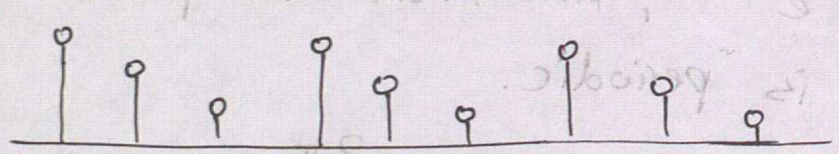
DT Complex Exponential



$\omega n T \Rightarrow \omega n \leftarrow \mathbb{Z}$, $n \in \{0, 1, 2, \cdots\}$

* definition of time periodicity in DT:

$$x[n] = x[n + N_0]$$



Frequency-periodicity of DT complex exp.

$$e^{j(\omega+2\pi)n} = e^{j\omega n} \underbrace{e^{j2\pi n}}_{=1} = e^{j\omega n}$$

* a DT signal (complex exp) will oscillate
  - $\Rightarrow$ slowly when ω is near even multiple of π
  - $\Rightarrow$ quickly " " " " odd " " "

MATLAB

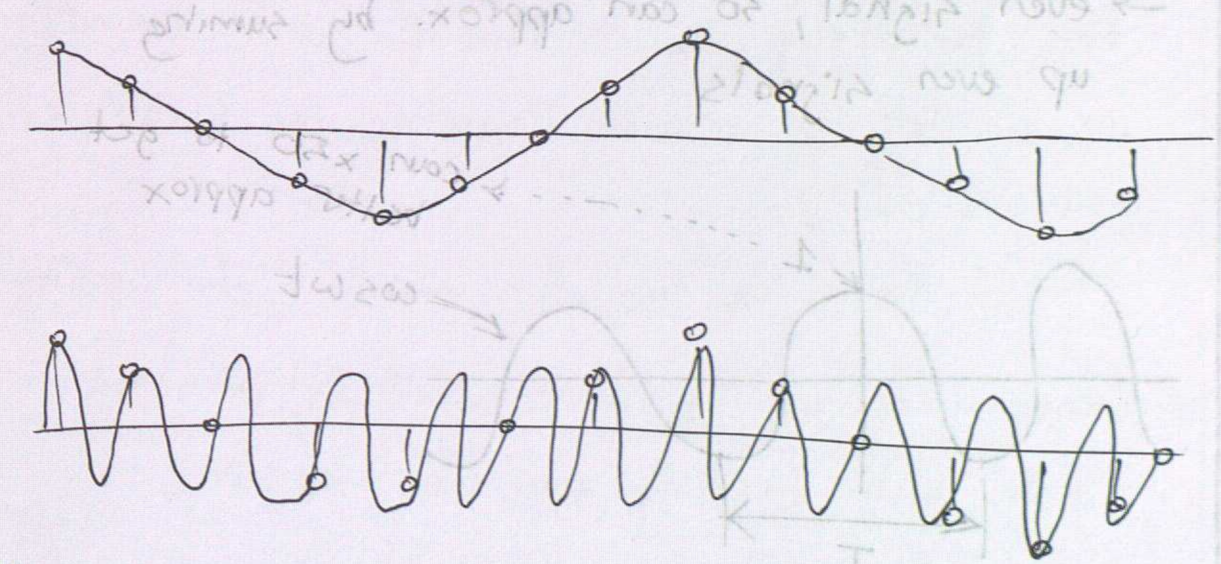
```
n = [-200 : 200] ;  → [-200, -199, ..., 199, 200]
x[n] = cos(w * n) ;
plot (n, x) ;
```

generates vector

explore for diff. values wide range, many pi's

Aliasing:

* different CT signals, when sampled, can produce the same DT signal



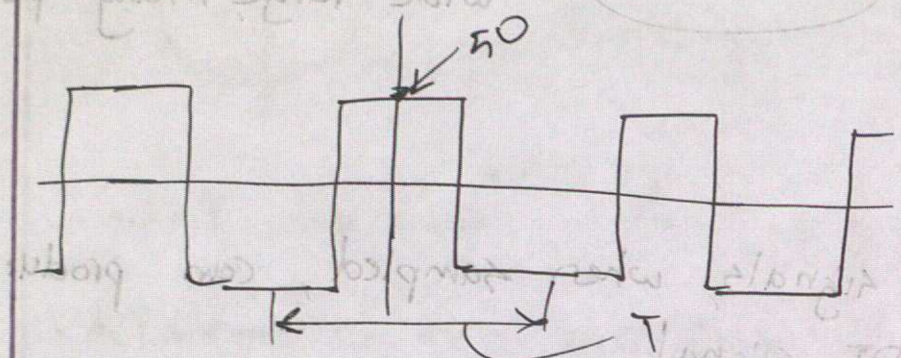
Summary

1) $e^{j\omega n}$ is periodic in n with period N iff $\omega = k\frac{2\pi}{N}$

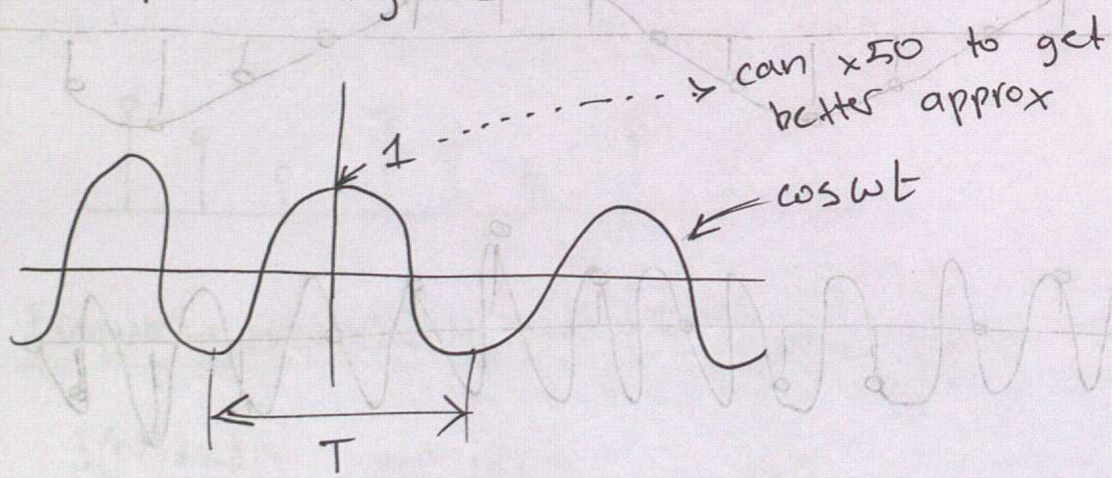
2) $e^{j\omega n}$ is periodic in ω with period 2π

Fourier Series

+ create weighted sum of basis function to represent a signal

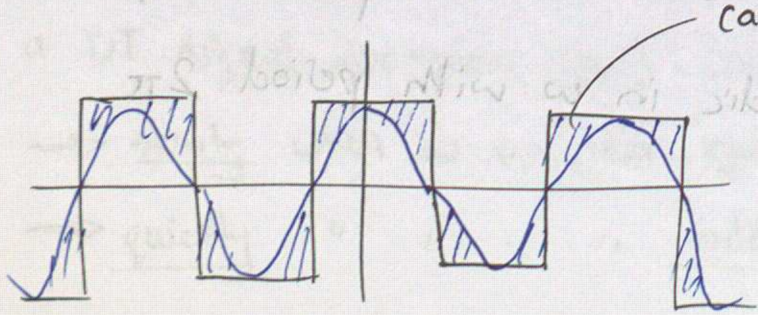


→ even signal, so can approx. by suming up even signals

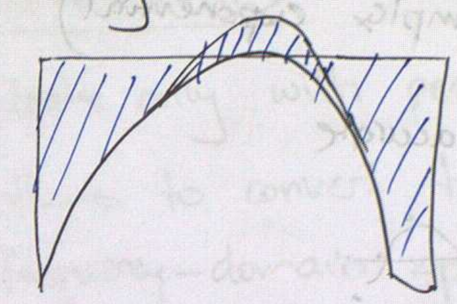


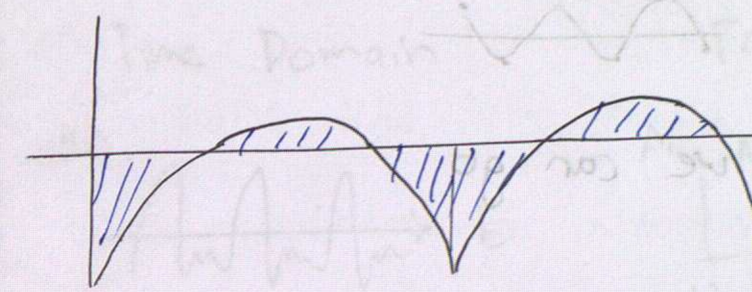
+ can scale cosine, but by what?

can mitigate error by making sale larger



+ plotting error



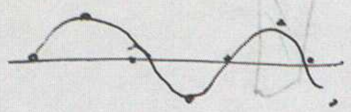


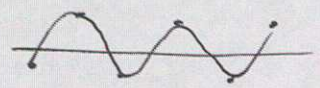
+ if we have a fundamental frequency, ω_0, any of these will fit original signal: $\omega_0, 2\omega_0, 3\omega_0 \ldots$

Week 2: Lecture 2 Jan 16, 2025

Frequency Periodicity (DT complex exponential)

$\omega \approx 0$, (very small) $\rightarrow$ DT accurate

$\omega = \frac{\pi}{2} = 90^\circ \longrightarrow$ 

$\omega = \pi = 180 \longrightarrow$ 

$\downarrow$

this is the fastest we can go

Key Takeaways:

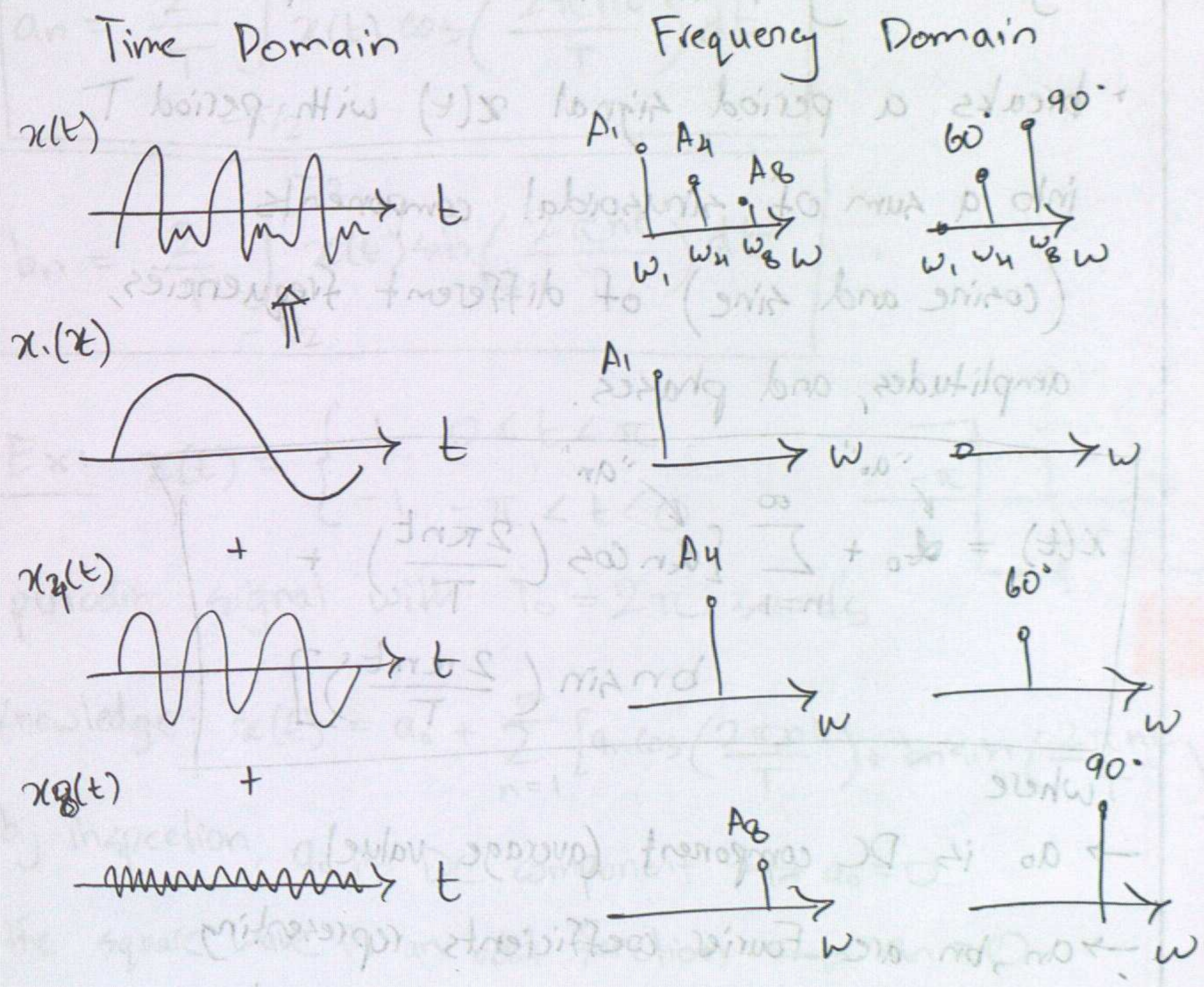
- time periodicity is conditional and depends on whether the angular frequency is a rational multiple of 2π
  - $\Longrightarrow$ if that's the case, then periodic ... else, aperiodic
- frequency periodicity is inherent to all DT signals with frequency spectrum repeating every 2π radians regardless of ω

Aliasing

- is when CT signal is under-sampled (at a rate less than twice its highest frequency) ... Nyquist rate
- results in high frequency information to be incorrectly represented in the sampled data.

Fourier Series

- deals only with periodic signals
- allows to convert time-domain signals to frequency-domain spectral representation



- 1, 4, 8 are harmonic numbers

ω_0 : DC offset

ω_1 : fundamental

$\omega_2, \omega_3 \dots$: harmonic

Fourier Series

- allows for most practical cases to compose / decompose a periodic signal by summing smaller sinusoidal components
- breaks a period signal $x(t)$ with period T into a sum of sinusoidal components (cosine and sine) of different frequencies, amplitudes, and phases.

$$\boxed{x(t) = \underset{\text{"a}_0\text{"}}{a_0} + \sum_{n=1}^{\infty} \left[\underset{\text{"a}_n\text{"}}{a_n} \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right) \right]}$$

where

→ a_0 is DC component (average value)

→ a_n, b_n are Fourier coefficients representing amplitudes of cosine and sine terms

→ n is Harmonic numbers (integer multiples of fundamental frequency)

and these coefficients are calculated by:

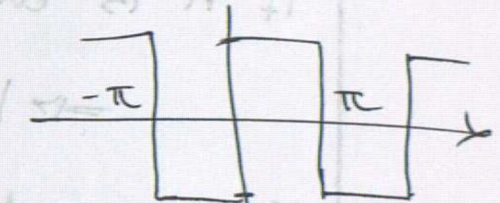
$$\boxed{a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt}$$

a_n and b_n quantify the contribution of each cosine or sine.

$$\boxed{a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos\left(\frac{2\pi n t}{T}\right)dt}$$

$$\boxed{b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin\left(\frac{2\pi n t}{T}\right)dt}$$

Ex: $x(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & -\pi < t < 0 \end{cases}$



periodic signal with $T_0 = 2\pi$ seconds

Knowledge: $x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right)\right]$

by inspection, a_0 is DC component $\Rightarrow a_0 = 0$

the square wave is an odd function $\Rightarrow a_n = 0$

$$\Rightarrow x(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{T}\right)$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin\left(\frac{2\pi n t}{T}\right)dt = \frac{2}{2\pi}\int_{-\pi}^{\pi} x(t)\sin(nt)\,dt$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin(nt)\,dt + \int_{0}^{\pi}(1)\sin(nt)\,dt\right]$$

$$b_n = \frac{2}{n\pi}\left(1-\cos(n\pi)\right)$$

$\Longrightarrow$ note $\cos(n\pi) = (-1)^n$

$$\Longrightarrow b_n = \frac{2}{n\pi}\left(1-(-1)^n\right)$$

if n is even:

$$\Longrightarrow b_n = \frac{2}{n\pi}(1-1) = 0$$

if n is odd: $n_{odd} = 2k+1$, $k = 0,1,2,3\cdots$

$$\Longrightarrow b_n = \frac{2}{n\pi}(1--1) = \frac{4}{n\pi}$$

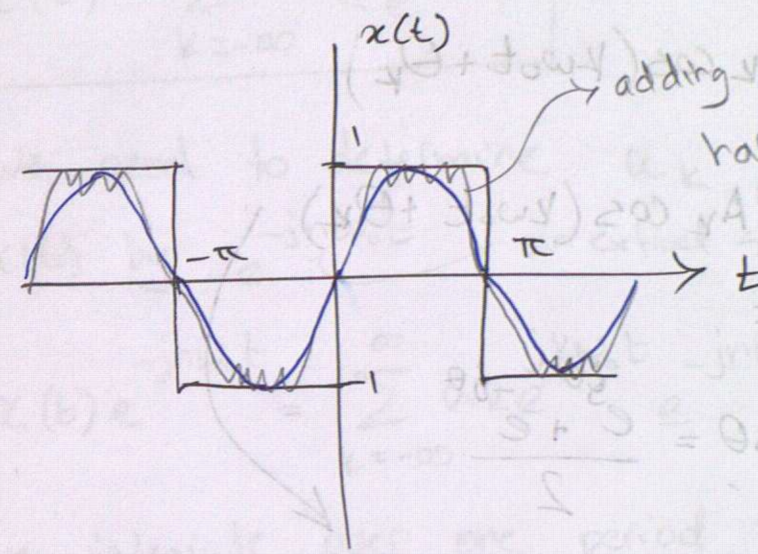
$$\therefore b_n = \begin{cases} 4/n\pi, & n_{odd} \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore x(t) = \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)} \sin\left((2k+1)t\right)$$

Week 2: Lecture 3 Jan 17, 2025

Expande form

$$x(t) = \frac{4}{\pi}\Big(\underbrace{\sin(t)}_{\text{fundamental}} + \underbrace{\tfrac{1}{3}\sin(3t)}_{\text{third harmonic}} + \underbrace{\tfrac{1}{5}\sin(5t)}_{\text{fifth harmonic}} + \cdots\Big)$$



Application:

- Filtering
- Analyzing ckt responses to periodic input signals
- modulation and demodulation

Ex: for $T_o > 0$ and $0 < \tau \leq T_o$

$$x(t) = \begin{cases} 1 & -\frac{\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$$

periodic signal

Complex Exponential Fourier Series

* Recall: A periodic signal can be composed by adding cosine and sine functions at different frequencies

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

* consider only: $A_k \cos(k\omega_0 t + \theta_k)$

where $\omega_0 = \frac{2\pi}{T_0}$

+ Euler's $\Rightarrow$ $\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$

$$\Rightarrow \frac{A_k e^{j(k\omega_0 t + \theta_k)} + A_k e^{-j(k\omega_0 t + \theta_k)}}{2}$$

now take out constants

$$= \underbrace{\frac{A_k}{2} e^{j\theta_k}}_{C_k,\ \alpha_k} e^{jk\omega_0 t} + \underbrace{\frac{A_k}{2} e^{-j\theta_k}}_{\alpha_{-k}} e^{-jk\omega_0 t}$$

$$\Rightarrow x(t) = \underbrace{\alpha_0}_{\text{DC/avg value}} + \underbrace{\sum_{k=1}^{\infty} \alpha_k e^{jk\omega_0 t}}_{\text{(+)ve frequency terms}} + \underbrace{\sum_{k=-1}^{-\infty} \alpha_k e^{jk\omega_0 t}}_{\text{(-)ve frequency terms}}$$

where $\alpha_0 = A_0$, $\underbrace{\alpha_k = \frac{A_k}{2} e^{j\theta_k},\ \alpha_{-k} = \frac{A_k}{2} e^{-j\theta_k}}_{\alpha_{\pm k} \text{ are complex numbers}}$

we can also write:

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega_0 t}}$$

+ we need to determine α_k, so we multiply $x(t)$ by $e^{-jn\omega_0 t}$ → can extract specific α_k for an "n"

$$x(t) e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

+ we integrate over one period

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{\infty} \alpha_k e^{j(k-n)\omega_0 t} dt$$

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{\infty} \int_0^T \alpha_k e^{j(k-n)\omega_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty} \alpha_k \underbrace{\int_0^T e^{j(k-n)\omega_0 t} dt}_{\text{on next page}}$$

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \int_0^T \cos((k-n)\omega_0 t)\,dt + j\int_0^T \sin((k-n)\omega_0 t)\,dt$$

Conditions:

⟹ if $k \neq n$, then both integrals equal zero ... $= 0$

⟹ if $k = n$, then $k-n=0$ and sin goes zero ... $= T$

plug back

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \sum_{k=-\infty}^{\infty} \alpha_k T = a_n T = \alpha_n T$$

can conclude

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega_0 t} \qquad \alpha_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}\,dt}$$

← exponential form of Fourier series

also called (synthesis equation)

"forward engineering" (assemble)

(analysis equation)

Reverse engineering (overhand)

Week 3: Lecture 1 Jan 22 2025

CT Exponential Form of FS

- represents a periodic signal as sum of complex exponentials
- CT periodic signal $x(t)$ with period T has

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j\omega_0 t}} \quad \text{where} \quad \omega_0 = \frac{2\pi}{T}$$

→ α_k are Fourier coefficients (complex numbers)

→ k is integer representing harmonic number

→ $$\boxed{\alpha_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}}$$

- combines sine and cosine terms into simple complex exponential ← the exponential form does this

⟶ simplifies multiplications and convolution

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_n\cos(k\omega_0 t) + b_n\sin(k\omega_0 t)\right)$$

- need to know orthogonality of sine and cosine

consider

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = \frac{1}{2}\int_{-T/2}^{T/2}[\cos((m-n)\omega_0 t) - \cos((m+n)\omega_0 t)]\,dt$$

* if $m \neq n \implies \sin(n'\omega_0 t)\Big|_{-T/2}^{T/2} - \sin(n''\omega_0 t)\Big|_{-T/2}^{T/2} = 0$

integral $= 0$

* if $m = n \implies (T)\Big|$

$$= \frac{1}{2}\int_{-T/2}^{T/2} \cos(0)\,dt - \frac{1}{2}\int_{-T/2}^{T/2} \cos(2n\omega_0 t)\,dt$$

$$= \frac{1}{2}\int_{-T/2}^{T/2} dt = \frac{1}{2}\left(\frac{T}{2}+\frac{T}{2}\right) = \boxed{\frac{T}{2} \text{ for } m=n}$$

consider

$$\int_{-T/2}^{T/2} \underbrace{\underbrace{\cancel{\cos}^{\sin}(m\omega_0 t)}_{\text{odd}}\ \underbrace{\cos(n\omega_0 t)}_{\text{even}}}_{\text{odd}}\,dt = 0$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)\,dt = 0$$

3 conditions: (will show up)

1) $\cos \cdot \cos$
2) $\cos \cdot \sin$
3) $\sin \cdot \sin$

Orthogonality Relations

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)\,dt = T/2$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)\,dt = 0$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = T/2$$

$\boxed{\text{if } m=n}$

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(\omega_0 n t) + b_n\sin(\omega_0 n t)\right]$$

$n=0 \implies x(t) = a_0$, want to find average over one period

integrate over one period

$$\boxed{\int_{-T/2}^{T/2} x(t)\,dt = \int_{-T/2}^{T/2} a_0\,dt = a_0\int_{-T/2}^{T/2} dt = a_0 T}$$

$$\Rightarrow \boxed{a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt}$$

- multiply both sides by $\cos(m\omega_0 t)$ and then integrate over one period

$$\Rightarrow \int_{-T/2}^{T/2} x(t)\cos(m\omega_0 t)\,dt = a_0 \int_{-T/2}^{T/2} \cos(m\omega_0 t)\,dt \leftarrow = 0$$

$$+ \sum_{n=1}^{\infty} \int_{-T/2}^{T/2} a_n \cos(n\omega_0 t)\cos(m\omega_0 t)\,dt$$

$$+ \sum_{n=1}^{\infty} \int_{-T/2}^{T/2} b_n \sin(n\omega_0 t)\cos(m\omega_0 t)\,dt \leftarrow = 0$$

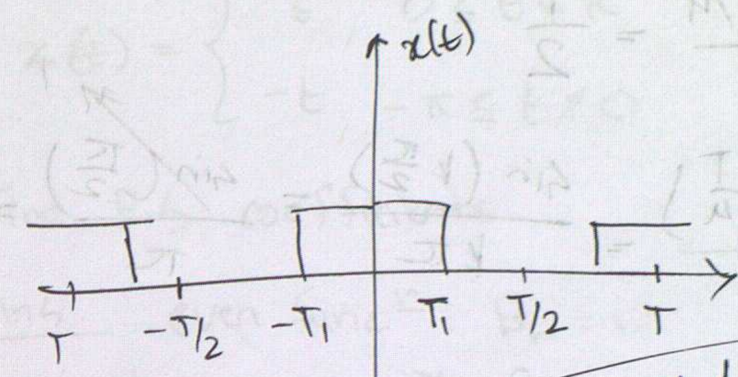
$$\Rightarrow \int_{-T/2}^{T/2} x(t)\cos(m\omega_0 t)\,dt = a_m \frac{T}{2}$$

can write for all values of n

$$\boxed{a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0 t)\,dt}$$

multiply by $\sin(m\omega_0 t)$ to extract b_n

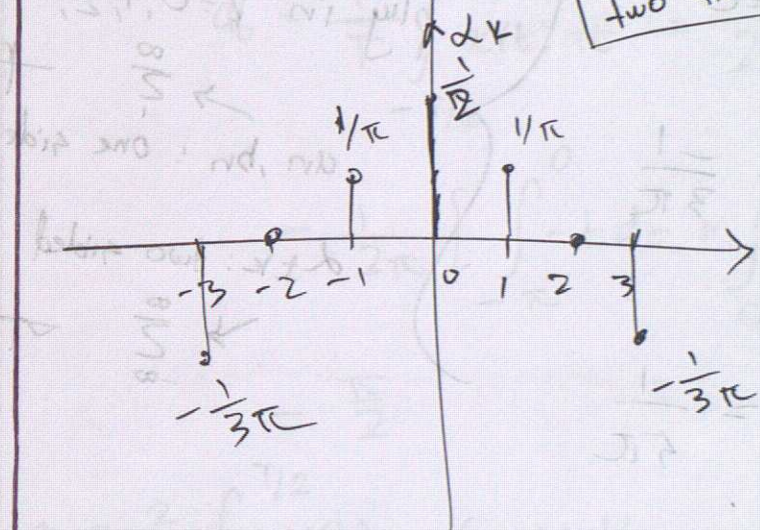
Ex: $x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$



periodic with period T

Find F.S coefficients

two-sided plot



$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\,dt$$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt$$

$$= \frac{1}{T}(T_1 + T_1) = \frac{2T_1}{T}$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} (1) e^{-jk\omega_0 t}\,dt = \frac{-1}{Tjk\omega_0} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}$$

$$a_k = \frac{-1}{Tjk\omega_0}\left[e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}\right] \qquad \omega_0 = \frac{2\pi}{T}$$

$$\cancel{= \frac{1}{Tjk\omega_0}} \qquad a_k = \frac{2}{k\omega_0 T}\sin(k\omega_0 T_1) = \frac{1}{k\pi}\sin(k\omega_0 T_1)$$

for $k \neq 0$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

from graph

Now let $T = 4T_1 \implies T_1 = \frac{T}{4}$

$$a_0 = \frac{2T_1}{T} = \frac{2T/4}{T} = \frac{1}{2}$$

$$a_{\pm 1} = \frac{\sin\left(k\frac{2\pi}{T}\cdot\frac{T}{4}\right)}{k\pi} = \frac{\sin\left(k\frac{\pi}{2}\right)}{k\pi} = \frac{\cancel{\sin\left(\frac{\pi}{2}\right)}^{\,1}}{\pi} = \frac{1}{\pi}$$

$$a_{\pm 2} = \frac{\sin\left(\frac{2\pi}{2}\right)}{2\pi} = 0$$

$$a_{\pm 3} = \frac{\sin\left(\frac{3\pi}{2}\right)}{3\pi} = \frac{-1}{3\pi}$$

$$a_{\pm 4} = 0$$

$$a_{\pm 5} = \frac{\sin\left(\frac{5\pi}{2}\right)}{5\pi} = \frac{1}{5\pi}$$

plug in $k = 0, 1, 2, \dots$

$\rightarrow \sum_{0}^{\infty}$

a_n, b_n : one sided

$\alpha_{\pm k}$: two sided

$\rightarrow \sum_{-\infty}^{\infty}$

---

if $x(t)$ is real $\Rightarrow C_n$ is conjugate symmetric

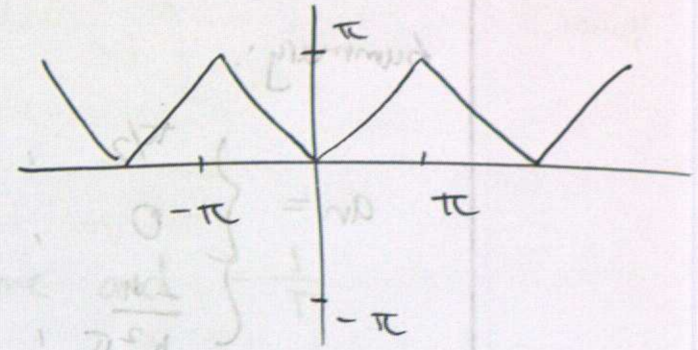
$\Downarrow$

$|C_n| \rightarrow$ Even

$\angle C_n \rightarrow$ odd

Week 3: Lecture 2 — Jan 23, 2025

<u>Ex:</u> $T = 2\pi$ periodic function



$$x(t) = \begin{cases} t, & 0 \le t < \pi \\ -t, & -\pi \le t < 0 \end{cases}$$

Find F.S coefficients

<u>Ans:</u> even func^n : $b_n = 0$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\,dt$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} -t\,dt + \int_{0}^{\pi} t\,dt\right]$$

$$= \frac{\pi}{2} \qquad \omega_0 = \frac{2\pi}{T} = 1$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{2}{2\pi}\left[\int_{-\pi/2}^{0} -t\cos(nt)\,dt + \int_{0}^{+\pi} t\cos(nt)\,dt\right]$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} t\cos(nt)\,dt \leftarrow \text{use a "by-parts"}$$

$$= \frac{2}{\pi}\left[\frac{\overbrace{\cos(n\pi)}^{(-1)^n}}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)$$

Summary:

$$a_n = \begin{cases} \pi/2 & , n=0 \\ 0 & , n \text{ even} \\ \frac{-4}{n^2\pi} & , n \text{ odd} \end{cases} \quad \text{and} \quad b_n = 0$$

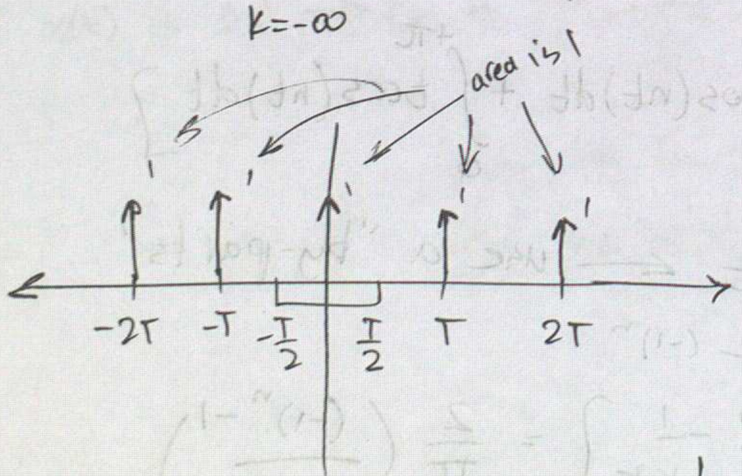
by F.S: (Fourier Series Approximation)

$$x(t) = \frac{\pi}{2} - \sum_{n=1,\text{odd}}^{\infty} \frac{4}{n^2\pi}\cos(nt)$$

Expanded Form

$$x(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(t) + \frac{1}{3^2}\cos(3t) + \frac{1}{5^2}\cos(5t) + \cdots\right)$$

Ex: $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$ and T is period



Find F.S. coefficients

$$\alpha_k = \frac{1}{T}\int_T x(t)e^{-j\omega_0 kt}\,dt$$

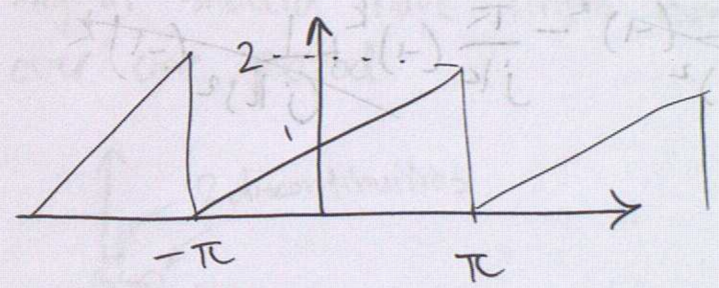
$$\alpha_k = \frac{1}{T}\underbrace{\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_0 t}\,dt}_{=1} = \frac{1}{T}$$

All F.S coeffients are the same and $= \frac{1}{T}$

and even $\alpha_0 = \frac{1}{T}$

Ex: Period of 2π, find F.S. coeffients

$$x(t) = \frac{1}{\pi}t + 1 \quad -\pi \leq t < \pi$$



← both cosine and sine present

$$\alpha_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$\alpha_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{t}{\pi}+1\right)e^{-jk\omega_0 t}\,dt$$

$$\alpha_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{t}{\pi}+1\right)dt = 1$$

$k \neq 0$ $\quad \alpha_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{t}{\pi}+1\right)e^{-jkt}\,dt \Rightarrow$ next page

$k \neq 0$

$$a_k = \frac{1}{2\pi^2}\underbrace{\int_{-\pi}^{\pi} t e^{-jkt}\,dt}_{\text{by parts}} + \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-jkt}\,dt$$

$$= \left[\frac{t}{-jk} e^{-jkt} - \frac{1}{(-jk)}\frac{1}{(-jk)} e^{-jkt}\right]_{-\pi}^{\pi}$$

$$= \frac{\pi}{-jk} e^{-jk\pi} - \frac{1}{(jk)^2} e^{-jk\pi} + \frac{-\pi}{jk} e^{+jk\pi} + \frac{1}{(jk)^2} e^{jk\pi}$$

$$= \frac{-\pi}{jk}(-1)^k - \cancel{\frac{1}{(jk)^2}(-1)^k} - \frac{\pi}{jk}(-1)^k + \cancel{\frac{1}{(jk)^2}(-1)^k}$$

$$= \frac{-2\pi}{jk}(-1)^k$$

plug back

$$a_k = \frac{1}{2\pi^2}\left(\frac{-2\pi}{jk}(-1)^k\right) + \frac{1}{2\pi}\left(\frac{1}{-jk} e^{-jkt}\Big|_{-\pi}^{\pi}\right)$$

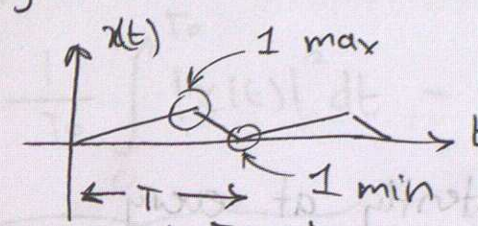
$$a_k = \frac{-(-1)^k}{jk\pi} \quad \text{for } \forall \alpha\ \mathbb{Z},\ \alpha \neq 0$$

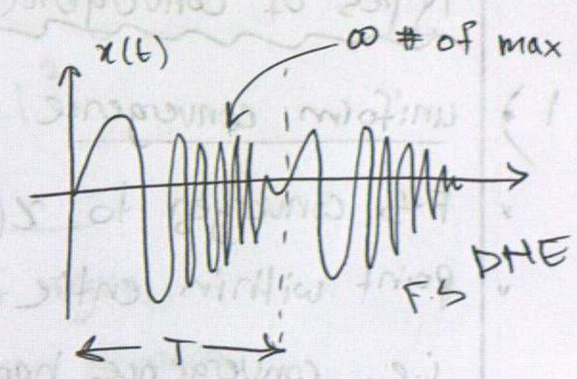
$$x(t) = 1 - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^k}{jk\pi} e^{jkt}$$

Dirishlet (dee-rish-lay) Conditions

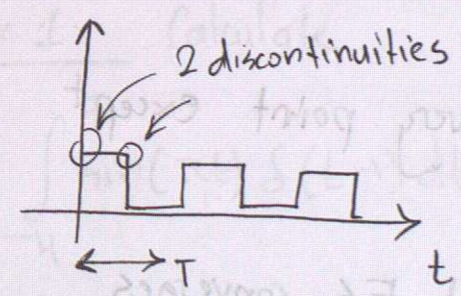
- are a set of conditions for existence of F.S.

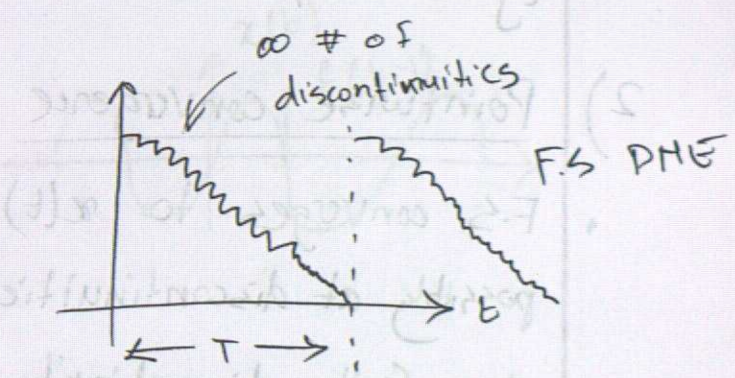
1) function $x(t)$ must be periodic with a finite period T
2) signal should have finite numbers of maximum and mins



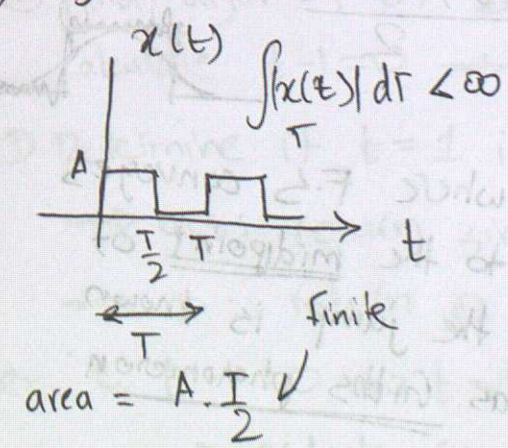


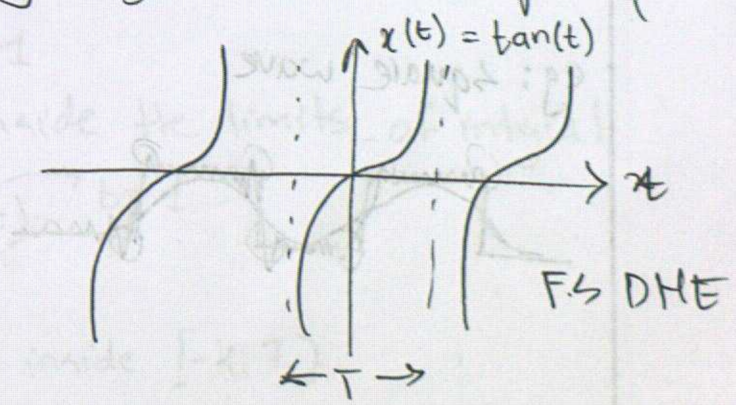
3) Signal should have finite numbers of discontinuities over one period





4) Signal should be absolutely integrable over one period





Convergence of CT F.S.

- explains how well F.S. approximates a periodic function $x(t)$

Types of Convergence

1) uniform convergence

- F.S converges to $x(t)$ consistently at every point within entire interval

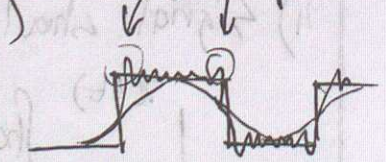
i.e., convergence happens uniformly at every point

eg: sine, cosine

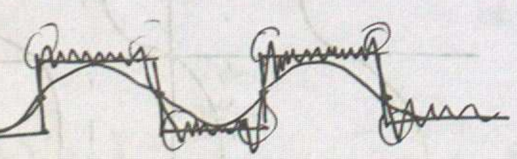
2) Pointwise convergence

- F.S converges to $x(t)$ at every point except possibly at discontinuities
- at a finite discontinuity (jump), F.S converges to the midpoint value $\frac{x(t^-) + x(t^+)}{2}$

gibb's phenomenon



eg: square wave



* where F.S converges to the midpoint of the jump is known as Gibbs phenomenon

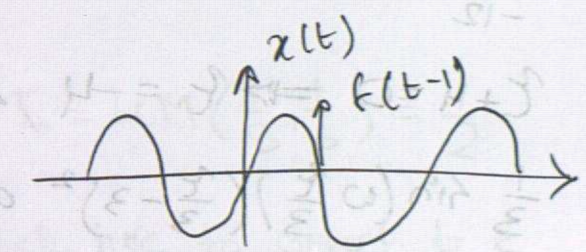
Parseval's Relation

- establishes a relation b/n total energy of $x(t)$ over one period to the sum of the squared magnitudes of Fourier coefficients

$$\underbrace{\frac{1}{T_0}\int_0^{T_0} |x(t)|^2 \, dt}_{\text{energy of signal per unit time}} = \underbrace{\sum_{k=-\infty}^{\infty} |a_k|^2}_{\text{energy of signal in frequency domain}}$$

aka: Average Power (time domain)

Ex 1: Calculate



$$\int_{-4}^{7} \sin(\pi t)\, \delta(t-1)\, dt$$

① Find variable of integration: t

② Find argument of $\delta()$: $t-1$

Calculate $t-1=0 \Rightarrow t=1$

③ Determine if $t=1$ is inside the limits of integral

$\Rightarrow$ yes: return $\sin(\pi t)$ → $t=1$

$\Rightarrow$ no: return 0

ex: t, $t-1=0 \Rightarrow t=1$, so inside $[-4, 7]$

$\Rightarrow$ evaluate $\sin(\pi t) = \sin(\pi) = 0$

Ex 2: $\int_0^2 \sin(\pi t)\,\delta(t-2.5)\,dt \; \boxed{=0}$

since $t-2.5=0 \Rightarrow t=2.5$ ← outside limits

Ex 3: $\int_{-4}^{7} \sin(\omega t)(t-3)^2\,\delta(3t+4)\,dt$

change of variables $\Rightarrow \tau = 3t \Rightarrow t = \frac{\tau}{3}$

$dt = \frac{1}{3}\,d\tau$

lower limit: $3(-4) = -12$

upper limit: $3(7) = 21$

$$\int_{-12}^{21} \sin\left(\omega \frac{\tau}{3}\right)\left(\frac{\tau}{3}-3\right)^2 \delta(\tau+4)\,\frac{d\tau}{3}$$

$\tau+4=0 \Rightarrow \tau=-4$, so inside limits

$\frac{1}{3}\sin\left(\omega\frac{\tau}{3}\right)\left(\frac{\tau}{3}-3\right)^2$ at $\tau=-4$

$\rightarrow \frac{1}{3}\sin\left(-\frac{4}{3}\omega\right)\left(-\frac{4}{3}-3\right)^2 = 6.256\sin\left(-\frac{4}{3}\omega\right)$

Tutorial 3 Jan 27, 2025

Fourier Series

$$x(t) = \sum_{k=-\infty}^{+\infty} \alpha_k e^{j\omega_0 k t}$$

↓ periodic → fourier coefficients

period $= T_0$

$\omega_0 = \frac{2\pi}{T_0}$ ← ω_0 Fundamental frequency

$\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}\,dt$ ← integral over 1 period (doesn't HAVE to be 0 to T_0)

2.b) $x(t) = \sin(16\pi t) + \cos(12\pi t)$

$\Rightarrow T_1 = \frac{1}{8},\; T_2 = \frac{1}{6} \Rightarrow T_0 = LCM\left(\frac{1}{8},\frac{1}{6}\right) = \frac{1}{2}$

$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{1/2} = 4\pi$, trick here to convert to e

$$x(t) = \frac{e^{j16\pi t} - e^{-j16\pi t}}{2j} + \frac{e^{j12\pi t} + e^{-j12\pi t}}{2} = \sum_k \alpha_k e^{j4\pi k t}$$

$4\pi = \omega_0$

$$= \frac{1}{2j}e^{j16\pi t} - \frac{1}{2j}e^{-j16\pi t} + \frac{1}{2}e^{j12\pi t} + \frac{1}{2}e^{-j12\pi t}$$

↓ $k=4$, ↓ $k=-4$, ↓ $k=3$, ↓ $k=-3$

$\alpha_4 = \frac{1}{2j}$, $\alpha_{-4} = -\frac{1}{2j}$, $\alpha_3 = \frac{1}{2}$, $\alpha_{-3} = \frac{1}{2}$

Find k by setting phase equal

eg: $j16\pi t = j4\pi k t$

$k = \frac{16\pi}{4\pi} = 4$

c) $x(t) = e^{-10|t|}$ for $-5 \le t \le 5 \longrightarrow T_0 = 10$

$t<0: e^{-10(-t)} = e^{10t}$

$t>0: e^{-10(t)} = e^{-10t}$

$$\alpha_k = \frac{1}{10}\int_{-5}^{5} x(t)\, e^{-j\frac{2\pi}{10}tk}\, dt$$

$$= \frac{1}{10}\left[\int_{-5}^{0} e^{10t} e^{-j\frac{2\pi}{10}tk} dt + \int_{0}^{5} e^{-10t} e^{-j\frac{2\pi}{10}kt} dt\right]$$

$$= \frac{1}{10}\left[\int_{-5}^{0} e^{(10-j\frac{2\pi}{10}k)t} dt + \int_{0}^{5} e^{-(10+j\frac{2\pi}{10}k)t} dt\right]$$

$$= \frac{1}{10}\left[\left(\frac{e^{(10-j\frac{2\pi}{10}k)t}}{10-j\frac{2\pi}{10}k}\right)_{-5}^{0} + \left(\frac{e^{-(10+j\frac{2\pi}{10}k)t}}{-(10+j\frac{2\pi}{10}k)}\right)_{0}^{5}\right]$$

Q2. i) $x(t) = \cos(2\pi t)\cos(2\pi^2 t)$

$T_1 = \frac{2\pi}{2\pi} = 1 \qquad T_2 = \frac{2\pi}{2\pi^2} = \frac{1}{\pi}$

$LCM(1, \frac{1}{\pi})$ is not rational number

$\therefore$ not periodic

ii) Find the relation b/n α_k and α_{-k} for odd func^n $x(t)$

$\boxed{-x(t) = x(-t)} \leftarrow$ odd $\Leftarrow$

$\boxed{\alpha_{-k} = -\alpha_k \Leftarrow \text{every odd function}} \rightarrow \therefore$ a)

$\boxed{-x(t) = x^*(t) \Longleftarrow \text{imaginary function}}$ → has only imaginary part

$\boxed{\alpha_{-k} = -\alpha_k^* \Leftarrow \text{for every imaginary function}}$

$a = 2j \longrightarrow (2j)^* = -2j = -(a)$

$a^* = -a$

$a = -a^*$

$\alpha_k = -\alpha_k^*$

function both odd and imaginary:

$$\left.\begin{array}{l} \text{odd} \rightarrow \alpha_{-k} = -\alpha_k \\ \text{im} \rightarrow \alpha_{-k} = -\alpha_k^* \end{array}\right\} \alpha_k = \alpha_k^*$$

Q.3 Parseval's theorem

$$\underbrace{\frac{1}{T_0}\int_0^T |x(t)|^2}_{\text{average energ of signal}} = \underbrace{\sum_{k=-\infty}^{+\infty} |\alpha_k|^2}_{\text{energy of its frequency components}}$$

$T_0 = \frac{1}{2}$

$$LHS = \frac{1}{1/2}\int_0^{1/2} |x(t)|^2 dt$$

$1 - 2\sin^2\theta = \cos 2\theta$

$2\cos^2\theta - 1 = \cos 2\theta$

$$LHS = 2\int_0^{1/2} (\sin 16\pi t + \cos 12\pi t)^2\, dt$$

$$= 2\int_0^{1/2} (\sin^2 16\pi t + \cos^2 12\pi t + 2\sin 16\pi t \cos 12\pi t)\, dt$$

$$= 2\int_0^{1/2} \left(\frac{1-\cos 32\pi t}{2} + \frac{\cos 24\pi t + 1}{2} + \sin 28\pi t + \sin 4\pi t\right) dt$$

$$= 2\left[t - \frac{\sin 32\pi t}{2(32\pi)} + \frac{\sin 24\pi t}{2(24\pi)} - \frac{\cos 28\pi t}{28\pi} - \frac{\cos 4\pi t}{4\pi} \right]_0^{1/2}$$

$$RHS = \frac{1}{4} + \frac{1}{4} + \left|\frac{j}{2}\right|^2 + \left|\frac{-j}{2}\right|^2 = 4 \cdot \frac{1}{4} = 1$$

Q 4. $x_1(t): T_0 = \frac{1}{2} \rightarrow$ odd, $Freq = \frac{k}{T_1} = \frac{k}{1/2} = 2k$ (only even frequencies)

$x_2(t): T_0 = 1 \rightarrow$ odd, $Freq = \frac{k}{1} = k$ (both even and odd frequencies)

$x_3(t): T_0 = \frac{1}{2}$, $Freq = 2k$

$x_4(t): T_0 = \frac{1}{2} \rightarrow$ odd, $Freq = 2k$

$x_5(t): T_0 = \frac{1}{2}$, $Freq = 2k$

$x_2(t) \rightarrow$ 2nd graph since contains both even and odd frequencies

if odd signal, $\alpha_k = -\alpha_{-k}$ and $\alpha_0 = 0$

$\therefore x_1(t), x_4(t) \rightarrow$ 3rd or 4th

$x_3(t), x_5(t) \rightarrow$ 1st or 5th

$x_3(t)$ looks like sine + DC

$x_5(t)$ looks like −sine + DC

$$x_3(t) = \frac{e^{j\theta} - e^{-j\theta}}{2j} = \frac{1}{2j}e^{j\theta} - \frac{1}{2j}e^{-j\theta}$$

for (+)ve freq, the multiplier is (−)ve

$$= -\frac{1}{2}je^{j\theta} + \frac{1}{2}je^{-j\theta}$$

$$x_5(t) = -\left(\frac{e^{j\theta} - e^{-j\theta}}{2j}\right) = -\frac{1}{2j}e^{j\theta} + \frac{e^{-j\theta}}{2j}$$

for (+)ve freq, the multiplier is (+)ve

$$= \frac{1}{2}je^{j\theta} - \frac{1}{2}je^{-j\theta}$$

$$x_1(t) = -\sin(\theta) = -\frac{(e^{j\theta} - e^{-j\theta})}{2j} = \frac{j}{2}(e^{j\theta} - e^{-j\theta})$$

↳ 4th graph

$\therefore x_4(t) \rightarrow$ 3rd graph

and $x_3(t) \rightarrow$ 1st graph

$x_5(t) \rightarrow$ 5th graph

$$x(t) = \sum_{k=-\infty}^{+\infty} \alpha_k e^{j\frac{2\pi}{T}kt} \rightarrow f = \frac{k}{T}$$

$$x_2(t) = \frac{3j}{2}e^{j2\pi(-3t)} + je^{j2\pi(-2t)} + \frac{-3j}{2}e^{j2\pi(3t)} + -je^{j2\pi(2t)}$$

$$x_3(t) = -\frac{3j}{2}e^{j2\pi(-2t)} + 2e^{j(2\pi 0)t} + \frac{3j}{2}e^{j2\pi(2t)}$$

Q.5 $x(t) \rightarrow a_k$

$y(t) \rightarrow b_k$

F. coeff's and $x(t), y(t)$ have same fundamental period T_0

$z(t) = \alpha x(t) + \beta y(t) \longrightarrow C_k$

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} z(t) e^{-j\omega t k}\, dt$$

$$= \frac{1}{T}\int \left(\alpha(x(t)) + \beta y(t)\right) e^{-j\omega_0 k t}\, dt$$

$$= \frac{\alpha}{T}\int x(t) e^{-j\omega_0 k t}\, dt + \frac{\beta}{T}\int y(t) e^{-j\omega_0 k t}\, dt$$

$$\boxed{C_k = \alpha a_k + \beta b_k}$$

Q.7 $x(t) \longrightarrow \alpha_k$

$y(t) = x(t-\tau) \longrightarrow \beta_k$

Let $u = t - \tau$

definition of F.S. coeff.

$$\beta_k = \frac{1}{T}\int_T y(t) e^{-j\omega_0 k t}\, dt$$

$$= \frac{1}{T}\int_T x(t-\tau) e^{-j\omega_0 k t}\, dt$$

$$= \frac{1}{T}\int_{-\frac{T}{2}-\tau}^{T/2-\tau} x(u) e^{-j\omega(u+\tau)k}\, du = \frac{e^{-j\omega \tau k}}{T}\int_{-T/2-\tau}^{T/2-\tau} x(u) e^{-j\omega_0 u k}\, du$$

$$\boxed{\beta_k = e^{-j\omega_0 \tau k} \cdot \alpha_k}$$

Q.6 $s(t) = 2x\left(t - \frac{\pi}{4}\right) - 1$

where we have $\underline{x(t) \rightarrow \alpha_k}$, and what of $s(t) \rightarrow \beta_k$?

$$y(t) = x\left(t - \frac{\pi}{4}\right) \Longrightarrow \alpha_k e^{-j\omega_0 \frac{\pi}{4} k} \quad ① \quad = \alpha_k e^{-j\frac{\pi}{2}k}$$

~~$s(t) = 2y(t) - 1$~~ $\quad z(t) = 2y(t) \longrightarrow 2\alpha_k e^{-j\frac{\pi}{2}k}$

$s(t) = z(t) - 1 \longrightarrow$

DC, so only affects $k=0$

$$\beta_k = \begin{cases} 2\alpha_0 - 1, & k = 0 \\ 2\alpha_k e^{-j\frac{\pi}{2}k}, & k \neq 0 \end{cases}$$

Week 4: Lecture 1 Jan 29, 2025

Discrete Time Fourier Series (DTFS)

- is a powerful tool used to analyze periodic DT signals in the frequency domain
- represents a periodic signal as sum of complex exponential components, each corresponding to a specific frequency
- For DT periodic $x[n]$ with period N with DTFS representation

$\omega_0 \to \Omega_0$ in some books

$$\boxed{x[n] = \sum_{k=0}^{N_0-1} \alpha_k e^{jk\omega_0 n}}$$

synthesis equation

where $\omega_0 = \frac{2\pi}{N_0}$ and $n = 0, 1, \cdots, N_0 - 1$

$$\boxed{\alpha_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-jk\omega_0 n}}$$

where $k = 0, 1, 2, \cdots, N_0 - 1$

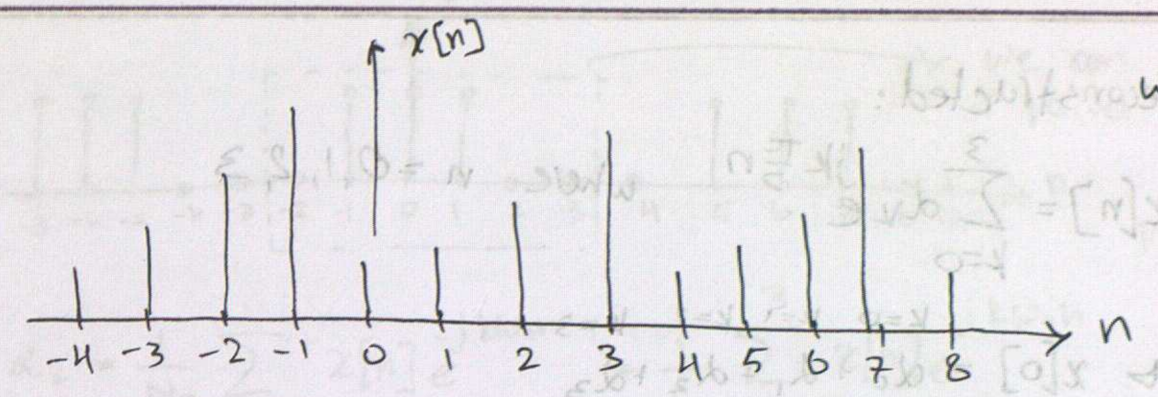
analysis equation

$\sum_{n=0}^{N_0-1} \downarrow \sum_{n=\langle N_0 \rangle} \downarrow$ window, ie any sequence of period N

Ex1: Given a periodic signal $x[n]$ with period $N_0 = 4$

$x[n] = \{1, 2, 3, 4\}, \; n = 0, 1, 2, 3$

find DTFS coefficients α_k and reconstruct the original signal



$\omega_0 = \frac{2\pi}{N_0} = \frac{2\pi}{4} = \pi/2$

$$x[n] = \sum_{k=0}^{N_0-1} \alpha_k e^{jk\omega_0 n} \quad \text{and} \quad \alpha_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n] e^{-jk\omega_0 n} \to k = 0, 1, 2 \cdots N_0 - 1$$

$$\alpha_k = \frac{1}{4}\sum_{n=0}^{3} x[n] e^{-jk\frac{\pi}{2}n} \to k = 0, 1, 2, 3$$

$k=0$: $\Rightarrow \alpha_0 = \frac{1}{4}\left[\underset{n=0}{\overset{x[0]}{1(1)}} + \underset{n=1}{\overset{x[1]}{2(1)}} + \underset{n=2}{\overset{x[2]}{3(1)}} + \underset{n=3}{\overset{x[3]}{4(1)}}\right] = \frac{10}{4} = 2.5$

$k=1$: $\Rightarrow \alpha_1 = \frac{1}{4}\left[1 + (2)e^{-j\frac{\pi}{2}} + (3)e^{-j\frac{2\pi}{2}} + (4)e^{-j\frac{3\pi}{2}}\right]$

$= \frac{1}{4}\left[1 + 2(-j) + 3(-1) + 4(j)\right] = -0.5 + 0.5j$

$k=2$: $\Rightarrow \alpha_2 = \frac{1}{4}\left[1 + 2(e^{-j(2)\frac{\pi}{2}(1)}) + 3(e^{-j(2)\frac{\pi}{2}(2)}) + 4(e^{-j(2)\frac{\pi}{2}(3)})\right] = -0.5$

$k=3$: $\Rightarrow \alpha_3 = \frac{1}{4}\left[1 + 2e^{-j\frac{3\pi}{2}} + 3e^{-j3\pi} + 4e^{-j\frac{9\pi}{2}}\right] = -0.5 - 0.5j$

$\alpha_k = \{2.5, -0.5 + 0.5j, -0.5, -0.5 - 0.5j\}$

Reconstructed:

$$x[n] = \sum_{k=0}^{3} \alpha_k e^{jk\frac{\pi}{2}n} \quad \text{where } n = 0,1,2,3$$

n=0

$$\Rightarrow x[0] = \underset{k=0}{\alpha_0} + \underset{k=1}{\alpha_1} + \underset{k=2}{\alpha_2} + \underset{k=3}{\alpha_3}$$

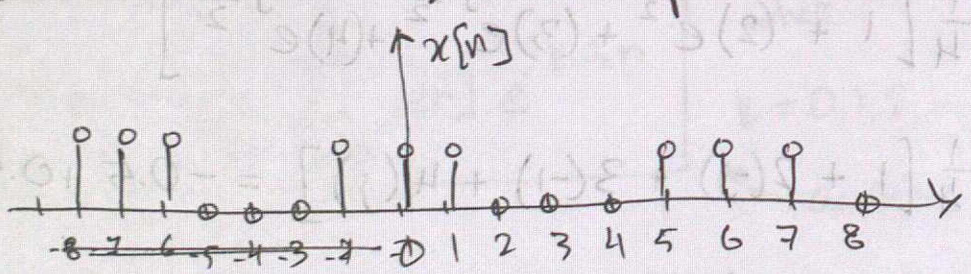
$$= 2.5 - 0.5 + 0.5j - 0.5 - 0.5 - 0.5j = 1$$

n=1

$$\Rightarrow x[1] = \underset{k=0}{2.5(1)} + \underset{k=1}{(-0.5+0.5j)e^{j(1)\frac{\pi}{2}(1)}} + \underset{k=2}{(-0.5)e^{j(2)\frac{\pi}{2}(1)}}$$

$$+ \underset{k=3}{(-0.5-0.5j)e^{j(3)\frac{\pi}{2}(1)}} =$$

Ex 2: Consider $x[n]$ with period $N_0 = 6$, find α_k



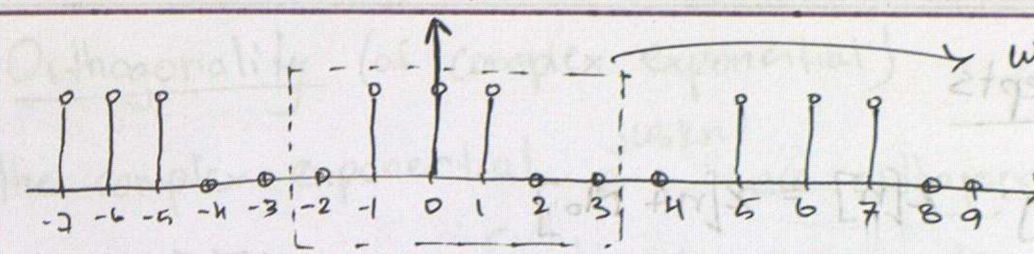
use general formula since some portions = 1 or way tho

$$\omega_0 = \frac{2\pi}{N_0} = \frac{2\pi}{6} = \frac{\pi}{3}, \quad \alpha_k = \frac{1}{6}\sum_{n=-2}^{3} x[n] e^{-jk\omega_0 n}$$

$$\alpha_k = \frac{1}{6}\Big[\underset{n=-2}{\overset{x[-2]}{0}} + \underset{n=-1}{(1)e^{-jk\omega_0(-1)}} + \underset{n=0}{(1)e^{-jk\omega_0(0)}} + \underset{n=1}{(1)e^{-jk\omega_0(1)}} + \underset{n=2}{0} + \underset{3}{0}\Big]$$

$$= \frac{1}{6}\left[e^{jk\omega_0} + 1 + e^{-jk\omega_0}\right] = \frac{1}{6} + \frac{1}{3}\cos(k\omega_0)$$

copy notes



we can pick any period 6 units long, so we pick $n \in [-2, 3]$

$$\alpha_k = \frac{1}{N_0}\sum_{n=\langle N_0\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{6}\sum_{n=-2}^{3} x[n]e^{-jk\omega_0 n}$$

$$\alpha_k = \frac{1}{6}\left[0 + (1)e^{-jk\omega_0(-1)} + (1)e^{-jk\omega_0(0)} + (1)e^{-jk\omega_0(1)} + 0 + 0\right]$$

$$= \frac{1}{6}\left[e^{jk\omega_0} + 1 + e^{-jk\omega_0}\right] = \frac{1}{6} + \frac{1}{6}\cdot 2\left(\frac{e^{jk\omega_0} + e^{-jk\omega_0}}{2}\right)$$

$$= \frac{1}{6} + \frac{1}{3}\cos(\omega_0 k) = \frac{1}{6} + \frac{1}{3}\cos\left(\frac{\pi}{3}k\right)$$

$k=0 \Rightarrow \alpha_0 = \frac{1}{6} + \frac{1}{3}\cos(0) = \frac{1}{2}$ $\quad \Rightarrow \frac{1}{2}, \frac{1}{3}, 0, -\frac{1}{6}, 0 \ldots$ and repeat

$k=\pm 1 \Rightarrow \alpha_{\pm 1} = \frac{1}{6} + \frac{1}{3}\cos\left(\frac{\pi}{3}\right) = \frac{1}{3}$

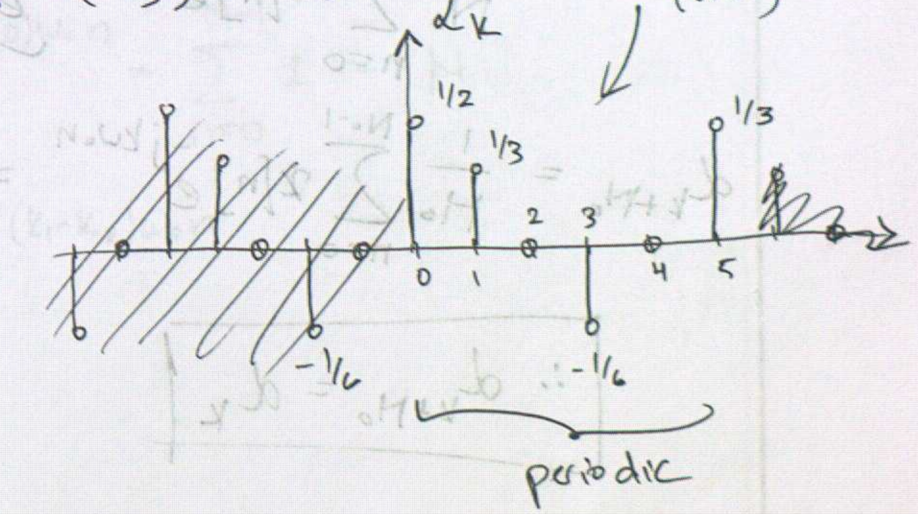
$k=\pm 2 \Rightarrow \alpha_{\pm 2} = \frac{1}{6} + \frac{1}{3}\cos\left(\frac{2\pi}{3}\right) = 0$

$k=\pm 3 \Rightarrow \alpha_{\pm 3} = \frac{1}{6} + \frac{1}{3}\cos\left(\frac{3\pi}{3}\right) = -\frac{1}{6}$

$k=\pm 4 \Rightarrow \alpha_{\pm 4} = 0$

$k=\pm 5 \Rightarrow \alpha_{\pm 5} = \frac{1}{3}$

symmetric about y axis (even)



Week 4: Lecture 2 Jan 30, 2025

term test ☆

Key Concepts

- periodicity: $x[n] = x[n+N_0]$

$$\alpha_k = \alpha_{k+N_0} \quad X[k] = X[k+N_0]$$

- proof: need to show $\alpha_k = \alpha_{k+N_0}$ for $\forall k$ are periodic with Period N_0

Knowing $\alpha_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-j\omega_0 kn}$

Consider α_{k+N_0} plug in "$k+N_0$" for "k"

$$\alpha_{k+N_0} = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-j(k+N_0)\omega_0 n}$$

$$= \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-jk\omega_0 n}e^{-jN_0\omega_0 n} \qquad \omega_0 = \frac{2\pi}{N_0}$$

$$= \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-jk\omega_0 n}\underbrace{e^{-j2\pi n}}_{=1}$$

$$\alpha_{k+N_0} = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-jk\omega_0 n} = \alpha_k$$

$$\boxed{\therefore \alpha_{k+N_0} = \alpha_k}$$

Orthogonality (of complex exponential)

- the complex exponential $e^{j\omega_0 kn}$ are orthogonal over period N_0.
- note: DTFS basis functions are complex exponentials of the form:

$$\phi_k[n] = e^{jk\omega_0 n} \qquad k = 1, 2, 3, \dots, N-1$$

- to prove orthogonality, need to show that the inner product of 2 different basis functions over one period N is

$$\sum_{n=0}^{N-1}\phi_{k_1}[n]\,\phi^*_{k_2}[n] = \begin{cases} N & k_1 = k_2 \\ 0 & k_1 \neq k_2 \end{cases}$$

if we put $\frac{1}{N_0}\sum \dots$ then $k_1 = k_2$ would result in 1 instead of N

$$\sum_{n=0}^{N-1} e^{+jk_1\omega_0 n}e^{-jk_2\omega_0 n} = \sum_{n=0}^{N-1} e^{j(k_1-k_2)\omega_0 n}$$

if $k_1 = k_2 \Rightarrow \sum_{n=0}^{N-1} e^{j(0)\omega_0 n} = \sum_{n=0}^{N-1} 1 = N$

if $k_1 \neq k_2 \Rightarrow \sum_{n=0}^{N-1} e^{j(k_1-k_2)\omega_0 n}$

can be $10-20=-10$ or $20-1=19$ or any integer "m"

Tips

$$\sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r}$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

for $|r|<1$

$$\sum_{n=a}^{b} e^{j\omega n} = e^{j\omega a} \boxed{\sum_{n=0}^{b-a} e^{j\omega n}}$$

now in this form

← apply first formula

$$\sum_{n=0}^{N-1} e^{j(k_1-k_2)\omega_0 n} = \frac{1-\left(e^{j(k_1-k_2)\omega_0}\right)^N}{1-e^{j(k_1-k_2)\omega_0}}$$

$$= \cancel{\sum} \Rightarrow \frac{1-e^{j(k_1-k_2)\omega_0 N}}{1-e^{j(k_1-k_2)\omega_0}} = \frac{1-1}{1-e^{j(k_1-k_2)\omega_0}} = 0$$

$$e^{j(k_1-k_2)\omega_0 N} = e^{j(k_1-k_2)\frac{2\pi}{\cancel{N}}\cancel{N}} = 1$$

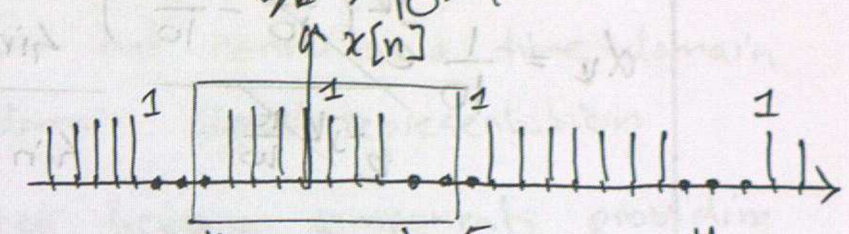
Can conclude that

$$\sum_{n=0}^{N_0-1} e^{j(k_1-k_2)\omega_0 n} = \begin{cases} 1 & k_1 = k_2 \\ 0 & k_1 \neq k_2 \end{cases}$$

- orthogonality ensures that each coeffient α_k captures only the contribution of the corresponding frequency without interference from other frequencies

Ex 1: For the DTFS, show that $\cancel{\alpha_k = \frac{1}{10} \sin}$ $N_0 = 10$

$$\alpha_k = \frac{1}{10}\,\frac{\sin\left(k\frac{7\pi}{10}\right)}{\sin\left(\frac{k\pi}{10}\right)}$$



→ choose this period

$$\alpha_k = \frac{1}{N_0}\sum_{n=\langle N_0\rangle} x[n]e^{-jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{10} = \frac{\pi}{\cancel{2}} = \frac{\pi}{5}$$

$$\alpha_k = \frac{1}{10}\sum_{n=-4}^{5} x[n]e^{-jk\omega_0 n} = \frac{1}{10}\sum_{-3}^{3} e^{-jk\omega_0 n}$$

← "0 everywhere else, so just ignore it"

Use $\displaystyle\sum_{n=a}^{b} e^{j\omega_0 n} = e^{j\omega_0 a}\sum_{n=0}^{b-a} e^{j\omega_0 n}$

$$\alpha_k = \frac{1}{10} e^{-j\omega_0 k(-3)} \sum_{n=0}^{3-(-3)} e^{-jk\omega_0 n}$$

$N = 6+1 = 7$

$$\alpha_k = \frac{1}{10} e^{jk\omega_0(3)} \sum_{n=0}^{6} e^{-jk\omega_0 n} = \frac{1}{10} e^{jk\omega_0(3)} \left(\frac{1-\left(e^{-jk\omega_0}\right)^7}{1-e^{-jk\omega_0}}\right)$$

$$\alpha_k = \frac{1}{10} e^{jk\frac{3\pi}{5}} \frac{1-e^{-jk\frac{7\pi}{5}}}{1-e^{-jk\frac{\pi}{5}}}$$

$$= \frac{1}{10} e^{jk\frac{3\pi}{5}} \left(\frac{e^{-jk\frac{7\pi}{10}}\left(e^{jk\frac{7\pi}{10}} - e^{-jk\frac{7\pi}{10}}\right)}{e^{-jk\frac{\pi}{10}}\left(e^{jk\frac{\pi}{10}} - e^{-jk\frac{\pi}{10}}\right)}\right) \cdot \frac{2j}{2j}$$

$$\alpha_k = \frac{1}{10} \frac{e^{jk\left(\frac{6\pi}{10} - \frac{7\pi}{10}\right)}}{e^{-jk\frac{\pi}{10}}} \frac{\sin\left(\frac{k7\pi}{10}\right)}{\sin\left(\frac{k\pi}{10}\right)}$$

$$\alpha_k = \frac{1}{10} \frac{\sin\left(\frac{k7\pi}{10}\right)}{\sin\left(\frac{k\pi}{10}\right)} \quad //$$

Week 4: Lecture 3 — Jan 31, 2025

Continuous Time Fourier Transform

- is a mathematical operation that converts a time-domain signal into its frequency-domain ~~signal~~ representation
- decomposes signals into their frequency components, providing insight into how a system behaves
- Fourier Series is a special case of Fourier Transform
- Historically, FS was developed/understood first, then was generalized by FT.
- FT is for non-periodic signals and is given by:

FT pairs as follows

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega}$$ synthesis equation inverse FT

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$ analysis equation FT

$$x(t) \xleftrightarrow{FT} X(\omega)$$

- FT can be extended to periodic signals where FT consists of a train of impulses in frequency domain given by

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega_0 t} \xleftrightarrow{FT} X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \alpha_k \, \delta(\omega - k\omega_0)$$

FS representation of a periodic signal

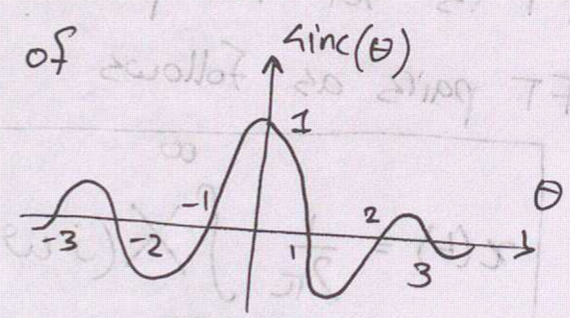
equally spaced delta functions

Sinc Function

- is a function of the form of

$$\boxed{\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}}$$

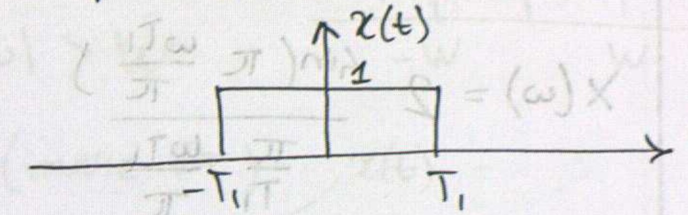
→ normalized



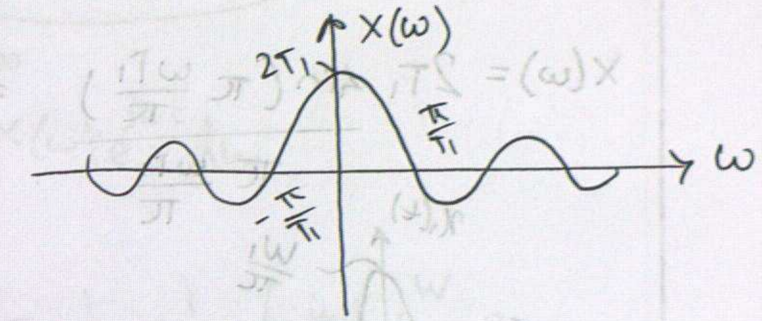
non-normalized $= \pm\pi, \pm 2\pi, \ldots$

Ex 1: FT of a rectangular pulse, find $X(\omega)$

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & \text{otherwise} \end{cases}$$



Knowledge:



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$= \int_{-T_1}^{T_1} (1) e^{-j\omega t}\, dt$$

$$= \frac{-1}{j\omega}\left(e^{-j\omega t} \Big|_{-T_1}^{T_1} \right) = \frac{-1}{j\omega}\left(e^{-j\omega T_1} - e^{j\omega T_1} \right)$$

$$= \frac{2}{\omega}\left(\frac{e^{j\omega T_1} - e^{-j\omega T_1}}{2j} \right) = \frac{2\sin(\omega T_1)}{\omega}$$

$\omega = 0$? $\Rightarrow$ L'hopital's : ~~2ωcos~~ $\left.\frac{2T_1 \cos(\omega T_1)}{1}\right|_{\omega=0} = 2T_1$

Zeroes? $\Rightarrow \frac{2\sin(\omega T_1)}{\omega} = 0, \ \omega \neq 0$

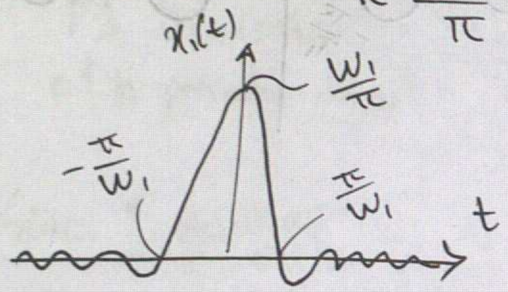
$$2\sin(\omega T_1) = 0 \Rightarrow \sin(\omega T_1) = 0$$

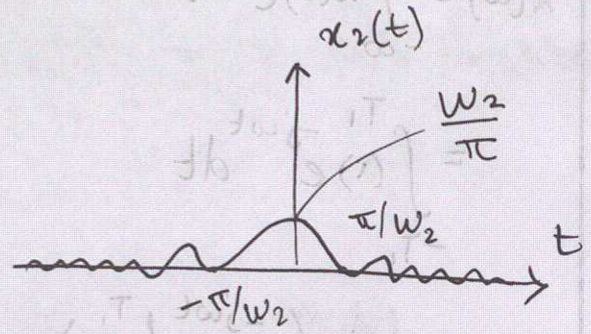
$$\omega T_1 = k\pi \Rightarrow \omega = \frac{\pi}{T_1} k$$

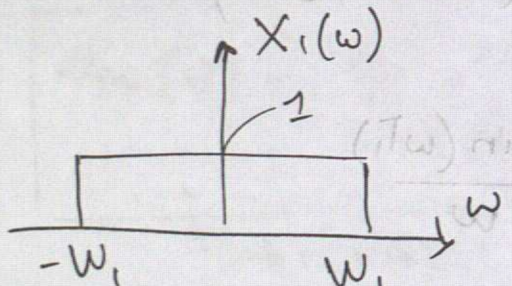
- express as sinc function (normalized)

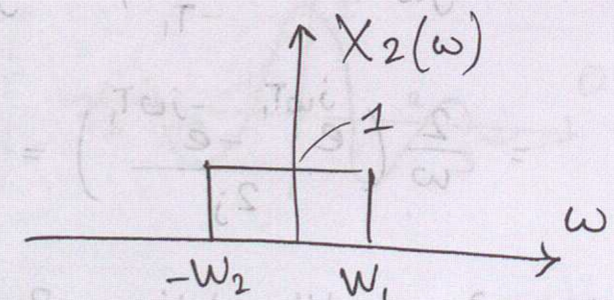
$$X(\omega) = 2\,\frac{\sin\left(\pi\frac{\omega T_1}{\pi}\right)}{\frac{\pi}{T_1}\frac{\omega T_1}{\pi}} \qquad \text{Let } \theta = \frac{\omega T_1}{\pi}$$

$$X(\omega) = 2T_1\,\frac{\sin\left(\pi\frac{\omega T_1}{\pi}\right)}{\pi\frac{\omega T_1}{\pi}} = 2T_1 \operatorname{sinc}(\theta)$$





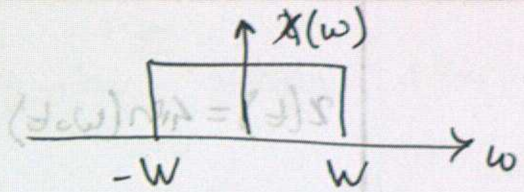




- W increases $\Rightarrow X(\omega)$ becomes broader

- Given

$$X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$



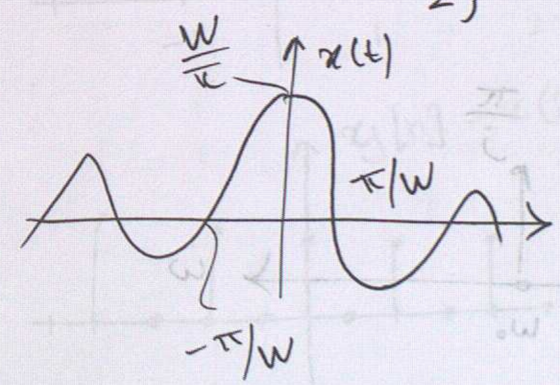
or $X(\omega)$, find original (inverse FT) $x(t)$

knowledge: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} (1) e^{j\omega t}\, d\omega = \frac{1}{2\pi}\frac{1}{jt}\left[e^{j\omega t}\Big|_{-W}^{W}\right]$$

$$= \frac{1}{\pi t}\left[\frac{e^{jWt} - e^{-jWt}}{2j}\right] = \frac{\sin(Wt)}{\pi t}$$

math, express as sinc function



$$x(t) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)$$

<u>Ex:</u> Given $x(t) = \sin(\omega_0 t)$, apply FT and find $X(\omega)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \xleftrightarrow{FT} X(\omega) = 2\pi\sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

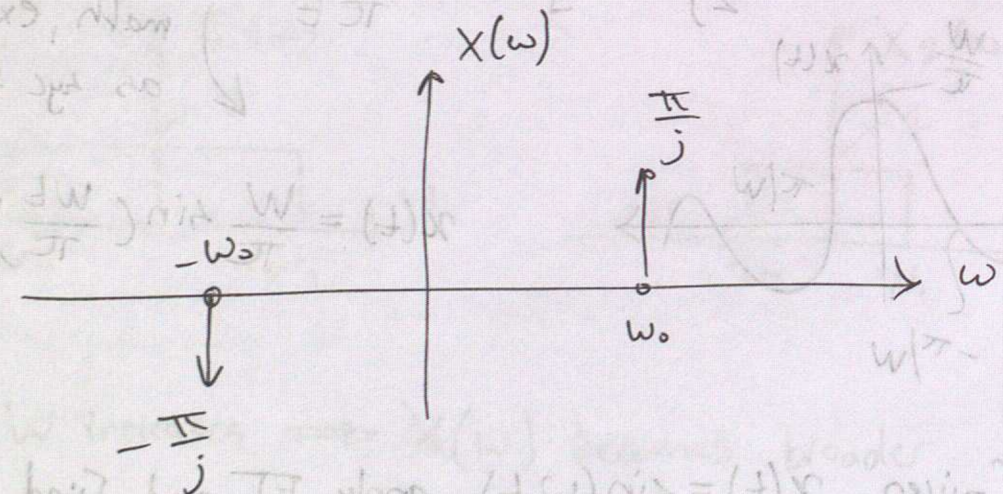
$$x(t) = \sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

$\alpha_1 = \frac{1}{2j}$ and $\alpha_{-1} = \frac{-1}{2j}$

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \alpha_k \delta(\omega - k\omega_0)$$

$$= 2\pi\left[\left(\frac{1}{2j}\right)\delta(\omega - \omega_0) - \frac{1}{2j}\delta(\omega + \omega_0)\right]$$

$$= \frac{\pi}{j}\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]$$



Week 5 Tutorial 4 — Feb 3, 2025

Discrete Time FS

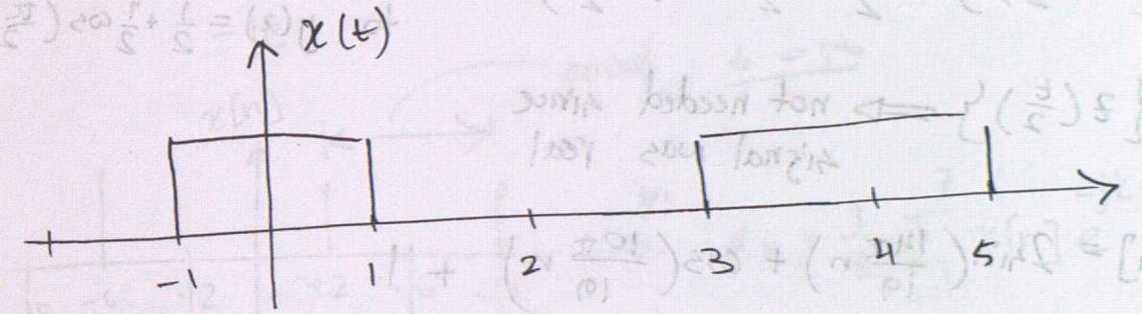
- $x[n]$ with period N_0

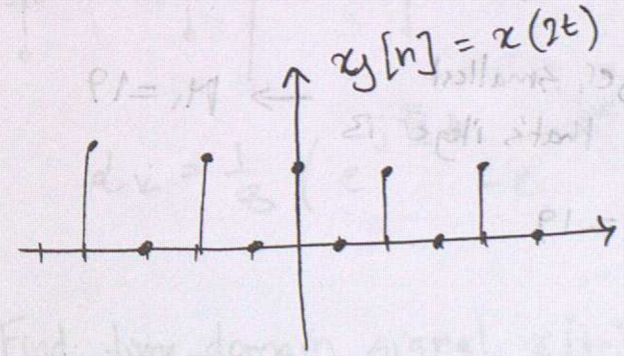
$$\alpha_k = \frac{1}{N_0} \sum_{n=0}^{N_0 - 1} x[n] e^{-j\frac{2\pi}{N_0}kn} \Longleftarrow \text{FS coefficient}$$

- α_k are periodic with period $N_0 \Longrightarrow \alpha_{k+N_0} = \alpha_k$

$$x[n] = \sum_{k=0}^{N_0-1} \alpha_k$$

Q1 i)





ii) $N_0 = 2$

iii) $\alpha_k = \frac{1}{2} \sum_{0}^{1} y[n] e^{-j\frac{2\pi}{2}kn} = \frac{1}{2}\left[e^{-j2k(0)} + 0\right] = \frac{1}{2}[1+0] = \frac{1}{2}$

in this case, $\alpha_k = \frac{1}{2}$ for every k

iv) $z(t) = \sum_{k=0}^{N_0-1} \alpha_k e^{j\frac{2\pi}{N_0}kt}$

$= \frac{1}{2} e^{j\frac{2\pi}{2}t(0)} + \frac{1}{2} e^{j\frac{2\pi}{2}t(1)}$

$= \frac{1}{2} + \frac{1}{2} e^{j\pi t}$

$= \frac{1}{2} + \frac{1}{2}(\cos \pi t + j \sin \pi t)$

• reduce sampling period to increase accuracy of our approximation → T_s

• making T_s closer to T_0 of CT increases number of α_k terms

$Re\{z(\frac{t}{2})\} = \frac{1}{2} + \frac{1}{2}\cos(\frac{\pi t}{2})$ ← real part is similar to $y(x) = \frac{1}{2} + \frac{1}{2}\cos(\frac{\pi}{2}x)$

$Im\{z(\frac{t}{2})\}$ ⟹ not needed since signal was real

Q.2 i) $x[n] = 2\sin(\frac{14\pi}{19}n) + \cos(\frac{10\pi}{19}n) + 1$

$N_1 = \frac{2\pi}{14\pi/19} = \frac{19}{7}$, not integer, smallest multiple that's integer is → $N_1 = 19$

$N_2 = \frac{2\pi}{10\pi/19} = \frac{19}{\cancel{10}}$ → $N_2 = 19$

$N_0 = LCM(19, 19) = 19$

$x[n] = \cancel{2}\left(\frac{e^{j\frac{14\pi}{19}n} - e^{-j\frac{14\pi}{19}n}}{\cancel{2}j}\right) + \left(\frac{e^{j\frac{10\pi}{19}n} + e^{-j\frac{10\pi}{19}n}}{2}\right) + 1$

$= \sum_{k=-9}^{9} \alpha_k e^{j\frac{2\pi}{19}nk}$

only 5 non-zero terms

why from $k=-9$ to 9?

$\alpha_k \Rightarrow \sum_{k=-9}^{9} \alpha_k e^{j\frac{2\pi}{19}nk} = -je^{j\frac{14\pi}{19}n} + je^{-j\frac{14\pi}{19}n} + \frac{1}{2}e^{j\frac{10\pi}{19}n} + \frac{1}{2}e^{-j\frac{\pi}{19}n} + 1$

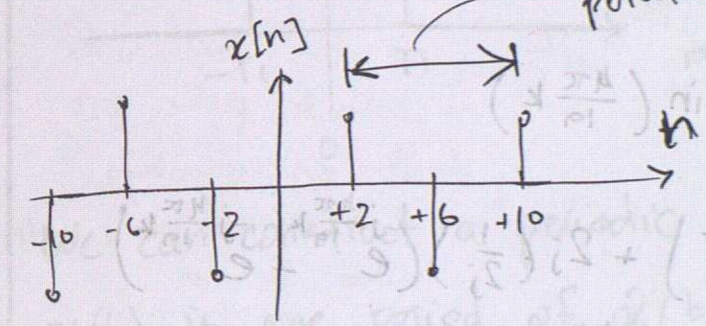
$\alpha_{\pm 5} = \frac{1}{2}, \ \alpha_7 = -j, \ \alpha_{-7} = j, \ \alpha_0 = 1$

ii) $x[n] = \sum_{m=-\infty}^{+\infty} (-1)^m [\delta(n-2m) + \delta(n+3m)]$

$\alpha_k = \frac{1}{12}[2e^0 + 1e^{-j\frac{4\pi k}{12}} - 1e^{-j\frac{6\pi k}{12}} + 1e^{-j\frac{8\pi k}{12}} + 1e^{-j\frac{16\pi k}{12}}$

iii) $-1e$

period $= 8 = N_0$



$\alpha_k = \frac{1}{8}\sum_{n=0}^{7} x[n] e^{-jk\frac{2\pi}{8}n}$

$\alpha_k = \frac{1}{8}\left(e^{-j\frac{\pi}{2}k} - e^{-j\frac{3\pi k}{2}}\right)$

Q3 i) Find time-domain signal $x[n]$:

1) find period of α_k, which is period of $x[n]$ time-dom

2) use synthesis equation for $x[n]$ or use Euler def of cos/sin

$$\alpha_k = \frac{1}{2}\left(e^{j\frac{8\pi k}{21}} + e^{-j\frac{8\pi}{21}k}\right) = \frac{1}{21}\sum x(n)e^{-j\frac{2\pi}{21}nk}$$

$$\frac{x(4)}{21} = \frac{1}{2} \Rightarrow x(4) = \frac{21}{2}$$

since $N_0 = 21$

$$\frac{x(-4)}{21} = \frac{1}{2} \Rightarrow x(-4) = \frac{21}{2}$$

$N_0 = \frac{2\pi}{8\pi/21} = \frac{21}{4} \Rightarrow 21$ is nearest integer multiple of $21/4$

$$x[n] = \begin{cases} 21/2 & n = 4, -4 \\ 0 & -10 \le n \le 10,\ n \ne 4,\ n \ne -4 \end{cases}$$

$\hookrightarrow$ over one period

ii) $\alpha_k = \cos\left(\frac{10\pi}{19}k\right) + 2j\sin\left(\frac{4\pi}{19}k\right)$

$$= \frac{1}{2}\left(e^{j\frac{10\pi}{19}k} + e^{-j\frac{10\pi}{19}k}\right) + 2j\left(\frac{1}{2j}\right)\left(e^{j\frac{4\pi}{19}k} - e^{-j\frac{4\pi}{19}k}\right)$$

$$= \frac{1}{2}e^{j\frac{10\pi}{19}k} + \frac{1}{2}e^{-j\frac{10\pi}{19}k} + e^{j\frac{4\pi}{19}k} - e^{-j\frac{4\pi}{19}k}$$

$$= \frac{1}{19}\sum x[n]e^{-j\frac{2\pi}{19}nk}$$

$$\frac{x(5)}{19} = \frac{1}{2} \qquad \frac{x(-5)}{19} = \frac{1}{2}$$

$$\frac{x(2)}{19} = -1 \qquad \frac{x(-2)}{19} = 1$$

$$\Rightarrow x[n] = \begin{cases} 19, & n = -2 \\ -19, & n = 2 \\ \frac{19}{2}, & n = \pm 5 \\ 0, & \text{else} \end{cases}$$

Q4) i(i) # of terms in approx: $2K+1$

in general, number of terms: N_0

Week 5 : Lecture 1 Feb 5, 2025

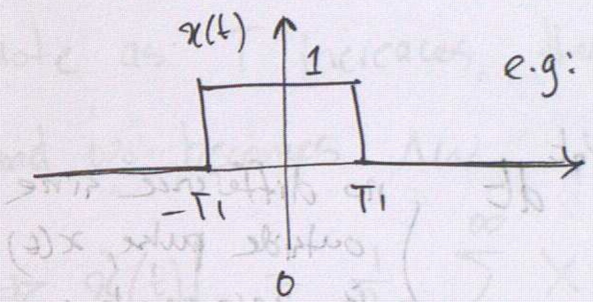
Representations of aperiodic CTFT signals

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$x(t) \underset{\text{IFT}}{\overset{\text{FT}}{\rightleftharpoons}} X(j\omega)$$

Proof: Consider a non-periodic signal

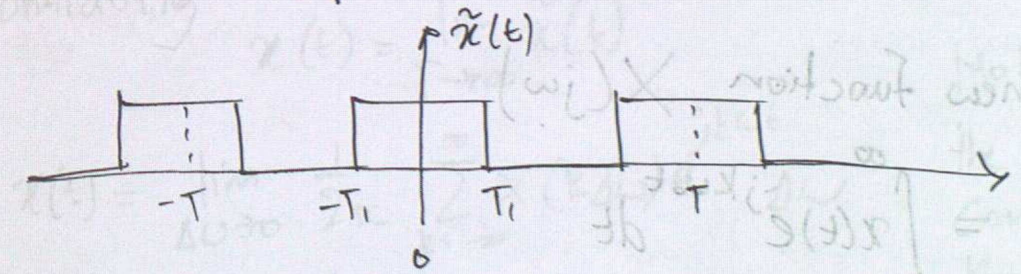


e.g: $x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & \text{otherwise} \end{cases}$

non periodic

periodic

We can construct a periodic signal $\tilde{x}(t)$ where $x(t)$ is one period of $\tilde{x}(t)$



such that $\lim_{T\to\infty} \tilde{x}(t) = x(t)$

Write FS for $\tilde{x}(t)$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega_0 t} \quad , \quad \alpha_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} \, dt$$

Note that $x(t) = \tilde{x}(t)$ for $|t| < T/2$

$x(t) = 0$ otherwise

Replace $\tilde{x}(t)$ by $x(t)$

$$\Rightarrow \alpha_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} \, dt$$

no difference since outside pulse, $x(t)$ is zero anyways, so can make bounds $-\infty$ to ∞

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} \, dt$$

Define new function $X(j\omega)$

$$X(\omega_0) = \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} \, dt$$

$$\Rightarrow \alpha_k = \frac{1}{T} X(\omega_0)$$

Knowing $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega_0 t}$, sub for α_k

$$\Rightarrow \tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t} \quad , \text{ also } \omega_0 = \frac{2\pi}{T_0}$$

$X(\omega_0)$

$$\Rightarrow \frac{1}{T} = \frac{\omega_0}{2\pi}$$

$$\Rightarrow \tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega_0) e^{jk\omega_0 t} \omega_0$$

Note as T increases, then $\omega_0 = \frac{2\pi}{T}$ decreases

and ω_0 becomes $\Delta\omega$

$$\Rightarrow \tilde{x}(t)\Big|_{T\to\infty} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(\underbrace{k\Delta\omega}_{\omega}) e^{jk\Delta\omega t} \underbrace{\Delta\omega}_{d\omega}$$

Considering $x(t) = \lim_{T\to\infty} \tilde{x}(t)$

$$x(t) = \lim_{\Delta\omega\to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\Delta\omega) e^{jk\omega t} \Delta\omega$$

$\vdots$

Note, at the limit, the spectrum becomes continuous rather than discrete

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega \quad \text{— synthesis}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad \text{— periodic}$$

$$x[n] \underset{\text{IFT}}{\overset{\text{FT}}{\rightleftarrows}} X(e^{j\omega})$$

aperiodic — periodic

take notes

Ex 1: $x(t) = A\cos(\omega_0 t + \theta)$ periodic with $T_0 = \frac{2\pi}{\omega_0}$

$$\cos^2(\omega t) = \frac{1+\cos(2\omega t)}{2}$$

$$P = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\,dt$$

$$= \frac{1}{\frac{2\pi}{\omega_0}}\int_0^{2\pi/\omega_0} A^2\frac{1}{2}\left(1+\cos(2\omega_0 t + 2\theta)\right)dt \quad \rightarrow 0$$

$$= \frac{A^2\omega_0}{2\pi}\int_0^{2\pi/\omega_0}\frac{1}{2}\,dt = \frac{A^2\omega_0}{2\pi}\left(\frac{1}{2}\left(\frac{2\pi}{\omega_0}\right)\right) = \frac{A^2}{2}$$

Find energy

$$\|x\|_2^2 = \int_{-\infty}^{\infty}|x(t)|^2\,dt = E$$

$$E = \int_{-\infty}^{\infty}A^2\cos^2(\omega_0 t+\theta)\,dt = A^2\int_{-\infty}^{\infty}\frac{1+\cos(2\omega_0 t + 2\theta)}{2}\,dt$$

$E = \infty \implies$ Power signal

Week 5: Lecture 3 — Feb 6, 2025

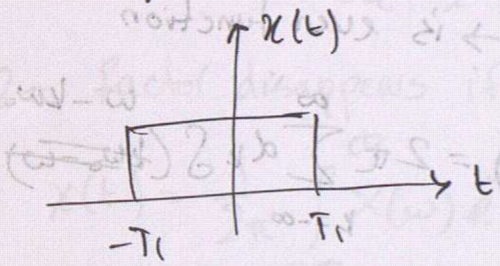
Fourier Series

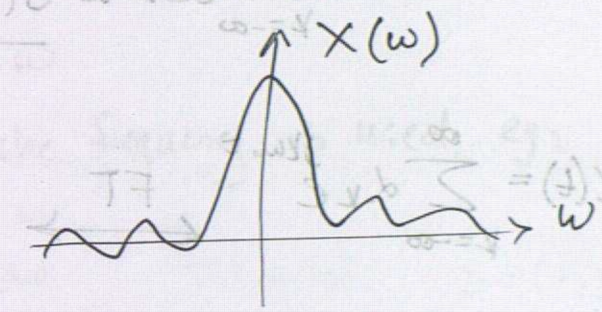
* represents periodic signals as harmonically related frequencies separated by increments of fundamental frequency (ω_0).

$$x(t) = \sum_{k=-\infty}^{\infty}\alpha_k e^{jk\omega_0 t}\,dt$$

$$\alpha_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$$

* Fourier Transform for aperiodic signals require a continuum of frequencies





$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}\,d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty}x(t)e^{-j\omega t}\,dt$$

Proof: generalize FT

* can apply FT to a ~~periodic~~ broader class of signals that are periodic

Consider a periodic signal $x(t)$, write as FS representation

$$x(t) = \sum_{k=-\infty}^{\infty}\alpha_k e^{jk\omega_0 t} \longrightarrow F\{x(t)\} = X(\omega)$$

$$\mathcal{F}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad \text{, plug } x(t)$$

$$X(\omega) = \int_{-\infty}^{\infty} \Big[\sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega_0 t} \Big] e^{-j\omega t}\, dt \quad \leftarrow \text{multiply by } \frac{2\pi}{2\pi}$$

$$= \sum_{k=-\infty}^{\infty} \alpha_k\, 2\pi \underbrace{\Big[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j(k\omega_0 - \omega)t}\, dt \Big]}_{\text{this is delta function}}$$

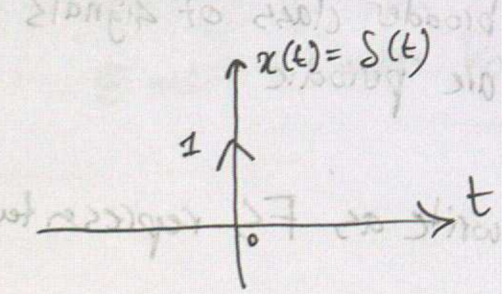
$$\Rightarrow X(\omega) = \sum_{k=-\infty}^{\infty} \alpha_k\, 2\pi\, \underline{\delta(k\omega_0 - \omega)}$$

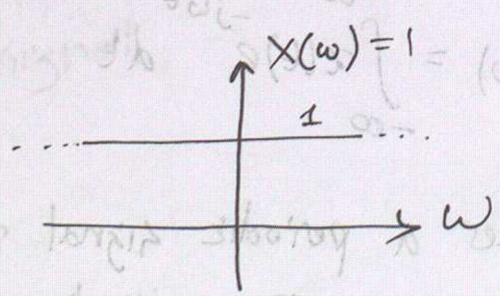
↳ is even function

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega_0 t} \xleftrightarrow{FT} X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \alpha_k\, \delta(\omega - k\omega_0)$$

Ex 1: Consider $x(t) = \delta(t)$, find FT

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = e^{0} = 1$$



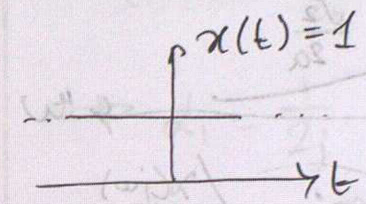


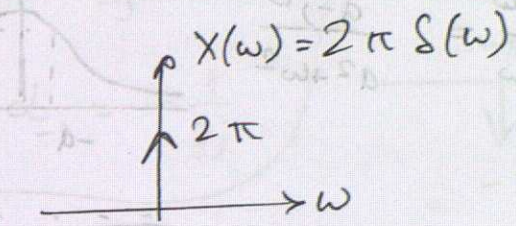
- this means dirac delta function in time-dom corresponds to constant function in freq-dom

Apply Duality Property

$$\delta(t) \xleftrightarrow{FT} 1$$

$$1 \xleftrightarrow{FT} 2\pi\, \delta(\omega)$$





- 2π factor is due to angular frequency (ω) instead of cyclic ~~fre~~ frequency, (f) Note: $2\pi f = \omega$
- 2π factor disappears if cyclic frequency is used, eg:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

becomes $$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

Ex 2: FT of exponential function (right-sided decaying exponential): $x(t) = e^{-at} u(t)$, $a > 0$

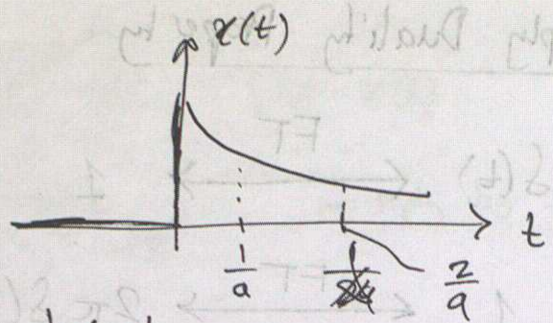
Knowledge: $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(\omega) = \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt = \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty} = \frac{1}{a+j\omega}$$

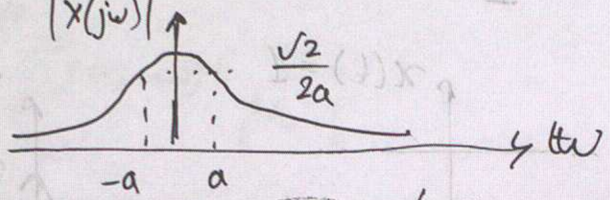
for $a > 0$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

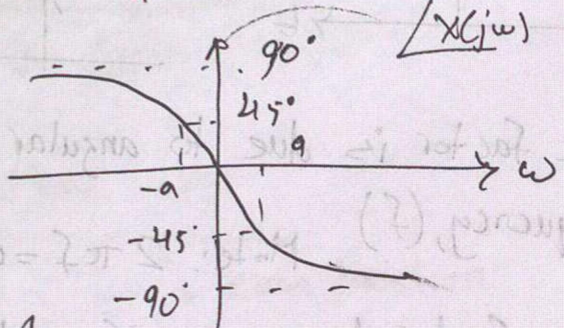
$$\angle X(j\omega) = -\arctan\left(\frac{\omega}{a}\right)$$



$$\frac{1}{a+j\omega} \cdot \frac{a-j\omega}{a-j\omega} = \frac{a-j\omega}{a^2+\omega^2}$$

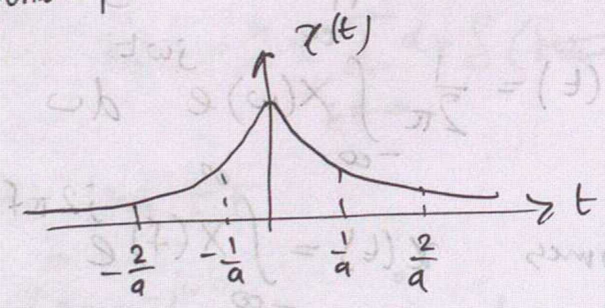


$$\frac{Im}{Re} = \frac{\frac{-\omega}{a^2+\omega^2}}{\frac{a}{a^2+\omega^2}}$$



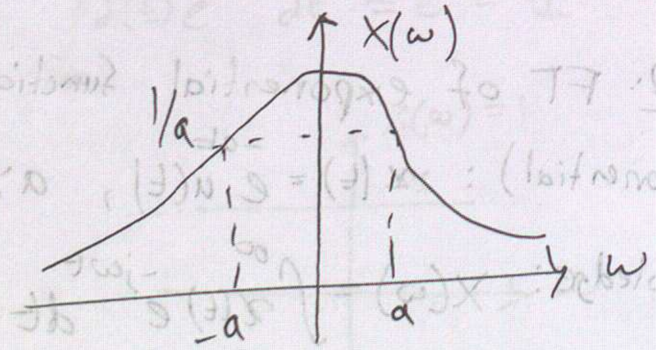
Ex 3: 2-sided decaying exponential

$x(t) = e^{-a|t|}$ for $a>0$



$$X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt$$

$$= \frac{2a}{a^2+\omega^2}$$

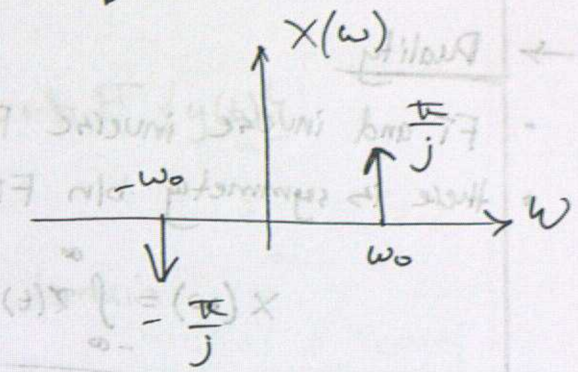


Ex 4: $x(t) = \sin(\omega_0 t)$, find $X(\omega)$

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega_0 t} \xleftrightarrow{FT} X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \alpha_k \delta(\omega - k\omega_0)$$

$$\sin(\omega_0 t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

$$\Longrightarrow \alpha_1 = \frac{1}{2j}, \quad \alpha_{-1} = -\frac{1}{2j}$$



$$X(\omega) = \underbrace{2\pi\left(\frac{1}{2j}\right)\delta(\omega-\omega_0)}_{k=1} + \underbrace{2\pi\left(\frac{-1}{2j}\right)\delta(\omega+\omega_0)}_{k=-1}$$

$$X(\omega) = \frac{\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

Ex 5: do same for

Week 5: Lecture 3 Feb 7, 2025

Properties of Fourier Transform

$$F^{-1}\{X(j\omega)\} = x(t) \xleftrightarrow{FT} X(j\omega) = F\{x(t)\}$$

Duality

- FT and inverse inverse FT are much the same machine
- there is symmetry b/n FT and IFT

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$x(t) \rightleftarrows X(\omega)$$

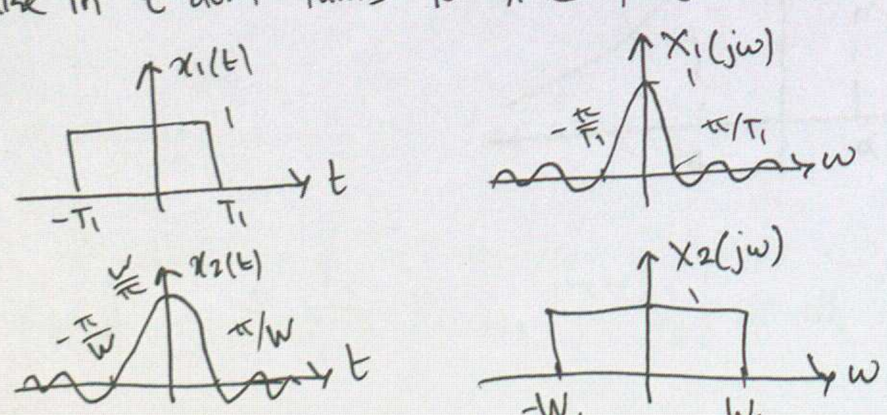
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, dt$$

- if there is a time frequency, then a frequency to time may also exist.
- consider pulse

$$x_1(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \xleftrightarrow{FT} X_1(j\omega) = \frac{2\sin(\omega T_1)}{\omega}$$

$$x_2(t) = \frac{\sin(\omega t)}{\pi t} \longleftrightarrow X_2(j\omega) = \begin{cases} 1 & |\omega| < W_1 \\ 0 & |\omega| > W_1 \end{cases}$$

- pulse in t-dom turns to sinc function in ω-dom



Linearity

- if $x(t) \xleftrightarrow{FT} X(\omega)$ and $y(t) \xleftrightarrow{FT} Y(\omega)$
- then $ax(t) + by(t) \xleftrightarrow{FT} aX(\omega) + bY(\omega)$
- Another way

$$F\{ax(t) + by(t)\} = aF\{x(t)\} + bF\{y(t)\}$$

Time Shifting

- introduces a phase shift in frequency domain
- If $x(t) \xleftrightarrow{FT} X(\omega)$
- then $x(t-t_0) \xleftrightarrow{FT} X(\omega)e^{-j\omega t_0}$
- consider $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, dt$ and sub $x(t-t_0)$

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-t_0)}\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t_0}[X(j\omega)e^{j\omega t}]\, d\cdots$$

Other Properties : if $x(t) \xleftrightarrow{FT} X(\omega)$

1) Time scaling : $x(at) \xleftrightarrow{FT} \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$

   time scaled by a, frequency scaled by $\frac{1}{a}$

2) Differentation : $\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(j\omega)$

   time differentiation is replaced by multiplication by jω in frequency

3) Integration : $\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \underbrace{\pi X(0)\delta(\omega)}_{\text{constant of integration}}$

   time ~~diff~~ integration is replaced by division by jω in frequency

Convolution Property

$$x(t) * h(t) \xleftrightarrow{FT} X(\omega)H(\omega)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Parseval's Relation

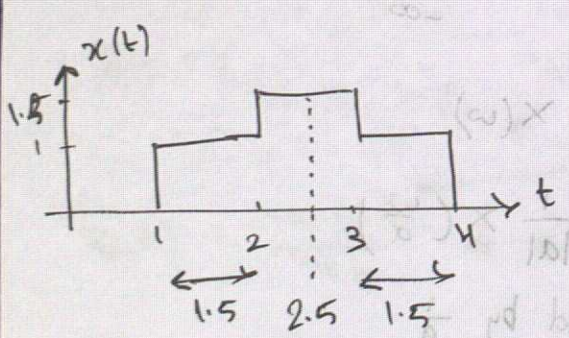
If $x(t) \xleftrightarrow{FT} X(j\omega)$,

then $$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

midterm 2

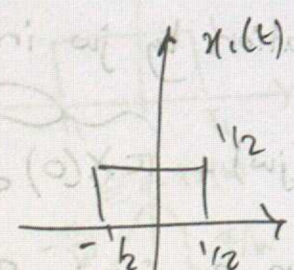
Ex 1: Find FT for $x(t)$ and knowing $x_c(t) \begin{cases} 1 & |t|<T_1 \\ 0 & \text{otherwise} \end{cases}$

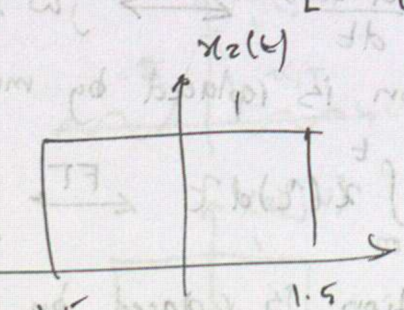
$$X_c(\omega) = \frac{2\sin(\omega T_1)}{\omega}$$



$$\therefore x(t) = \frac{1}{2}\left(x_1(t-2.5)\right) + x_2(t-2.5)$$

$$X(\omega) = \left[\frac{\sin(\omega/2)}{\omega} + \frac{2\sin(\omega\frac{3}{2})}{\omega}\right] e^{-j\omega(2.5)}$$



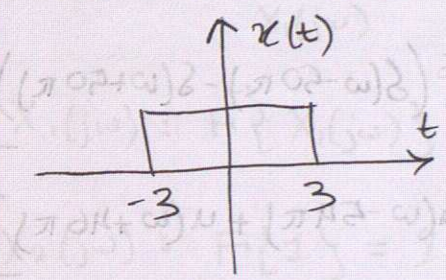


Week 6: Tutorial 5 — Feb 10, 2025

- CTFT: $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$
- ICTFT: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$

eg: $F\{e^{-at}u(t)\} = \frac{1}{a+j\omega} = X(j\omega)$

5.1.i) $x(t) = u(t+3)u(t-3)$



$$X(j\omega) = \frac{\sin(\omega\frac{\tau}{2})}{\omega/2} = \frac{\sin(3\omega)}{\frac{\omega}{2}} \quad //$$

$$\frac{\tau}{2} = +3 \Rightarrow \tau = 6$$

ii) $x(t) = \underbrace{\sin(4\pi t)}_{x_1(t)}\underbrace{\sin(50\pi t)}_{x_2(t)}$ $\qquad F\{x(t)\} = \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$

$$X_1(j\omega) = \frac{\pi}{j}\left(\delta(\omega-4\pi) - \delta(\omega+4\pi)\right)$$

$$X_2(j\omega) = \frac{\pi}{j}\left(\delta(\omega-50\pi) - \delta(\omega+50\pi)\right)$$

$$X(j\omega) = \frac{1}{2\pi}\left[\frac{\pi}{j}\left(\delta(\omega-4\pi)-\delta(\omega+4\pi)\right) * \frac{\pi}{j}\left(\delta(\omega-50\pi)-\delta(\omega+50\pi)\right)\right]$$

$$= \frac{-\pi}{2}\left[\delta(\omega-54\pi) - \delta(\omega+46\pi) - \delta(\omega-46\pi) + \delta(\omega+54\pi)\right] \quad //$$

iii) $x(t) = \underbrace{\frac{\sin(4\pi t)}{\pi t}}_{x_1(t)} \underbrace{\sin(50\pi t)}_{x_2(t)}$

$$X_2(j\omega) = \frac{\pi}{j}\left[\delta(\omega - 50\pi) - \delta(\omega + 50\pi)\right]$$

$$X_1(j\omega) = u(\omega + 4\pi) - u(\omega - 4\pi)$$

$$X(j\omega) = \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$$

$$= \frac{1}{2\pi}\left[\left(u(\omega+4\pi) - u(\omega - 4\pi)\right) * \frac{\pi}{j}\left(\delta(\omega - 50\pi) - \delta(\omega + 50\pi)\right)\right]$$

$$= \frac{1}{2j}\left[u(\omega - 46\pi) - u(\omega + 54\pi) - u(\omega - 54\pi) + u(\omega + 46\pi)\right] \quad //$$

5.2. i) $X(j\omega) = \frac{e^{-j\omega 3}}{2+j\omega}$, knowing $\mathcal{F}\{e^{-at}\} = \frac{1}{a+j\omega}$

$$x(t) = \mathcal{F}^{-1}\left\{\frac{e^{-j\omega 3}}{2+j\omega}\right\} = e^{-2(t-3)} u(t-3) \quad //$$

time shif comes from $e^{-j\omega z}$ in ω-dom

ii) $X(j\omega) = \frac{j\omega}{2+j\omega}$, knowing $\frac{d}{dt}x'(t) = j\omega X(j\omega)$

$$x'(t) \Rightarrow \frac{d}{dt}\left(e^{-2t}u(t)\right) = (-2)e^{-2t}u(t) + e^{-2t}\delta(t)$$

$$\therefore x(t) = \frac{d}{dt}x'(t) = -2e^{-2t}u(t) + e^{-2t}\delta(t) \quad //$$

$$x(t) = \frac{d}{dt}x'(t) = j\omega X(j\omega)$$

iii) $X(j\omega) = \underbrace{\frac{j\omega}{2+j\omega}}_{\text{same as ii), so denote with } x^{prev}(t)} e^{-j\omega 3}$ $\longrightarrow$ function has e^{-at}, a differentiation, and shif

$x(t) = x^{prev}(t-3)$ $\longleftarrow$ just shift by 3 since we have $e^{-j\omega 3}$

$$x(t) = e^{-2(t-3)}\left(\delta(t-3) - 2u(t-3)\right) \quad //$$

iv) $X(j\omega) = \underbrace{\frac{2\sin(\omega)}{\omega}}_{X_1(j\omega)} \underbrace{\sum_{k=-\infty}^{\infty}\left(\frac{\pi}{5}\right)\delta(\omega - 0.2\pi k)}_{X_2(j\omega)} \Rightarrow x(t) = x_1(t) *$

$X_1(j\omega)$: $\mathcal{F}^{-1}\{X_1(j\omega)\} = x_1(t) = u(t+1) - u(t-1)$

$X_2(j\omega)$: $\mathcal{F}\{1\} = 2\pi\delta(\omega)$, and ω-shift implies multiplication by $e^{j\omega_0 t}$ in t-dom

$$\mathcal{F}\{e^{j\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$$

$$x_2(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{10} e^{j0.2\pi k t}$$

$\longrightarrow$ write as summation of $\delta(t)$

$$\boxed{\sum_{n=-\infty}^{+\infty}\delta(t - nT_0) = \frac{1}{T_0}\sum_{k=\infty}^{+\infty} e^{j\frac{2\pi}{T_0}kt}}$$

since $T_0 = 10$ and $\frac{2\pi}{10} = 0.2\pi$

$$= \sum_{n=-\infty}^{+\infty}\delta(t - 10n)$$

$$\Rightarrow x_1(t) * x_2(t) = \sum_{n=-\infty}^{+\infty}\left(u(t+1-10n) - u(t-1-10n)\right) \quad //$$

5.3. i) $x[n] = (\frac{1}{2})^{n-1} u[n-1]$, looks like $a^n u[n]$, but shifted n

$a^n u[n]$, $|a| < 1 \xrightarrow{DTFT} \frac{1}{1-e^{-j\omega} a}$

$$\boxed{\begin{array}{l} X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} \leftarrow \text{DTFT} \\ x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n}\, d\omega \leftarrow \text{I}\cdot\text{DTFT} \end{array}}$$

$(\frac{1}{2})^n u[n] \xrightarrow{DTFT} \frac{1}{1-\frac{1}{2}e^{-j\omega}}$

$(\frac{1}{2})^{n-1} u[n-1] \xrightarrow{DTFT} \frac{e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$

$$\boxed{\begin{array}{l} x[n] \xrightarrow{FT} X(e^{j\omega}) \\ x[n-n_0] \xrightarrow{FT} e^{-j\omega n_0} X(e^{j\omega}) \end{array}}$$

$$\boxed{\begin{array}{l} x[n] \xrightarrow{FT} X(e^{j\omega}) \\ nj \cdot x[n] \longrightarrow \frac{d}{d\omega}\left[X(e^{j\omega})\right] \end{array}}$$

ii) $x[n] = (n+1) a^n u[n]$, $|a| < 1$

$y[n] \triangleq a^n u(n) \xrightarrow{DTFT} \frac{1}{1-ae^{-j\omega}} \triangleq Y(e^{j\omega})$

$x[n] = ny[n] + y[n] \Longrightarrow X(e^{j\omega}) = \frac{1}{j}\frac{d}{d\omega}\left[Y(e^{j\omega})\right] + Y(e^{j\omega})$

$x[n] = \frac{1}{j}\frac{d}{d\omega}\left[\frac{1}{1-ae^{-j\omega}}\right] + \left(\frac{1}{1-ae^{-j\omega}}\right)$ //

5.4. i) $x[n] = (\frac{1}{2})^n u[n-4]$

$\frac{1^n}{2^n} = \frac{(1)^{n-4}}{(2)^{n-4}} \Longrightarrow \frac{1}{2^n} = \frac{1}{2^{n-4}}$

☆ $= 2^{-4} \cdot (\frac{1}{2})^{n-4} u[n-4]$

$2^{n-4} = 2^n$

?? 2^{-4}

$\therefore X(e^{j\omega}) = 2^{-4} \cdot \left(\frac{e^{-j\omega 4}}{1-\frac{1}{2}e^{-j\omega}}\right)$ //

$$\boxed{\begin{array}{l} x[n] \xrightarrow{DTFT} X(e^{j\omega}) \\ x[n] e^{j\omega_0 n} \xrightarrow{DTFT} X(e^{j(\omega-\omega_0)}) \end{array}}$$

ii) $x[n] = (\frac{1}{4})^n u[n] \cos[\frac{1}{2}\pi n]$

$= (\frac{1}{4})^n u[n] \cdot \left(\frac{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}}{2}\right)$

$= \frac{1}{2}(\frac{1}{4})^n u[n] e^{j\frac{\pi}{2}n} + \frac{1}{2}(\frac{1}{4})^n u[n] e^{-j\frac{\pi}{2}n}$

$X(e^{j\omega}) = \frac{1}{2}\left(\frac{1}{1-\frac{1}{4}e^{-j(\omega-\frac{\pi}{2})}}\right) + \frac{1}{2}\left(\frac{1}{1-\frac{1}{4}e^{-j(\omega+\frac{\pi}{2})}}\right)$ //

5.5 i) $\omega_s = 2\omega_b$

ω_s: Nyquist frequency; ω_b: largest frequency in signal

$\omega_b = 5000\pi$

$\therefore \omega_s = 10000\pi$ //

ii) $\sin^2(500\pi t) = \frac{1-\cos(1000\pi t)}{2}$

$\omega_b = 1000\pi$

$\therefore \omega_s = 2000\pi$ //

iii) $105\sin(200\pi t)\cos(400\pi t) = 105\left(\sin(600\pi t) - \sin(200\pi t)\right)$

$\omega_b = 600\pi$, $\omega_s = 1200\pi$ //

5.6. i) $\alpha_k = \frac{1}{5}\sum_{n=0}^{4} x[n] e^{-j\frac{2\pi}{5}kn} = \dots$

$$\alpha_k = \begin{cases} 0 & k=0 \\ 15.47 & k=1,4 \\ 40.54 & k=2,3 \end{cases}$$

ii) $x[n] = \sum_{k=0}^{4} \alpha_k e^{j\frac{2\pi}{5}kn} = \dots =$ sum of complex expon.

$= \dots =$ sum of cosine functions $= \dots$

$x[n] = a + b\cos(2\pi f_1 n) + c\cos(2\pi f_2 n)$ $\quad f_1 = 0.2$

$a=0$, $b = 2\alpha_1 = 30.46$, $c = 2\alpha_2 = 81.04$, ~~f=0.4~~, $f_2 = 0.4$

iii) $x(t) = A\cos(11000.4t) + B\cos(20000.2t)$

Q: ⊘ how 11000.4n to 0.4n ??

$T_s = 1 \Rightarrow x[n] = x(t)|_{t=nT_s} = A\cos(\underbrace{11000.4n}_{0.4n}) + B\cos(\underbrace{20000.2n}_{0.2n})$

comparing with ii)

$A = 81.04$ and $B = 30.46$

iv) $f_1 = 15000.6$, $f_2 = 4000.2$, $f_3 = 7000$, observe above $f^* = 11000.4$

$f^* = f_1 - f_2 \Rightarrow \cos(f^* t) = \cos((f_1 - f_2)t) = \cos(f_1 t)\cos(f_2 t) + \sin(f_1 t)\sin(f_2 t)$ //

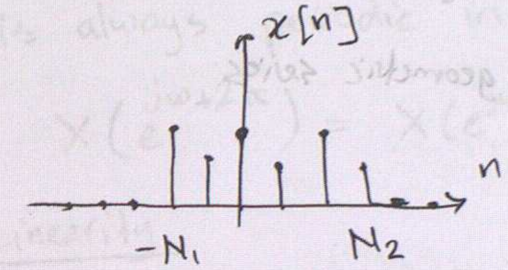
⊘ write

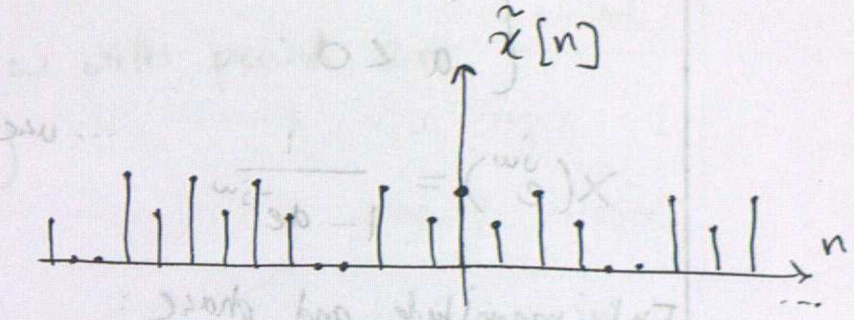
Week 6: Lecture 1 Feb 12, 2025

Discrete Time Fourier Transform

Consider discrete aperiodic signal:

$$x[n] = \begin{cases} \text{non zero} & -N_1 \le n \le N_2 \\ 0 & \text{otherwise} \end{cases}$$





construct a periodic signal $\tilde{x}[n]$ for which $x[n]$ is a period for

Write DTFS: how about no?

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega \quad \text{— synthesis}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad \text{— decomposition}$$

☆ write

$$x[n] \underset{\text{IFT}}{\overset{\text{FT}}{\rightleftarrows}} X(e^{j\omega})$$

non-periodic $\qquad$ periodic

Ex 1: $x[n] = a^n u[n]$ $|a|<1$, Find DTFS

Note:

$$\text{is} \begin{cases} |a|<1 \\ |a|>1 \\ a<0 \end{cases} \longrightarrow x[n] = \begin{cases} \text{decays} \\ \text{explodes} \\ \text{oscillates} \end{cases}$$

$$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$ ... use geometric series

* Find magnitude and phase:

Consider $1-ae^{-j\omega}$ and knowing $e^{-j\omega} = \cos\omega - j\sin\omega$

$$\Rightarrow 1-ae^{-j\omega} = 1-a(\cos\omega - j\sin\omega)$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{(1-a\cos\omega)^2 + (a^2\sin^2\omega)}} = \frac{1}{\sqrt{1-2a\cos\omega + a^2}}$$

$$\angle X(e^{j\omega}) = \tan^{-1}\left(\frac{Im}{Re}\right) = \tan^{-1}\left(\frac{a\sin\omega}{1-a\cos\omega}\right)$$

☆ write

Week 6: Lecture 2 Feb 13, 2025

Properties of DTFT

- provides further insight into transform
- reduces complexity

Periodicity

- is always periodic in ω with period 2π

$$X(e^{j\omega+2\pi}) = X(e^{j\omega})$$

Linearity

If $x[n] \longleftrightarrow X(e^{j\omega})$ and $y[n] \longleftrightarrow Y(e^{j\omega})$

Time Shifting

$$x[n-n_0] \longleftrightarrow X(e^{j\omega})e^{-j\omega n_0}$$

Frequency Shifting

$$x[n]e^{j\omega_0 n} \longleftrightarrow X(e^{j(\omega-\omega_0)})$$

Conjugation

$$x^*[n] \longleftrightarrow X^*(e^{-j\omega})$$

~~Differentation~~ **Differencing**

$$x[n]-x[n-1] \longleftrightarrow (1-e^{-j\omega})X(e^{j\omega})$$

Parseval's

Parseval's Relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 \, d\omega$$

Convolution

If input $x[n] \longleftrightarrow X(e^{j\omega})$

Impulse Response $h[n] \longleftrightarrow H(e^{j\omega})$

Output $y[n] \longleftrightarrow Y(e^{j\omega})$

$$y[n] = x[n] * h[n] \longleftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Ex 1: Find DTFT for $x[n] = \delta[n] + 2\delta[n-1]$

Knowing $\mathcal{F}\{\delta[n]\} = X(e^{j\omega}) = 1$

Time shifting: $x[n-n_0] = e^{-j\omega n_0} X(e^{j\omega})$

$$\mathcal{F}\{\delta[n-1]\} = e^{-j\omega}$$

from linearity: $\underline{\underline{X(e^{j\omega}) = 1 + 2e^{-j\omega}}}$

Ex 2: Find DTFT for $x[n] = (\delta[n] + \delta[n-1]) * (a^n u[n])$

Consider $y[n] = \delta[n] + \delta[n-1]$ $\qquad 0 < a < 1$

take FT of each: $1 \qquad e^{-j\omega} \implies Y(e^{j\omega}) = 1 + e^{-j\omega}$

for $h[n] = a^n u[n] \longleftrightarrow H(e^{j\omega}) = \dfrac{1}{1-ae^{-j\omega}}$ for $|a| < 1$

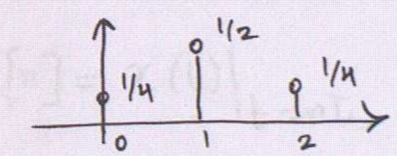
Using Convolution: $X(e^{j\omega}) = Y(e^{j\omega})H(e^{j\omega})$

$$\underline{\underline{X(e^{j\omega}) = \frac{1+e^{-j\omega}}{1-ae^{-j\omega}} \text{ for } |a|<1}}$$

Ex 3: Sum of impulses

$$x[n] = \frac{1}{4}\delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2]$$

plot



$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2]\right)e^{-j\omega n}$$

$$= \frac{1}{4}e^{-j\omega(0)} + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j\omega(2)}$$

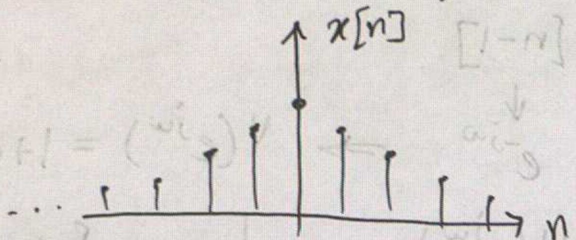
$$= \frac{1}{4} + \frac{1}{2}e^{j\omega} + \frac{1}{4}e^{-j2\omega}$$

$$= \frac{1}{4}e^{-j\omega}\left(e^{j\omega} + 2 + e^{-j\omega}\right)$$

$$X(e^{j\omega}) = \frac{1}{4}e^{-j\omega}(2 + 2\cos\omega)$$

Ex 8.4: DTFT of $x[n] = a^{|n|}$ $\quad |a| < 1$

plot



$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

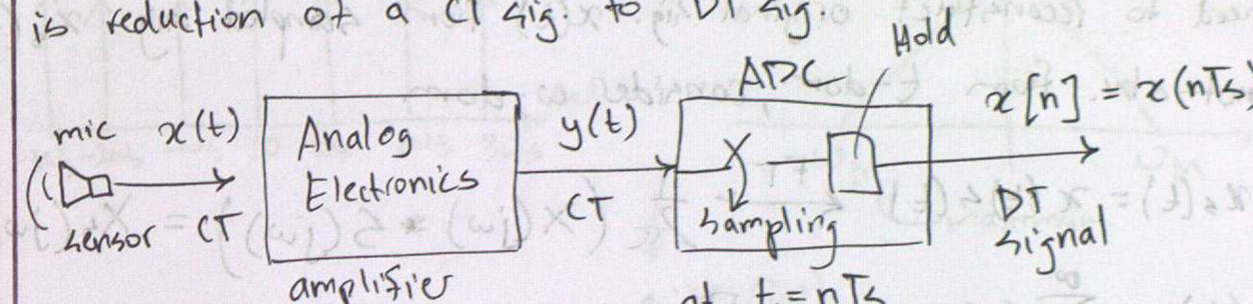
$$= \sum_{n=-\infty}^{-1} (a e^{j\omega})^{-n} + \sum_{n=0}^{\infty} (a e^{-j\omega})^{n} \quad \text{let } m = -n$$

☆ write

Week 6: Lecture 3 Feb 14, 2025

Sampling

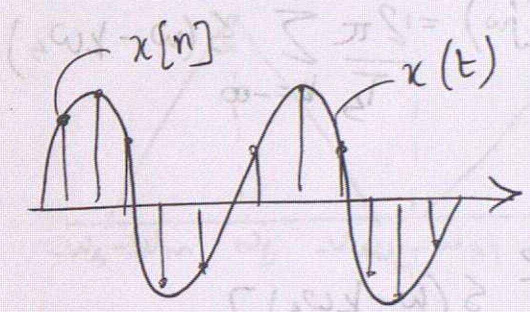
- is the process of converting a CT signal into a DT signal by taking periodic measurements
- is reduction of a CT sig. to DT sig.



at $t = nT_s$

$F_s = \frac{1}{T_s}$

↑ sampling rate/frequency



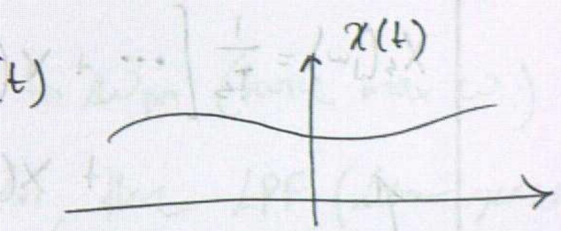
$$x[n] = x(t)\big|_{t = nT_s} = x(nT_s)$$

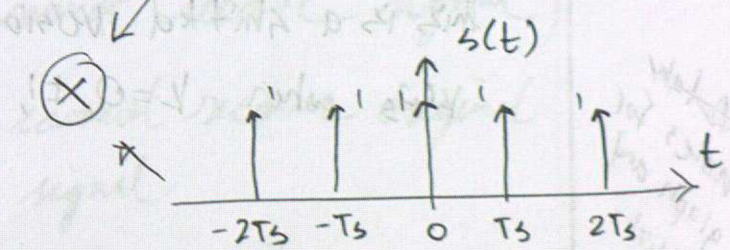
- you multiply the input $x(t)$ with an infinite shifted dirac delta to "grab" the $x(t)$ values every T_s time.

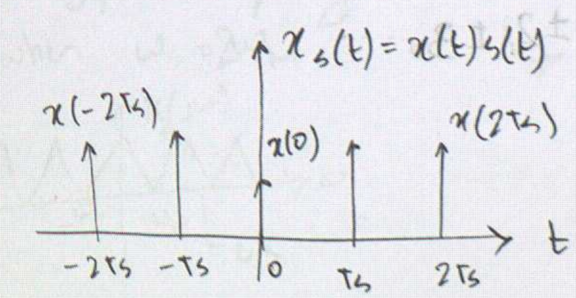
Math Model

$$x(t) \longrightarrow \otimes \longrightarrow x_s(t) = x(t)s(t)$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$







$$x_s(t) = x(t)s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s)$$

Reconstruction

- need to reconstruct original sig. $x(t)$ from sampled sig. $x_s(t)$
- not obv. from t-dom, consider ω-dom

$$x_s(t) = x(t)s(t) \overset{FT}{\longleftrightarrow} \frac{1}{2\pi}\left(X(j\omega) * S(j\omega)\right) = X_s(j\omega)$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s) \overset{FT}{\longleftrightarrow} S(j\omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

(periodic)

Can write

$$X_s(j\omega) = \frac{1}{\cancel{2\pi}}\left[X(j\omega) * \frac{\cancel{2\pi}}{T_s}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)\right]$$

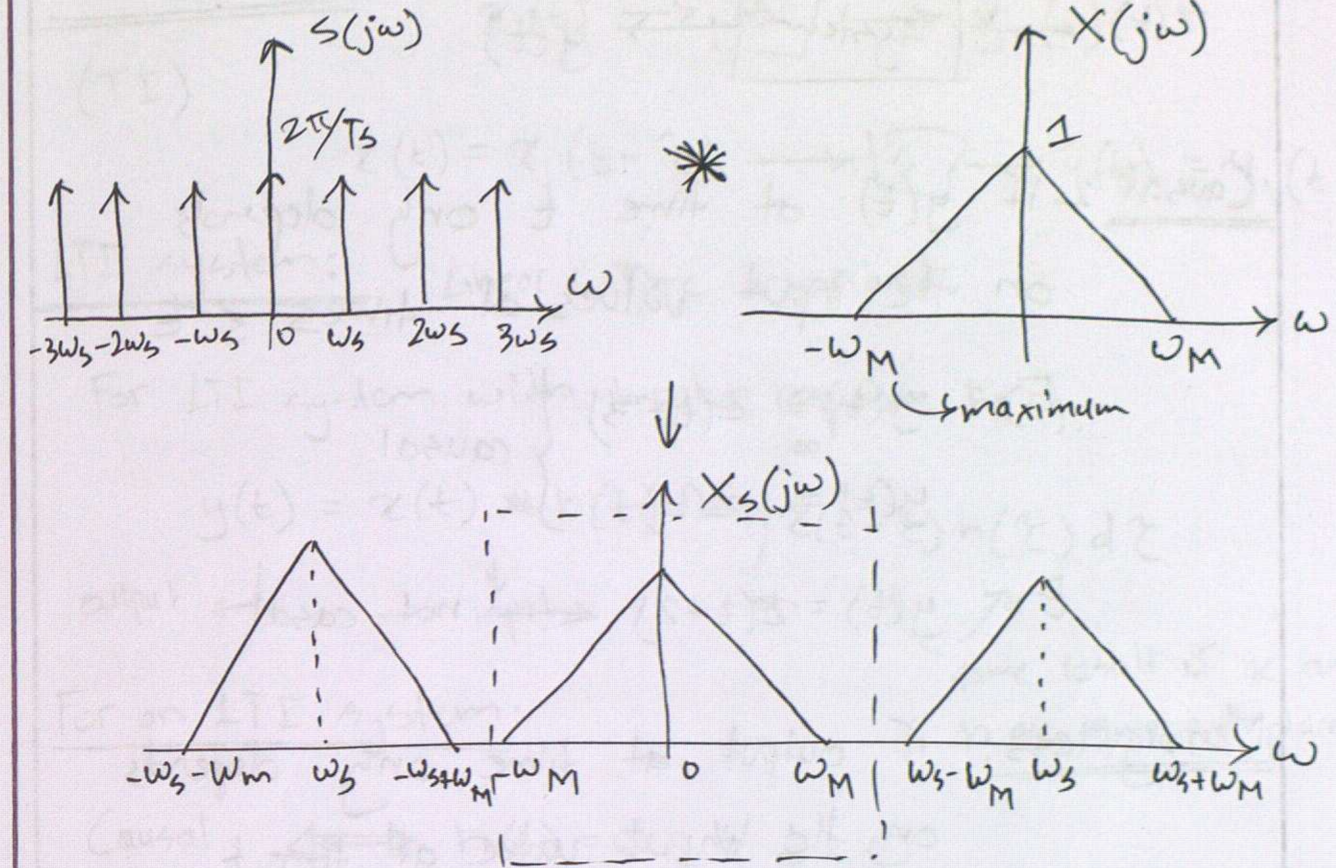
$$= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_s(j\omega) = \frac{1}{T_s}\Big[\cdots + X(j(\omega+2\omega_s)) + X(j(\omega+1\omega_s)) + X(j\omega) + X(j(\omega-\omega_s)) + X(j(\omega-2\omega_s)) + \cdots\Big]$$

this is a shifted version of $X(j\omega)$ centred about $k\omega_s$ when $k = 0, \pm1, \pm2, \pm3, \ldots$

~~Make~~ notes for graph and Nyquist frequency

Graphical Representation



applying a low pass filter, can recover signal

Note: $\omega_s = \frac{2\pi}{T_s}$ and if $T_s\uparrow$ $\omega_s\downarrow$

Q: How far can ω_s decrease

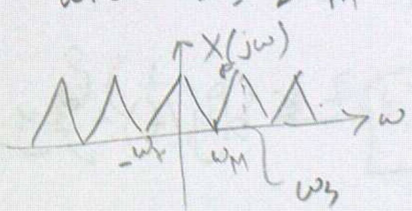
A: ω_s must be greater than or equal to $2\omega_M$ (twice max ω)

$\boxed{\omega_s \geq 2\omega_M}$ → if $\omega_s < 2\omega_M$, then LPF (low pass filter cannot recover signal)

↳ cannot recover original signal

Nyquist Frequency

when $\omega_s = 2\omega_M$



Week 7: Tutorial 6 Feb 24, 2025

$$x(t) \longrightarrow \boxed{\text{system}} \longrightarrow y(t)$$

Causal: if $y(t)$ at time t only depends on the input values at times $\leq t$

Ex: $\left.\begin{array}{l} y(t) = x(t-3) \\ y(t) = x(t) \end{array}\right\}$ causal

Ex: $y(t) = x(t+2) \longleftarrow$ not casal

Memory less: if output at time only depends on the input value at time t

Ex: $\left.\begin{array}{l} y(t) = (x(t))^3 \\ y(t) = x(t) \end{array}\right\}$ memory-less

Ex: $y(t) = x(t-2) \longleftarrow$ not memory-less

Linear:

$$x_1(t) \longrightarrow \boxed{S} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{S} \longrightarrow y_2(t)$$

$$\alpha x_1(t) + \beta x_2(t) \longrightarrow \boxed{S} \longrightarrow \alpha y_1(t) + \beta y_2(t)$$

Time invariant (TI):

$$x_1(t) \longrightarrow \boxed{S} \longrightarrow y_1(t)$$

$$x(t) = x_1(t-\tau) \longrightarrow \boxed{S} \longrightarrow y(t) = y_1(t-\tau)$$

LTI system: Linear + Time invariant

For LTI system with impulse response $h(t)$:

$$\underbrace{y(t)}_{\text{output}} = \underbrace{x(t)}_{\text{input}} * h(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

same result if x and h are swapped places

For an LTI system:

Causal $\Longleftrightarrow h(t) = 0, \ \forall\, t < 0$

$$\int_{-\infty}^{+\infty} |h(t)|\, dt < +\infty \Longrightarrow \text{BIBO}$$ "bounded input bounded output"

Some Properties

$$\int_{-\infty}^{\infty} \delta(\tau - a) f(\tau)\, d\tau = f(a)$$

$$\delta(\tau - a) = \delta(a - \tau)$$

$$\delta(t-a) * f(t) = f(t-a)$$

Q1. i) $y(t) = \int_{t-3}^{t} x(\tau)\,d\tau$ ← causal since $y(t)$ depends only of $t < \tau < t-3$

- not memoryless since $y(t)$ depends on
- linearity:
$$\begin{cases} y_1(t) = \int_{t-3}^{t} x_1(\tau)\,d\tau \\ y_2(t) = \int_{t-3}^{t} x_2(\tau)\,d\tau \end{cases} \leftarrow \text{I}$$

$x(t) = \alpha x_1(t) + \beta x_2(t)$

$$y(t) = \int_{t-3}^{t} (\alpha x_1(\tau) + \beta x_2(\tau))\,d\tau$$

$$= \alpha \int_{t-3}^{t} x_1(\tau)\,d\tau + \beta \int_{t-3}^{t} x_2(\tau)\,d\tau$$

using I

$$= \alpha y_1(t) + \beta y_2(t)$$

∴ system is linear

- time invariance

$x_1(t) \longrightarrow y_1(t)$

$x(t) = x_1(t - t_0) \longrightarrow ?$

$$y(t) = \int_{t-3}^{t} x(\tau)\,d\tau = \int_{t-3}^{t} x_1(\tau - t_0)\,d\tau$$ ← change of variable

$u \triangleq \tau - t_0 \implies y(t) = \int_{t-t_0-3}^{t-t_0} x_1(u)\,du = y_1(t - t_0)$

∴ system is time invariant.

ii) $y(t) = \int_{t-3}^{t+1} x(\tau)\,d\tau$ ← not causal ($y(t)$ depends on $t+1$)
not memoryless

- linear and time invariant (same as i) except for upper bound)

iii) $y(t) = \cos(3t)\,x(t)$ ← both causal and memoryless

- also linear
$$\begin{cases} y_1(t) = \cos(3t)\,x_1(t) \\ y_2(t) = \cos(3t)\,x_2(t) \end{cases}$$

∴ Linear

$$y(t) = \cos(3t)(\alpha x_1(t) + \beta x_2(t))$$

$$= \alpha\cos(3t)x_1(t) + \beta\cos(3t)x_2(t) = \alpha y_1(t) + \beta y_2(t)$$

- time invariance: $y_1(t) = \cos(3t)x_1(t)$, $x(t) = x_1(t - t_0)$

$y(t) = \cos(3t)\cdot x_1(t - t_0)$

but $y_1(t - t_0) = \cos(3(t - t_0))\,x_1(t - t_0)$

not equal, so this is not time-invariant since $y(t) \neq y_1(t - t_0)$

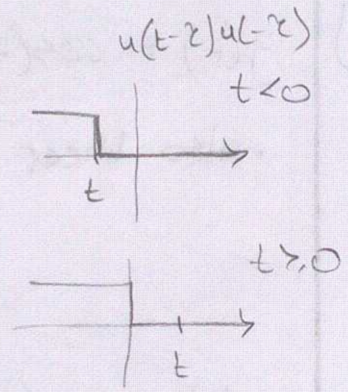
Q.2 i) $h(t) = \delta(t) - 3e^{-3t}u(t)$ and $x(t) = u(-t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,d\tau$$

$$= \int_{-\infty}^{+\infty} \left[\delta(t-\tau) - 3e^{-3(t-\tau)}u(t-\tau)\right]\left[u(-\tau)\right]d\tau$$

$$= \int_{-\infty}^{\infty} \delta(t-\tau)u(-\tau)\,d\tau - 3\int_{-\infty}^{\infty} e^{-3(t-\tau)}u(t-\tau)u(-\tau)\,d\tau$$

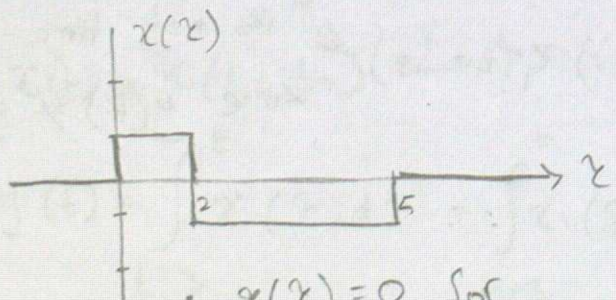
$$= u(-t) - 3\int_{-\infty}^{\infty} e^{-3(t-\tau)}u(t-\tau)u(-\tau)\,d\tau$$



$$= \begin{cases} u(-t) - 3\int_{-\infty}^{t} e^{-3(t-\tau)}\,d\tau, & t \leq 0 \\ u(-t) - 3\int_{-\infty}^{0} e^{-3(t-\tau)}\,d\tau, & t \geq 0 \end{cases}$$

Q.3 i)

$$\left.\begin{aligned} x(t) &= u(t) - 2u(t-2) + u(t-5) \\ h(t) &= e^{2t}u(1-t) \end{aligned}\right\} y(t) = h(t) * x(t)$$

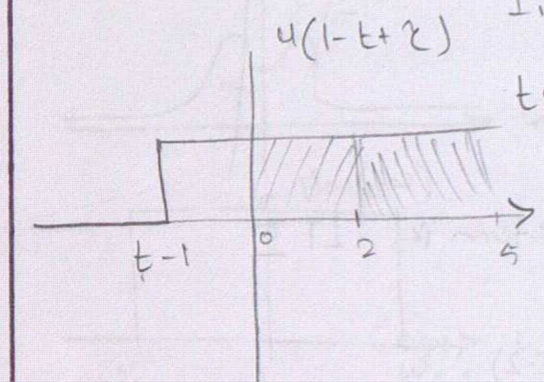


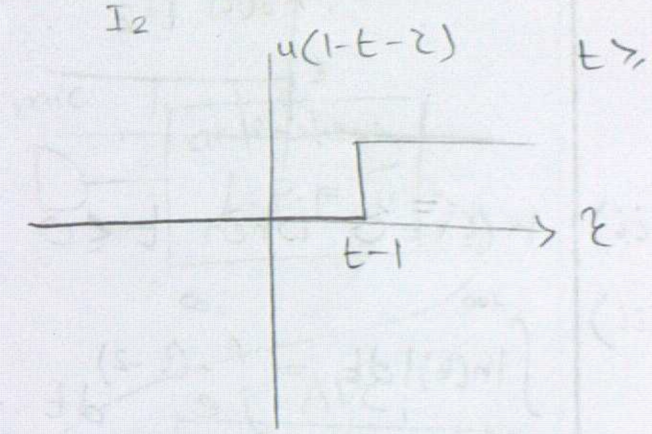
$\therefore x(\tau) = 0$ for $\tau < 0$ and $\tau > 5$

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,d\tau$$

$$= \int_{0}^{2} h(t-\tau)\,d\tau - \int_{2}^{5} h(t-\tau)\,d\tau$$

$$= \underbrace{\int_{0}^{2} e^{2(t-\tau)}u(1-t+\tau)\,d\tau}_{I_1} - \underbrace{\int_{2}^{5} e^{2(t-\tau)}u(1-t+\tau)\,d\tau}_{I_2}$$



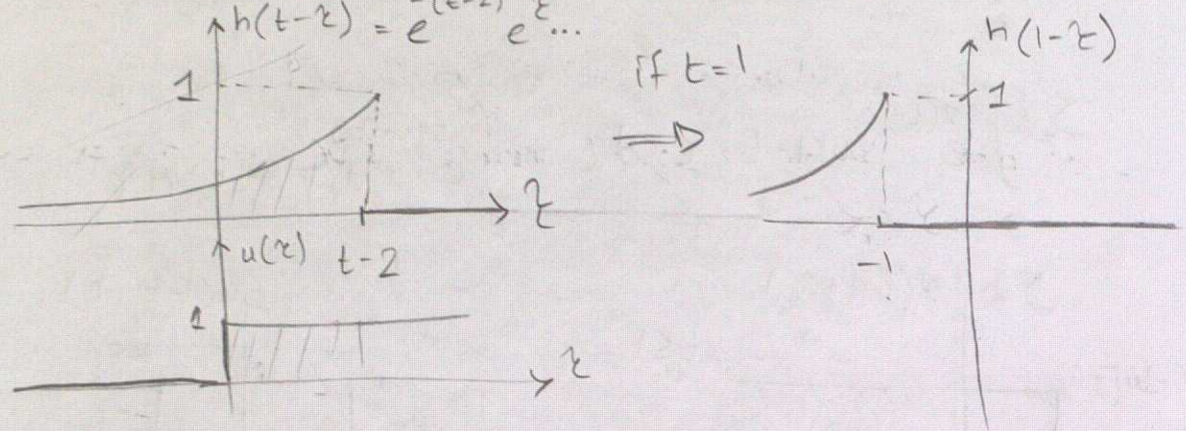


$$I_1 = \begin{cases} \int_{0}^{2} e^{2(t-\tau)}\,d\tau, & t \leq 1 \\ \int_{t-1}^{2} e^{2(t-\tau)}\,d\tau, & 1 \leq t \leq 3 \\ 0, & t \geq 3 \end{cases}$$

$$I_2 = \begin{cases} -\int_{2}^{5} e^{2(t-\tau)}\,d\tau, & t \leq 3 \\ -\int_{t-1}^{5} e^{2(t-\tau)}\,d\tau, & 3 \leq t \leq 6 \\ 0, & t \geq 6 \end{cases}$$

Q.4 $h(t) = e^{-(t-2)} u(t-2)$

i) $h(t-\tau)$ versus τ $\Rightarrow h(t-\tau) = e^{-(t-\tau-2)} u(t-\tau-2)$, for $t=1$



ii) $h(t)=0$ when $t \leq 0$ $\Rightarrow$ system is LTI

iii) $\int_{-\infty}^{+\infty} |h(t)| dt = \int_{2}^{+\infty} e^{-(t-2)} dt = \left(\frac{e^{-(t-2)}}{(-1)} \right)_{2}^{+\infty} = 1 < +\infty$

$\therefore$ system BIBO

iv) $x(t) = u(t)$, $y(t) = 0$ for all $t \leq t_1$, $\underline{t_1 = ?}$

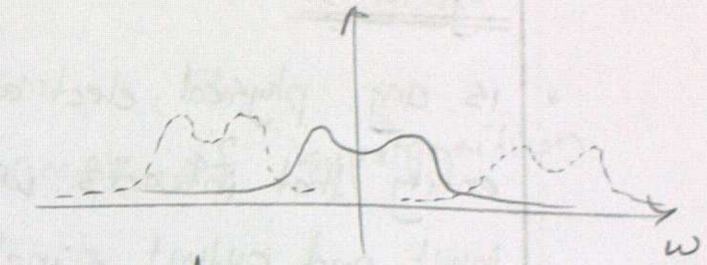
$y(t) = x(t) * h(t) = u(t) * h(t) = \int_{-\infty}^{+\infty} h(t-\tau) u(\tau) d\tau$

$$y(t) \begin{cases} 0, & t \leq 2 \\ \underbrace{\int_0^{t-2} e^{-(t-2)} e^{\tau} d\tau}_{1-e^{-(t-2)}}, & t \geq 2 \end{cases}$$

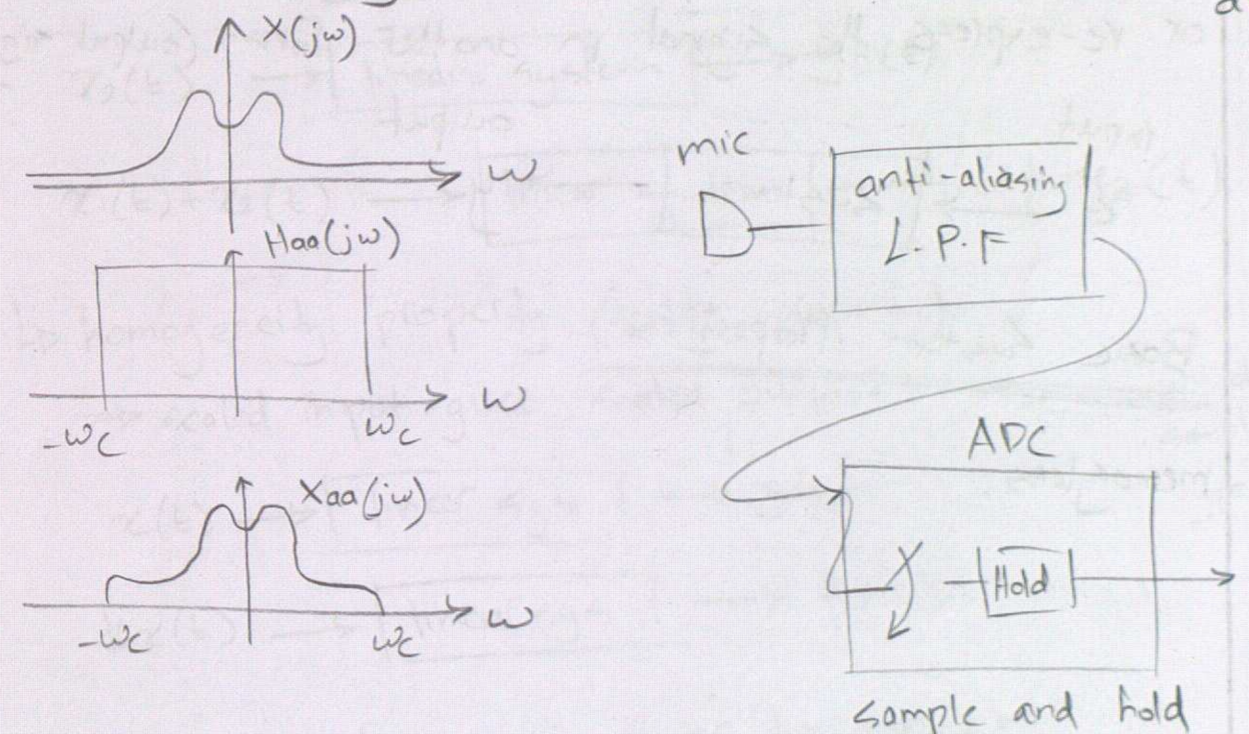
$\leftarrow$ output y is zero for all $t \leq 2$, $\therefore$ $\underline{t_1 = 2}$

Week 7: Lecture 1 Feb 26, 2025

Non-Band limited Signal



* aliasing always occurs
* all practical signals are non-band limited
* choose ω_s s.t. aliasing error is minimized
* place an anti-aliasing (L.P.F) before ADC to avoid excessive aliasing



Band limited Signal

Systems

* is any physical, electrical, mechanical, or mathematical entity that interacts with its environment through input and output signals.

* operates on a signal (input) to modify, manipulate, or re-express the signal in another form (output sig)

input $\qquad\qquad\qquad\qquad$ output

$$x \longrightarrow \boxed{\text{System}} \longrightarrow y$$

Basic System Properties

* memoryless:

Linearity

* linear systems
  - ↳ a system that satisfies principle of superposition
  - ↳ additivity property

$$x_1(t) \longrightarrow \boxed{\text{Linear system}} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{\text{linear system}} \longrightarrow y_2(t)$$

$$x_1(t) + x_2(t) \longrightarrow \boxed{\text{linear system}} \longrightarrow y_1(t) + y_2(t)$$

  - ↳ homogeneity property (scaling property)
    - ⇒ scaled input gives scaled output

$$x(t) \longrightarrow \boxed{\text{linear sys}} \longrightarrow y(t)$$

$$kx(t) \longrightarrow \boxed{\text{linear sys}} \longrightarrow ky(t)$$

· cont, discont
· sampling
· type of sys

* Note: linear systems are easier to analyze
  - ↳ can use tools like convolution, FT, Laplace transform

Ex: Is $y(t) = 3x(t)$ linear?

Let $y_1(t) = 3x_1(t)$ and $y_2(t) = 3x_2(t)$

$$x_1(t) \longrightarrow \boxed{3 \times \text{input}} \longrightarrow y_1(t) = 3x_1(t)$$

$x_1(t) + x_2(t) \longrightarrow \boxed{\text{should give}} \longrightarrow 3x_1(t) + 3x_2(t)$

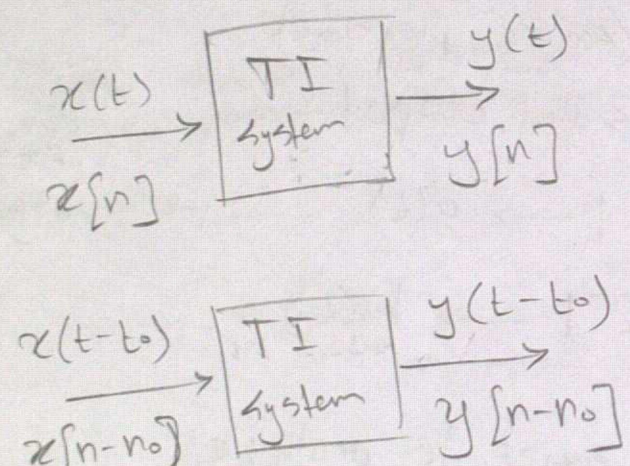
$\Rightarrow$ Additivity $\quad O(x_1(t) + x_2(t)) = 3(x_1(t) + x_2(t))$

$$= 3x_1(t) + 3x_2(t)$$

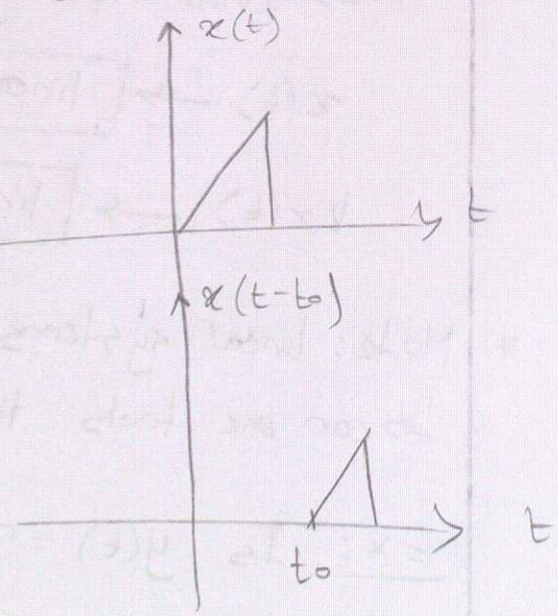
$y_1(t) + y_2(t) = 3x_1(t) + 3x_2(t)$ ← satisfies

Time Invariant System

* is when a time shift in input signal causes the same time shift in output signal



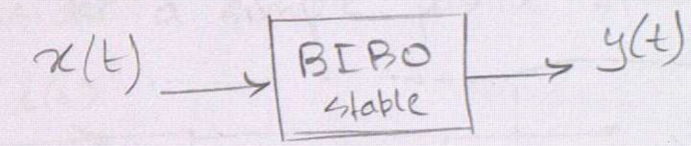
* system does not change over time



Stability

* bounded input: magnitude of signal never exceeds a finite value called "M"
  ↳ eg: $|x(t)| \le M$ for $-\infty < t < \infty$
* stable system: guarantees a bounded output for any bounded input signal

$\Rightarrow$ BIBO stable system

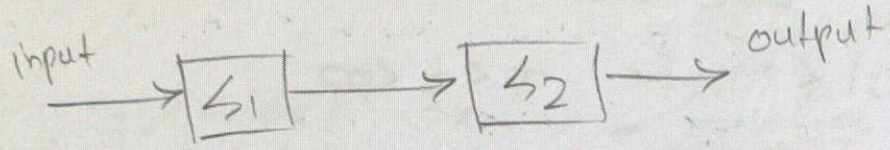


↳ if $|x(t)| \le M_x$, $-\infty < t < \infty$ then $|y(t)| \le M_y$, $-\infty < t < \infty$

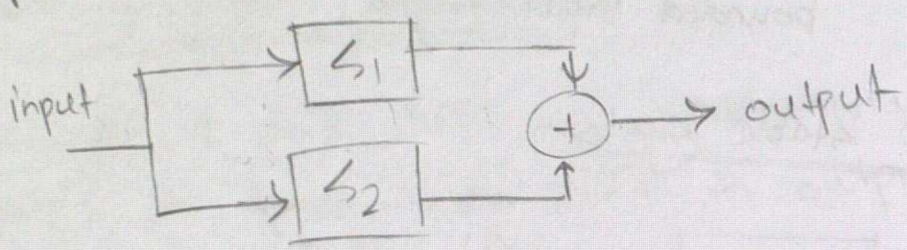
* a system is BIBO stable if a bounded input results in a bounded output.

Interconnection of Systems

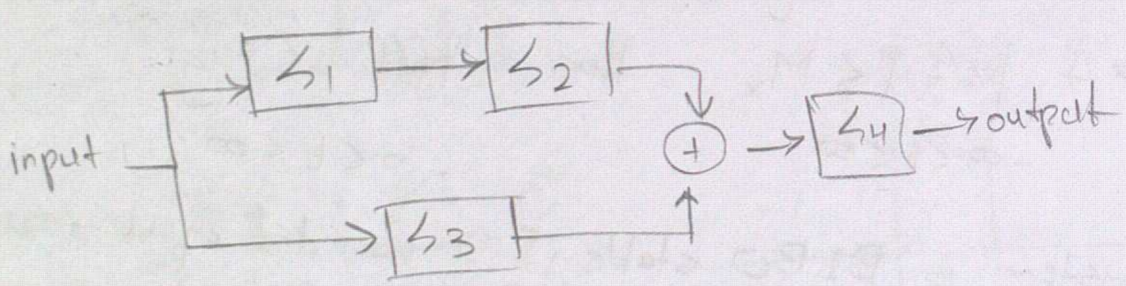
* series (cascade) interconnection



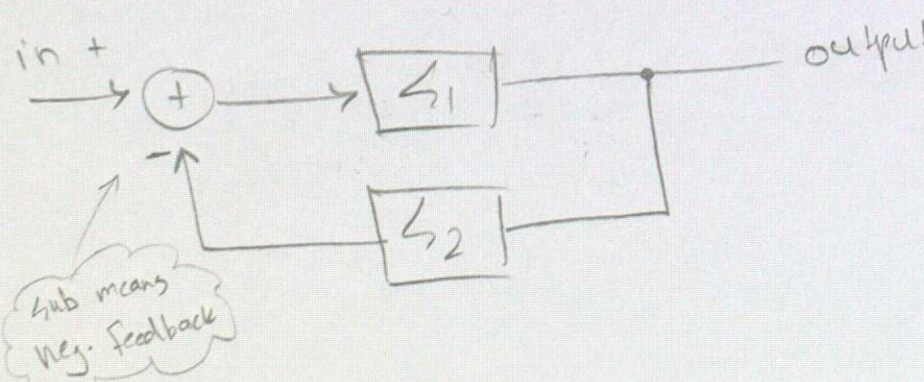
* parallel interconnection :



* combination :



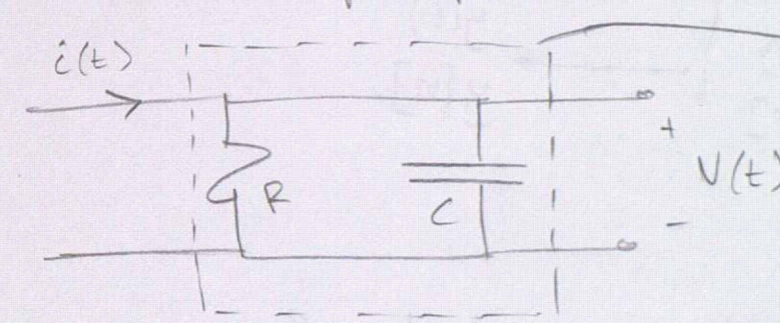
* feedback interconnection: output of one system is fed back into input eighter added or subtracted



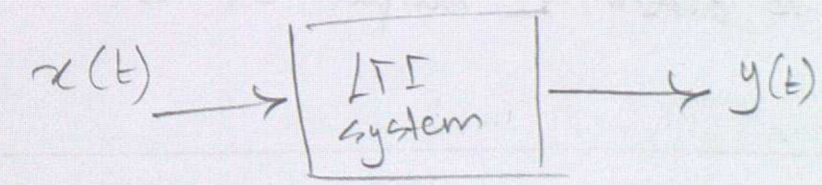
Week 7: Lecture 3 Feb 28, 2025

Linear Time Invariant (LTI) Systems

* most systems are linear and time invariant (LTI)
* this allows to use powerful tools to solve mathmatecal models
* since linear, can apply superposition
  - ↳ break signal into simpler ones and solve
* what is a mathematical model ?
  - ↳ nature filled with derivatives (eg: $\frac{dT}{dt}$ ← temperature, $i_c(t) = C\frac{dV_c(t)}{dt}$)
* Consider a simple passive filter (L.P.F.)



→ represent by a mathmatical model (LTI system)



* Solve: $i_R(t) = \frac{v(t)}{R}$, $i_c(t) = C\frac{dv(t)}{dt}$

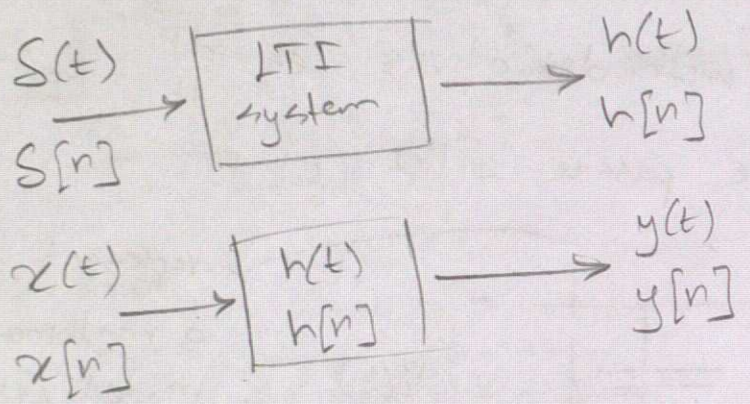
$$\underset{x(t)}{i(t)} = i_R(t) + i_c(t) = \frac{v(t)}{R} + C\frac{dv(t)}{dt} \quad \leftarrow y(t)$$

$$\Longrightarrow x(t) = \frac{y(t)}{R} + C\frac{d}{dt}y(t)$$

* how to solve $\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{C}x(t)$

* can solve output of LTI system for any arbitrary input knowing system response to impulse (delta function) $\delta(t)$ or $\delta[n]$

* this is called <u>impulse response</u> ($h(t)$ or $h[n]$)



Convolution

* provides tool to determine output of an LTI system

$$\frac{d}{dt}y(t) + \frac{1}{RC}y(t) = \frac{1}{C}x(t)$$

$$e^{\int \frac{1}{RC}dt} = e^{\frac{t}{RC}} \quad \text{integrating factor}$$

$$e^{t/RC}\frac{dy}{dt} + \frac{1}{RC}ye^{t/RC} = \frac{1}{C}x(t)e^{t/RC}$$

$$\frac{d}{dt}\left(e^{t/RC}y\right) = \frac{1}{C}x(t)e^{t/RC}$$

$$e^{t/RC}y = \int \frac{1}{C}x(t)e^{t/RC}dt' + K$$

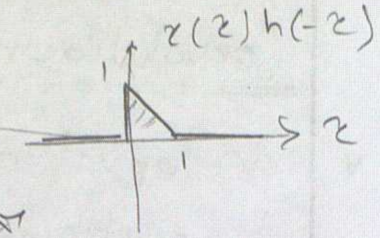
Convolutioon

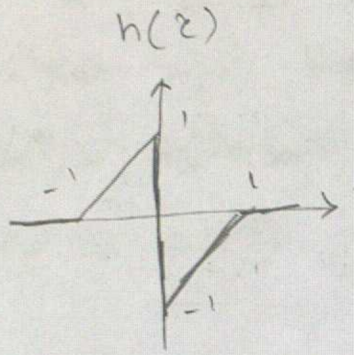
* an operation that describes how 2 sigs interact by combining them into a 3rd sig

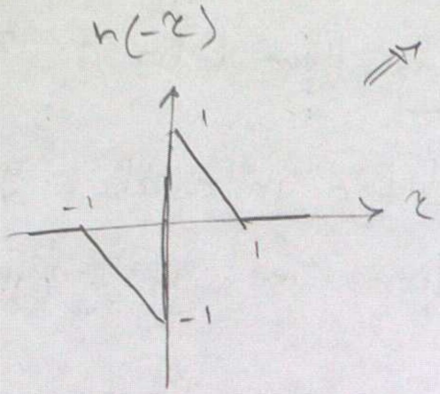
* CT convolution integral: $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

* DT convolution sum: $y[n] = x[n] * h[n] =$

Q.5: i) $y(0) = \int x(\tau)h(-\tau)d\tau = \frac{1}{2}$ area $= \frac{1}{2}$



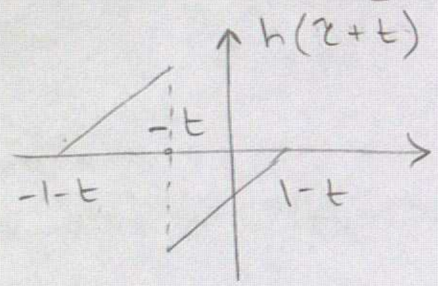




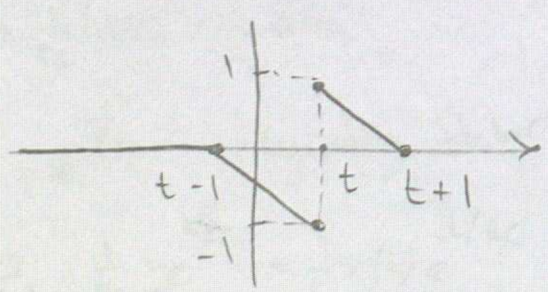
ii) We want to plot $x(\tau)\cdot h(t-\tau)$

We first need to sketch $h(t-\tau) = h(-\tau+t)$

1) shift to left by t (if $0 \le t \le 1$):



2) mirror image w.r.t vertical axis:

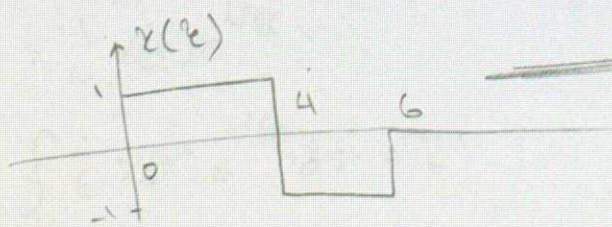


① $t+1 < -1$, $t < -1$

② $t-1 > 6$, $t > 7$

$$x(\tau)h(t-\tau) = \begin{cases} 0, & t \le -1 \quad ① \\ 0, & t \ge 7 \quad ② \end{cases}$$

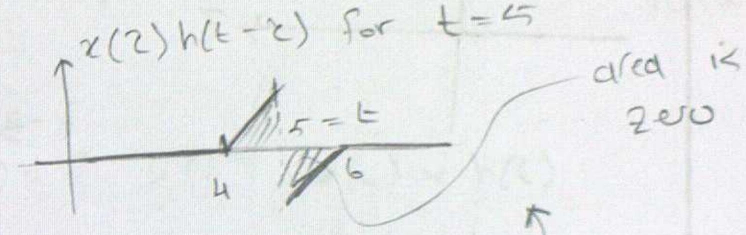
Recall



Also, $y(t) = \int x(\tau)h(t-\tau)d\tau = \{ 0, \ 1 \le t \le 3$

$t-1 \ge 0$, $t \ge 1$ ① $t+1 \le 4$, $t \le 3$ ②

$1 \le t \le 3$



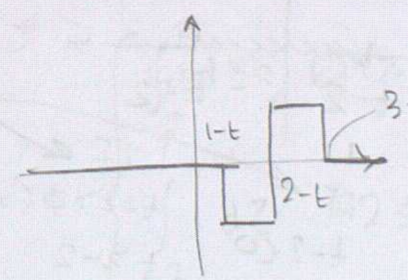
Also, $y(t) = \int x(\tau)h(t-\tau)d\tau = 0, \ t=5$

Q.6

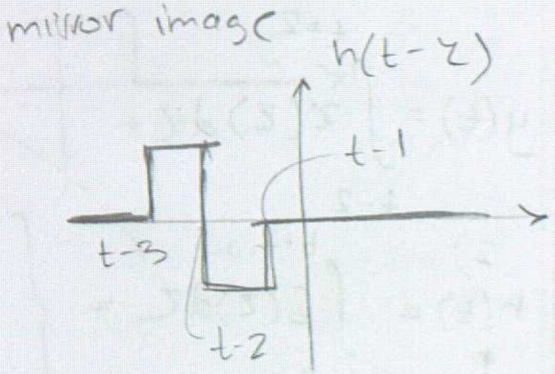
i) Is the LTI system BIBO?

$$\int_{-\infty}^{+\infty} |h(\tau)|d\tau = \int_1^2 |-1|d\tau + \int_2^3 |1|d\tau = 1+1 = \underbrace{2 < +\infty}_{\therefore \text{ BIBO}}$$

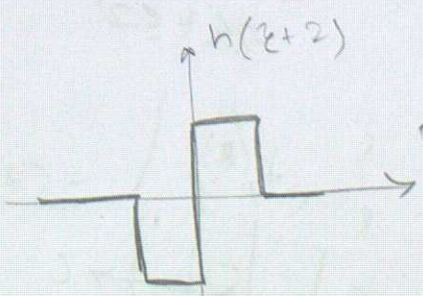
ii) shift $h(\tau)$ by t to left

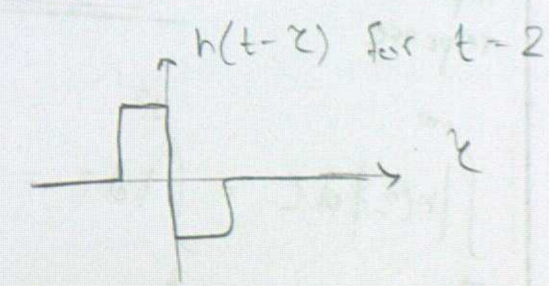


mirror image



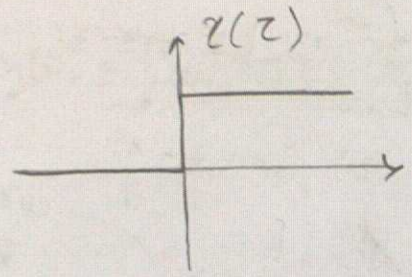
iii) if $t=2$, then $h(t-\tau)$ is

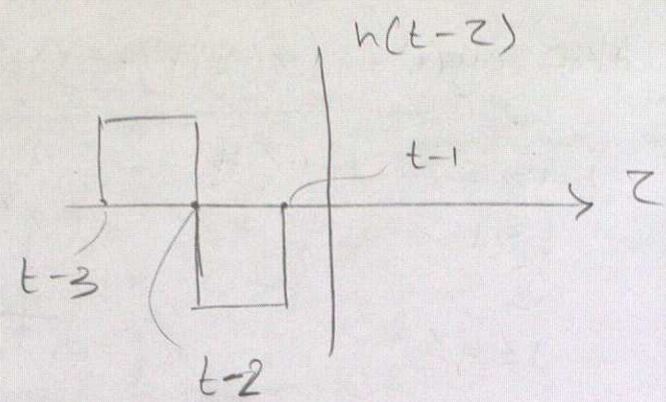




$y(2) = \int x(\tau)h(2-\tau)d\tau = \int u(\tau)h(2-\tau)d\tau = -1$

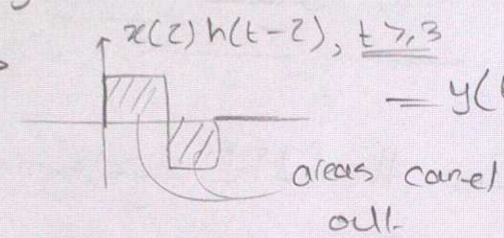
iv)





case ① : $t-1 \le 0$, no overlap, $y(t) = 0$

case ② : $t-3 \geq 0$, full overlap $\Rightarrow$



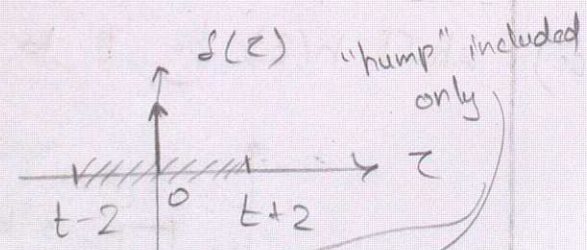
$= y(t) = 0$

$\therefore y(t) = 0 \begin{cases} t-1 \le 0 \Rightarrow t \le 1 \\ t-3 \geq 0 \Rightarrow t \geq 3 \end{cases}$

$t \le 1$

$y(t) = 0$ for $t \le T_1$ and $t \geq T_2$
for $T_1 = 1$ and $T_2 = 3$

Q.7 $\quad y(t) = \int_{t-2}^{t+2} x(\tau)\,d\tau$



i) $h(t) = \int_{t-2}^{t+2} \delta(\tau)\,d\tau = \begin{cases} 1, & -2 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$

impulse response

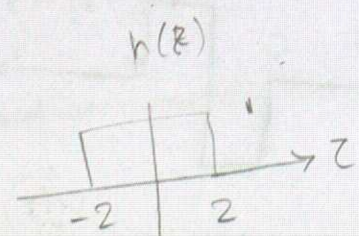
if $t-2 \le 0$, $t \le 2$

if $t+2 \geq 0$, $t \geq -2$

$-2 \le t \le 2$

ii) $\int_{-\infty}^{+\infty} |h(\tau)|\,d\tau = \int_{-2}^{+2} 1\,d\tau = \underbrace{4 \le +\infty}$

$\therefore$ BIBO

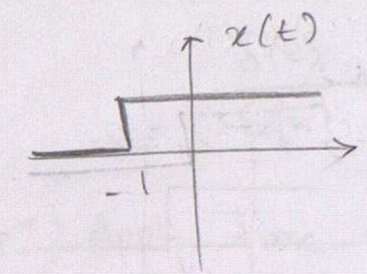


iii) Not causal since system depends on stuff at $t+2$ → future

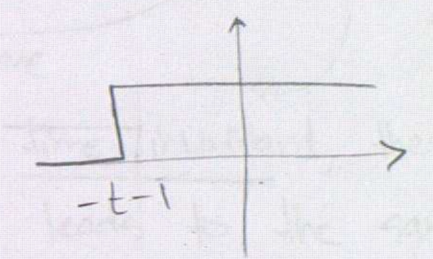
$y(t) = \int_{t-2}^{t+2} x(\tau)\,d\tau$ — $t+2$ → problem

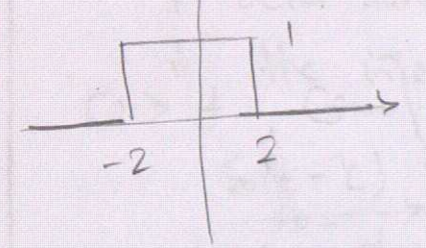
iv) $x(t) = u(t+1)$ $\quad y(t) = ?$ $\quad y(t) = x(t) * h(t)$

$= \int h(\tau)\,x(t-\tau)\,d\tau$

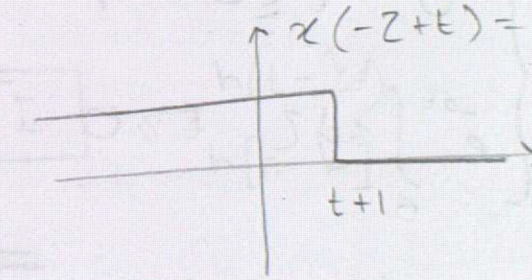


$x(\tau + t)$





$x(-\tau + t) = x(t-\tau)$



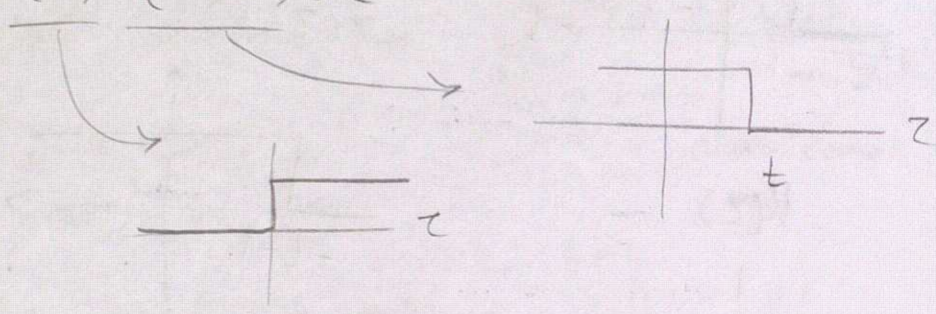
$y(t) = \begin{cases} \int_{-2}^{2} 1\,d\tau, & t+1 \geq 2 \\ \int_{-2}^{t+1} 1\,d\tau, & -2 \le t+1 \le 2 \\ 0, & t+1 \le -2 \end{cases}$

$y(t) = \begin{cases} 4, & t \geq 1 \\ t+3, & -3 \le t \le 1 \\ 0, & t \le -3 \end{cases}$

Q.8 $y(t) = \left(e^{-at}u(t)\right) * \left(e^{-at}u(t)\right)$

$$= \int_{-\infty}^{+\infty} \left(e^{-a\tau}u(\tau)\right)\cdot\left(e^{-a(t-\tau)}u(t-\tau)\right)d\tau$$

$$= e^{-at}\int_{-\infty}^{+\infty} u(\tau)\,u(t-\tau)\,d\tau$$



$$= \begin{cases} 0, & t \le 0 \\ e^{-at}\int_0^t 1\,d\tau, & t \geq 0 \end{cases} = \begin{cases} 0, & t \le 0 \\ te^{-at}, & t \geq 0 \end{cases}$$

Week 8: Lecture 1 March 5, 2025

Convolution Derivation

1) Define impulse response ($h(t)$ or $h[n]$)

↳ LTI system can be characterized completely by its impulse response to an impulse (delta) function

$$\delta(t),\ \delta[n] \longrightarrow \boxed{LTI} \longrightarrow h(t),\ h[n]$$

2) Apply time invariance

↳ Since system is time invariant, then shifting in delta function leads to the same shift in the impulse response

$$\delta(t-\tau),\ \delta[n-k] \longrightarrow \boxed{LTI} \longrightarrow h(t-\tau),\ h[n-k]$$

3) Apply scaling property of linearity

↳ Since system is linear, scaling the input by a constant α results in the output scaling by the same α

$$\alpha\delta(t-\tau),\ \alpha\delta[n-k] \longrightarrow \boxed{LTI} \longrightarrow \alpha h(t-\tau),\ \alpha h[n-k]$$

3-Q
- FT CT, DT
- properties
- sampling
- system

4) Apply input signal

↳ consider one instance of the input signal where $x(\tau)$ or $x[k]$ acts as the scaling factor

$$x(\tau)\delta(t-\tau) \longrightarrow \boxed{LTI} \longrightarrow x(\tau)h(t-\tau)$$
$$x[k]\delta[n-k] \qquad\qquad x[k]h[n-k]$$

5) Apply additivity property of linearity

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \qquad y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$x[n] \longrightarrow \boxed{LTI} \xrightarrow{y[n]} = x[n] * h[n]$$

↳ since system is linear, then the response to integral (or sum) of scaled impulse is the integral (or sum) of corresponding scaled impulse responses

Summary

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \longrightarrow \boxed{LTI} \longrightarrow y(t) = x(t) * h(t)$$
$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

Ex: Find $x(t) * h(t)$ for $x(t) = u(t)$ and $h(t) = e^{-t}u(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$
$$= \int_{-\infty}^{\infty} u(\tau)\, e^{-(t-\tau)}\, u(t-\tau)\,d\tau$$

limits $u(\tau) = \begin{cases} 1 & \tau \geq 0 \\ 0 & \tau < 0 \end{cases}$

$u(t-\tau) = \begin{cases} 1 & t-\tau \geq 0 \\ 0 & t-\tau < 0 \end{cases} \Longrightarrow \begin{matrix} \tau \leq t \\ \tau > t \end{matrix}$

the function is zero for $t < \tau < 0$

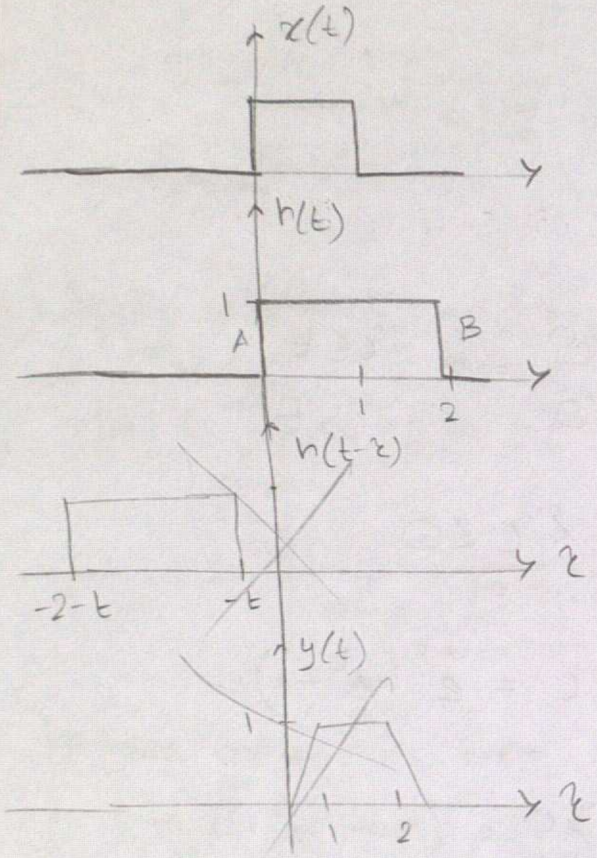
$$y(t) = \int_{0}^{t} e^{-(t-\tau)}\,d\tau = e^{-t}\int_{0}^{t} e^{\tau}\,d\tau = e^{-t}(e^{t}-1)$$

$\longrightarrow$ $y(t) = 1 - e^{-t}$ for $t \geq 0$

method 2: $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau = \int_{0}^{t} e^{-\tau}\cancel{u(\tau)}\,d\tau = 1 - e^{-t}$

Ex: Find $y(t) = x(t) * h(t)$

$$x(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} 1 & 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$



$$x(\tau) = \begin{cases} 1 & 0 \le \tau \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h(t-\tau) = \begin{cases} 1 & 0 \le t-\tau \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & t-2 \le \tau \le t \\ 0 & \text{otherwise} \end{cases}$$

$x(\tau)$ defined $0 \le \tau \le 1$

$h(t-\tau)$ defined $t-2 \le \tau \le t$

$\Rightarrow$ valid range for integration

$\max(0, t-2) \le \tau \le \min(1, t)$

- evaluate for different intervals

① $0 \le t \le 1$

$\max(0, t-2) \xrightarrow{\text{for } 0 \le t \le 1} \max(0, -2 \text{ to } -1) = 0$ ← lower limit

$\min(1, t) \xrightarrow{\text{for } 0 \le t \le 1} \min(1, 0 \text{ to } 1) = t$ ← upper limit

$$y(t) = \int_0^t 1 \times 1 \, d\tau = \underline{t}$$

② $1 \le t \le 2$

$\max(0, t-2) \xrightarrow{1 \le t \le 2} \max(0, -1 \text{ to } 0) = 0$

$\min(1, t) \xrightarrow{1 \le t \le 2} \min(1, 1 \text{ to } 2) = 1$

$$y(t) = \int_0^1 d\tau = 1$$

③ $2 \le t \le 3$

$\max(0, t-2) \xrightarrow{2 \le t \le 3} \max(0, 0 \text{ to } 1) = t-2$

$\min(1, t) \xrightarrow{2 \le t \le 3} \min(0, 2 \text{ to } 3) = 1$

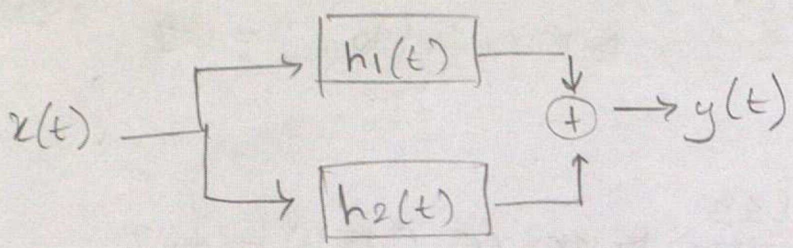
$$y(t) = \int_{t-2}^{1} d\tau = 1 - t + 2 = -t + 3$$

$$y(t) = \begin{cases} t & , \; 0 \le t \le 1 \\ 1 & , \; 1 \le t \le 2 \\ -t+3 & , \; 2 \le t \le 3 \\ 0 & , \; \text{otherwise} \end{cases}$$

Week 8: Lecture 2

LTI Systems

* parallel distributive property



$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

$$x(t) \rightarrow \boxed{h_1(t) + h_2(t)} \rightarrow y(t)$$

* idea is that maybe it is easier to split complicated convolutions into easier (or smaller ones) and then calculate them
* commutative property holds: $x(t) * h(t) = h(t) * x(t)$
* cascade (series) property (associative)

$$x(t) \rightarrow \boxed{h_1(t) * h_2(t)} \rightarrow y(t)$$

$$x(t) \rightarrow \boxed{h_2(t) * h_1(t)} \rightarrow y(t)$$

$$x(t) \rightarrow \boxed{h_2(t)} \rightarrow \boxed{h_1(t)} \rightarrow y(t)$$

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$

* can change order, maybe is easier to solve

Ex: Given $x[n] = (\frac{1}{2})^n u[n]$, $h[n] = u[n]$, find $y[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} (\tfrac{1}{2})^k u[k]\, u[n-k]$$

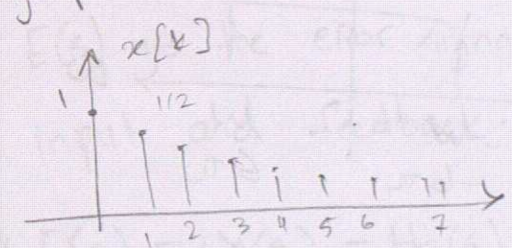
$k=0$ $n-k=0$, $k=n$

$$= \sum_{k=0}^{n} (\tfrac{1}{2})^k$$

* observe geometric series $\sum_{i=0}^{N} r^i = \frac{1-r^{N+1}}{1-r}$

$$y[n] = \frac{1-(\frac{1}{2})^{n+1}}{1-\frac{1}{2}} = 2(1-(\tfrac{1}{2})^{n+1})$$

* graphical:



Ex: sawtooth function w/ rectangular impulse response

$x(t) = t$ for $0 \le t \le 1 \Rightarrow x(\tau) = \tau$, $\boxed{0} \le \tau \le 1$

$h(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases} \Rightarrow h(t-\tau) = \begin{cases} 1 & 0 \le t-\tau \le 1 \\ 0 & \text{otherwise} \end{cases}$

rearrange: $\boxed{t-1} \le \tau \le t$

①: $0 \le t \le 1 \rightarrow \max(0, t-1)$ and $\min(1, t)$: $y(t) = \int_0^t \tau\, d\tau = \frac{1}{2}t^2$

②: $1 \le t \le 2 \rightarrow \max(0, t-1)$ and $\min(1, t)$: $y(t) = \int_{t-1}^{1} \tau\, d\tau$

$$\frac{1}{2} - \frac{(t-1)^2}{2}$$

Ex: Determine Nyquist rate:

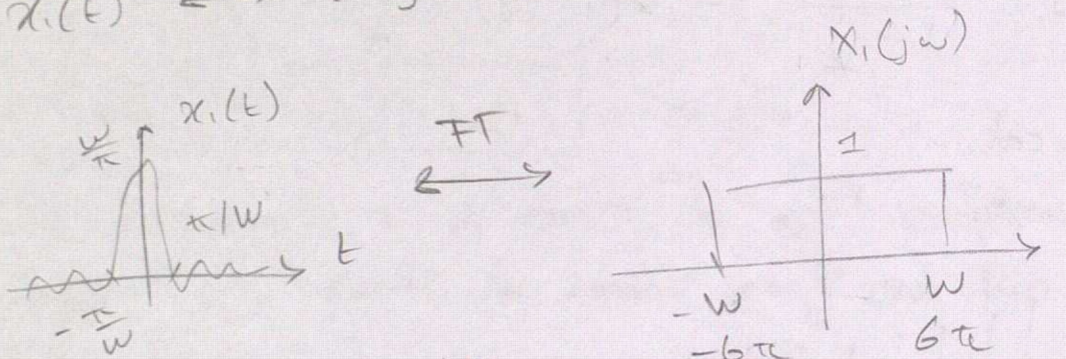
* sinc and box

$$x(t) = \left(\frac{\sin(6\pi t)}{\pi t}\right)^2$$

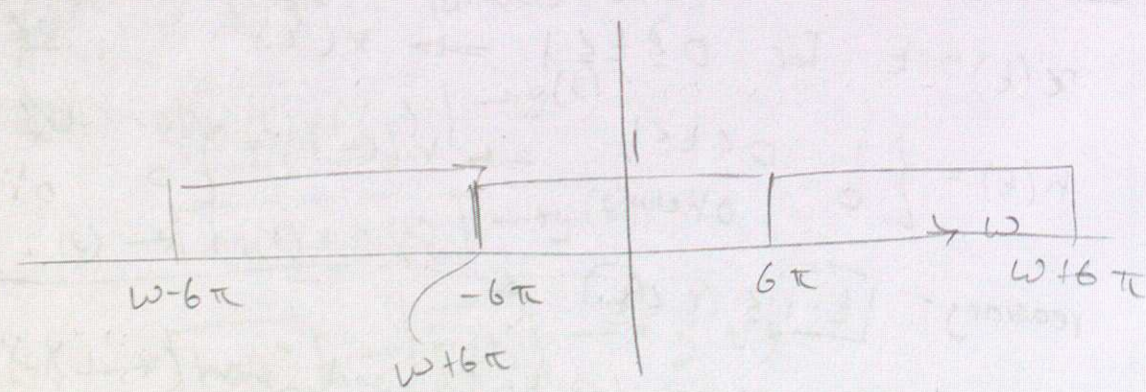
Let $x_1(t) = \frac{\sin(6\pi t)}{\pi t}$

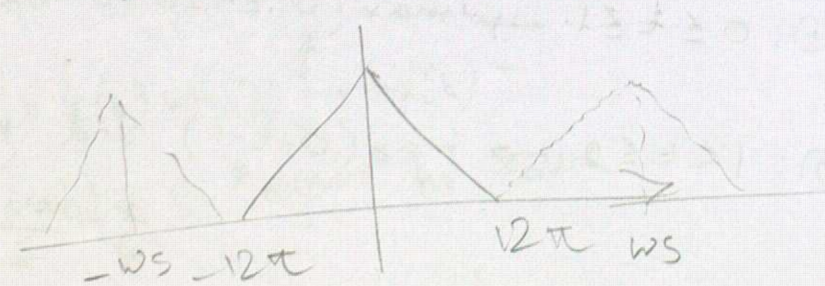
$$x(t) = x_1(t)x_1(t) = \frac{\sin(6\pi t)}{\pi t}\,\frac{\sin(6\pi t)}{\pi t}$$

$$x_1(t) \overset{FT}{\longleftrightarrow} x_1(j\omega)$$



$$x(t) = x_1(t)x_2(t) \overset{FT}{\longleftrightarrow} \frac{1}{2\pi}\left(X_1(j\omega) * X_2(j\omega)\right) = X(j\omega)$$

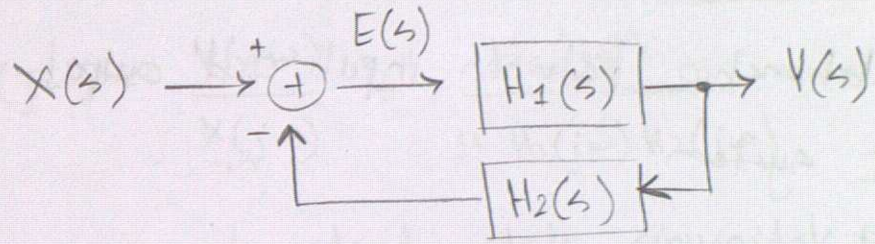




Week 8: Lecture 3 — Mar 7, 2025

Feedback Combination of LTI Systems

* LTI systems can be interconnected in a feedback config where a portion of output is fed back into input
* this modifies system's response



E(s) is the error signal that is the difference between input and feedback

$$E(s) = X(s) - H_2(s)Y(s)$$

and $Y(s) = H_1(s)E(s)$

$$Y(s) = H_1(s)\left[X(s) - H_2(s)Y(s)\right]$$

$$Y(s) = H_1(s)X(s) - H_1(s)H_2(s)Y(s)$$

$$Y(s) = \frac{X(s)H_1(s)}{1 + H_1(s)H_2(s)} \Rightarrow \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

* Note: feedback system modifies the response by denominator $1 + H_1(s)H_2(s)$ → if $H_2(s) = 0$ then no feedback: that is called open-loop system

* feedback can stabilize an unstable system or modify the gain

* clearly if feedback is <u>added</u> to input, the system becomes unstable

Transfer Function

* describes relationship between input and output of the LTI system

$$H(s) = \frac{Y(s)}{X(s)}$$

* describes system behaviour in Laplace Domain (complex frequency s)

$X(s)$ is $\mathcal{L}\{x(t)\}$, $Y(s)$ is $\mathcal{L}\{y(t)\}$

s is the complex frequency variable

$\boxed{s = \sigma + j\omega}$

real → damping

imaginary → oscillatory behaviour

* s focuses on general complex frequency that includes transient and steady state

* Note: for discrete time system, we have z-transform

$$H(z) = \frac{Y(z)}{X(z)}$$

Frequency Response

* is when s is equal to $j\omega$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{H_1(j\omega)}{1 + H_1(j\omega)H_2(j\omega)}$$

* focuses on only steady state sinusoidal inputs
* used to analyze frequency domain (one axis)
* magnitude response $|H(j\omega)|$ that shows gain
* phase response $\angle H(j\omega)$ that shows phase shift
* Helps in Bode plot and stability analysis

look for midterm

Methods to Analyze CT LTI Systems

1) Fourier Transform
2) Laplace Transform

Fourier Transform	Laplace Transform
* $H(j\omega) = \mathcal{F}\{h(t)\}$	$H(s) = \mathcal{L}\{h(t)\}$
* $j\omega$ (purely imaginary)	$s = \sigma + j\omega$ (complex plane)
* Frequency Response	any inputs (exponential, impulse steps ...)
* steady state sinusoidal inputs	Complex frequency Domain
	transtient and steady state

DE's and CT systems

* many physical phenomena are described by DE's

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 \overset{\text{output}}{y(t)} =$$

$$b_m \frac{d^m x(t)}{dt^m} + \dots + b_1 \frac{dx(t)}{dt} + b_0 \underbrace{x(t)}_{\text{input}}$$

* eg RC ckt: $RC\frac{dy(t)}{dt} + y(t) = x(t)$

* can we DE's to describe
  * causal CT LTI systems

Linear, inhomogeneous, constant-coefficient ODEs (LICC-ODE)

→ $a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \dots + a_0 y = f(t)$

* where $f(t)$ is forcing function

* note: if $f(t) = 0 \Rightarrow$ homogenous
  if $f(t) \neq 0 \Rightarrow$ inhomogenous

* Solution: $y(t) = y_h(t) + y_p(t)$

  homogenous $f(t) = 0$ — particular $f(t) \neq 0$

* Uniqueness: for homogenous solution, we can be multiple output solutions for the same input as:

  → if initial conditions are not known, then $\exists$ infinite possible solutions since the general solution has arbitrary constant which depend on initial conditions.

  → if initial conditions are known, then only $\exists$ one solutions

* note: physical systems require correct initial conditions.

  eg: cap voltage @ $t=0$ must be known

* Causality: a system is causal if the output at time t only depends on past and present inputs.

physical systems are <u>causal</u>

Week 9: Tutorial 7 Mar 10, 2025

* $x(t) \longrightarrow \underset{\text{LTI}}{\boxed{h(t)}} \longrightarrow y(t) = x(t) * h(t)$ in frequency domain, you multiply them

impulse response

$Y(j\omega) = X(j\omega) \cdot H(j\omega)$

* $x(t) = e^{j\omega_0 t} \longrightarrow \underset{\text{LTI}}{\boxed{h(t)}} \longrightarrow y(t) = e^{j\omega_0 t} \cdot \underbrace{H(j\omega_0)}_{\triangleq H(j\omega)|_{\omega=\omega_0}}$

$H(j\omega)$, frequency response (take FT of impulse response)

Q.1) $h(t) = \dfrac{4\sin(\omega_{co} t)}{\pi t}$

$\delta(t) \xleftrightarrow{FT} 1$ since $F\{\delta(t)\} = 1$

$\delta(t-n) \xleftrightarrow{FT} e^{-j\omega n}$ and shifting time means $\times$ by $e^{-j\omega n}$

i) $x(t) = \sum_{n=-\infty}^{+\infty} \delta(t-n)$

☆ $\delta \xrightarrow{FT} 1$?

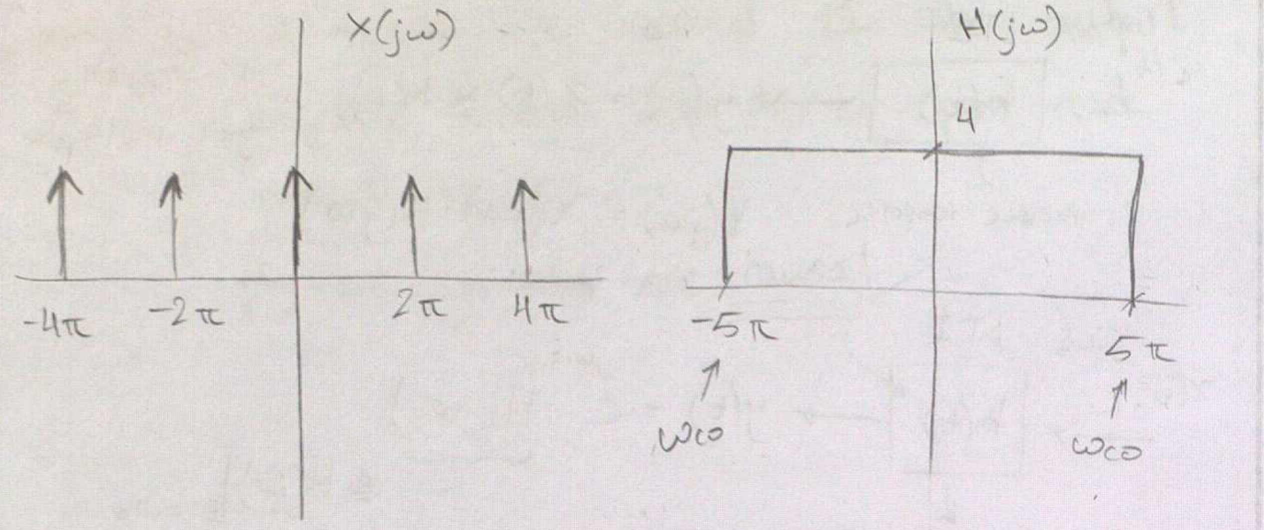
$X(j\omega) = F\{x(t)\} = \sum_{n=-\infty}^{+\infty} e^{\pm j\omega n} = 2\pi \sum_{n=-\infty}^{+\infty} \delta(\omega - 2\pi n)$

HW5 { summation of complex exponentials can be written as sum of deltas

ii) $\dfrac{\sin(\omega_c t)}{\pi t} \longleftrightarrow u(\omega+\omega_c) - u(\omega-\omega_c)$

$H(j\omega) = F\{h(t)\} = 4\left[u(\omega+\omega_{co}) - u(\omega-\omega_{co})\right]$ finding FT of h(t) using sinc function (turns into box)

☆ put 2π on top?



iii) $y(t) = ?$ $\quad Y(j\omega) = X(j\omega)\cdot H(j\omega) = 2\pi \cdot 4\big[\delta(\omega) + \delta(\omega - 2\pi) + \delta(\omega+2\pi) + \delta(\omega+4\pi) + \delta(\omega - 4\pi)\big]$

$\downarrow$

$2\pi\,\delta(\omega) \xleftrightarrow{FT} 1$

$2\pi\,\delta(\omega - \omega_0) \xleftrightarrow{FT} e^{j\omega_0 t}$

you only include what the deltas of $X(j\omega)$ reach while $\in [-5\pi, 5\pi]$

$\therefore\ y(t) = 4 + 4e^{j2\pi t} + 4e^{-j2\pi t} + 4e^{j4\pi t} + 4e^{-j4\pi t}$

(you took out the 2π since $F^{-1}\{2\pi\,\delta(\omega)\} = 1$

$\rightarrow y(t) = 4 + 8\cos 2\pi t + 8\cos 4\pi t$

iv) $h(t) = \dfrac{4\sin(\omega_{co} t)}{\pi t}$ → cut-off frequency

we had shifted deltas since $\omega_{co} = 5\pi$ included $\pm 2\pi, \pm 4\pi$ → corresponds to complex expon. in t-dom

$\therefore$ if $|\omega_{co}| < 2\pi \rightarrow y(t) = 4$ → our constant value

Q.2 $\quad H(j\omega) = \begin{cases} 10e^{-j0.0025\omega} & |\omega| \le 1000\pi \\ 0 & |\omega| > 1000\pi \end{cases}$

i) $x(t) = \underbrace{\cos(200\pi t)}_{x_1(t)} + \overbrace{\dfrac{2\sin(2000\pi t)}{\pi t}}^{x_2(t)}$ $\quad$ s.t. $y(t) = y_1(t) + y_2(t)$

because LTI, can apply superposition for each input

* $x_1(t) = \cos(200\pi t) = \dfrac{e^{j200\pi t} + e^{-j200\pi t}}{2}$ $\qquad e^{j\omega_0 t} \rightarrow \boxed{H(j\omega)} \rightarrow e^{j\omega_0 t}H(j\omega_0)$

1st expon: $\omega_0 = 200\pi \rightarrow H(j\omega_0) = 10e^{-j\frac{\pi}{2}} = -10j$

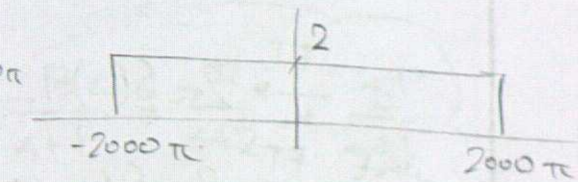
2nd expon: $\omega_0 = -200\pi \rightarrow H(j\omega_0) = 10e^{j\frac{\pi}{2}} = 10j$ $\Big\}$ recall this and can just multiply

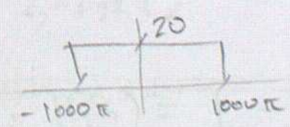
$y_1(t) = \dfrac{e^{j200\pi t}\cdot(-10j) + e^{-j200\pi t}\cdot(10j)}{2}$ $= 10\sin(200\pi t)$

$= 10\cos(200\pi t - 0.5\pi)$ sol^n handout

* $x_2(t) = \dfrac{2\sin(2000\pi t)}{\pi t} \Longrightarrow X_2(j\omega) = 2\big[u(\omega + 2000\pi) - u(\omega - 2000\pi)\big]$

$Y_2(j\omega) = H(j\omega)\cdot X_2(j\omega) = \begin{cases} 20e^{-j0.0025\omega}, & |\omega| \le 1000\pi \\ 0, & \text{otherwise} \end{cases}$





$\rightarrow y_2(t) = \dfrac{20\sin\left[1000\pi(t - 0.0025)\right]}{\pi(t - 0.0025)}$

we consider constant 20 height box and account for $e^{-j0.0025\omega}$ with a t-dom time shift

ii) $x(t) = \underbrace{x_1(t) + x_2(t)}_{\text{same as i)}} + \cos(3000\pi t)$

outside of non-zero $H(j\omega)$ range $\therefore = 0$

$$y_3(t) = \frac{e^{j3000\pi t} \cdot H(j\omega)|_{\omega=3000\pi} + e^{-j3000\pi t} \cdot H(j\omega)|_{\omega=-3000\pi}}{2}$$

$y_3(t) = 0 \quad \Rightarrow \quad y(t) = \underbrace{y_1(t) + y_2(t)}_{\text{same as i)}} + 0$

iii) $x(t) = \underbrace{\cos(200\pi t)}_{\text{found in i)}} + 2\delta(t)$ → output to delta input is the impulse response

☆ is h(t) shifted sinc fun.?

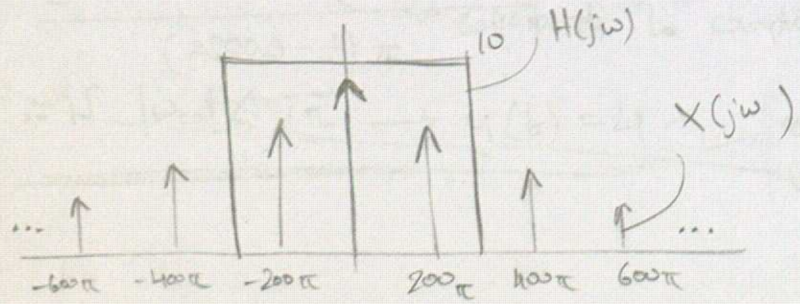
$y(t) = \underbrace{y_1(t)}_{\text{same as i)}} + 2h(t)$

Q.3) $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{j200\pi k t}$ $\quad a_k = \begin{cases} 1, & k=0 \\ \frac{1}{\pi k^2}, & k \neq 0 \end{cases}$

$1 \xrightarrow{FT} 2\pi\delta(\omega)$

$e^{j\omega_0 t} \xrightarrow{FT} 2\pi\delta(\omega-\omega_0)$

$\therefore X(j\omega) = \sum_{k=-\infty}^{+\infty} a_k 2\pi\delta(\omega - 200\pi k)$



iii) (v)

$Y(j\omega) = H(j\omega) \cdot X(j\omega) = 10 \sum_{k=-1}^{+1} a_k 2\pi\delta(\omega - 200\pi k)$ ← only 3 delta spikes within box

shifted deltas are complex exp. in t-dom

in jω-dom

$y(t) = 10 + \frac{10e^{j200\pi}}{\pi} + \frac{10e^{-j200\pi}}{\pi}$ → DC-value is $\underline{10}$

Q.4) $H(j\omega) = \frac{1-\omega^2+j\omega}{2-\omega^2+3j\omega} = C + \frac{C_1}{(\ )} + \frac{C_2}{(\ )}$ → order 1, at most order 2

① convert: $\boxed{s = j\omega} \Rightarrow \boxed{s^2 = -\omega^2}$

$H(s) = \frac{1+s^2+s}{2+s^2+3s}$ num / dem

② if order(num) ≥ order(den), divide

$H(s) = \frac{s^2+s+1}{s^2+3s+2} = \frac{s^2+3s+2-2s-1}{s^2+3s+2} = 1 - \frac{2s+1}{s^2+3s+2}$

can apply PFD $\frac{a}{(\ )} + \frac{b}{(\ )}$

③ partial fraction decomposition

$\frac{2s+1}{s^2+3s+2} = \frac{2s+1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1} = \frac{3}{s+2} + \frac{-1}{s+1}$

$\therefore H(s) = 1 - \frac{3}{s+2} + \frac{1}{s+1}$

④ convert to jω-dom

$H(j\omega) = 1 - \frac{3}{j\omega+2} + \frac{1}{j\omega+1} \xrightarrow{IFT} h(t) = \delta(t) - 3e^{-2t}u(t) + e^{-t}u(t)$

Q.6) $H(j\omega) = 2 + \dfrac{e^{-j\omega} - e^{-j3\omega}}{2+3j\omega-\omega^2} = 2 + \dfrac{e^{-j\omega}}{2+3j\omega-\omega^2} - \dfrac{e^{-j3\omega}}{2+3j\omega-\omega^2}$

i) $h(t) = ?$ • 2nd, third term, can ignore expon. (just t-shift)

2nd term: der. same as Q.4: $\dfrac{1}{2+3j\omega-\omega^2} = \dfrac{a}{j\omega+2} + \dfrac{b}{j\omega+1}$

$\xrightarrow{IFT} a e^{-2t}u(t) + b e^{-t}u(t)$ time shift by 1 $e^{-j\omega}$

$a e^{-2(t-1)}u(t-1) + b e^{-(t-1)}u(t-1)$

3rd term: same thing but shift by 3

$\rightarrow a e^{-2(t-3)}u(t-3) + b e^{-(t-3)}u(t-3)$

$\therefore h(t) = 2\delta(t)$ + 2nd term + 3rd term

ii) $G(j\omega) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} H(j(\omega-\gamma))\dfrac{1}{2+j\gamma}\,d\gamma$ def. of convolution (this time in jω-dom)

☆ convolution and 2π factor

$= \dfrac{1}{2\pi} H(j\omega) * \dfrac{1}{2+j\omega}$

$g(t) = h(t)\cdot e^{-2t}u(t)$

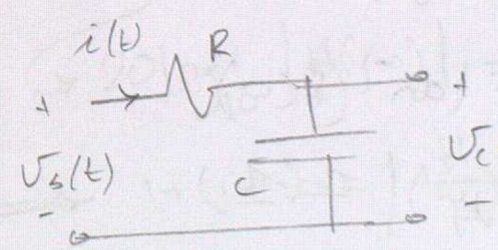
Week 9: Lecture 1 Mar 12, 2025

LTI system with exponential input

- LTI with complex expon. input has complex expon. output with the same frequency where amplitude is scaled and phase is shifted.
- Note: $e^{j\omega t}$ (or e^{st}) are eigen functions of LTI systems
  ↳ means if $e^{j\omega t}$ (or e^{st}) are inputs to an LTI system, then output is also the same exponential scaled by system frequency response

$\underbrace{H(j\omega)}_{\text{eigenvalues}} e^{j\omega t}$ or $\underbrace{H(s)}_{\text{eigenvalues}} e^{st}$

e.g. RC ckt



$\begin{cases} i(t) = C\dfrac{dv(t)}{dt} \\ i(t) = \dfrac{v_s(t) - v_c(t)}{R} \end{cases} \Longrightarrow RC\dfrac{dv_c(t)}{dt} + v_c(t) = v_s(t)$

- Given $h(t) = \dfrac{1}{RC}e^{-\frac{t}{RC}}u(t)$ ↳ impulse response
- Find frequency response

$H(j\omega) = \displaystyle\int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty}\frac{1}{RC}e^{-\frac{t}{RC}}e^{-j\omega t}dt$

☆ make sure you know how to graph phase and mag

$$H(j\omega) = \int_0^\infty \frac{1}{RC} e^{-(\frac{1}{RC}+j\omega)t} = \frac{1}{1+RCj\omega}$$

small c constant

⟹ Can write $x(t) = e^{-ct}u(t)$

⟹ $X(j\omega) = \frac{1}{c+j\omega}$, $c>0$

$$|H(j\omega)| = \frac{1}{\sqrt{1^2+(RC\omega)^2}} = \frac{1}{\sqrt{1+(RC)^2\omega^2}}$$

$$\angle H(j\omega) = \tan^{-1}\left(\frac{Im}{Re}\right)$$

$$\frac{1}{1+RCj\omega}\cdot\frac{1-RCj\omega}{1-RCj\omega} = \frac{1-RCj\omega}{1^2+(RC)^2\omega^2} = \frac{1}{1+(RC)^2\omega^2} - j\frac{RC\omega}{1+(RC)^2\omega^2}$$

$Re\{\dots\} = \frac{1}{1+(RC)^2\omega^2}$ $Im\{\dots\} = \frac{-RC\omega}{1+(RC)^2\omega^2}$

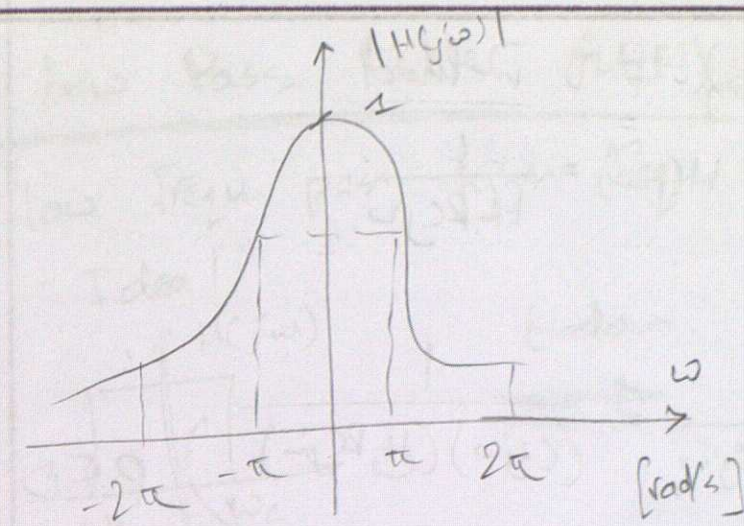
$$\angle H(j\omega) = \arctan(-RC\omega) = -\tan^{-1}(RC\omega)$$

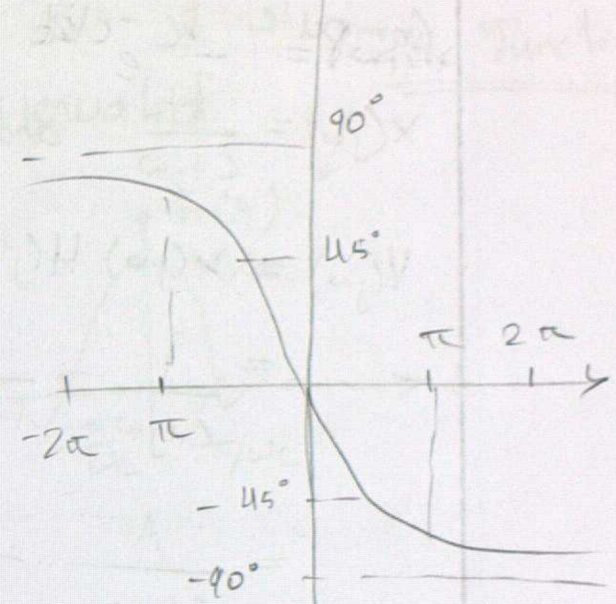
* Note:

mag: low freq $(\omega\to0)$ ⟹ $|H(j\omega)| \simeq 1$; high freq $(\omega\to\infty)$ ⟹ $|H(j\omega)| \simeq 0$

phase: low freq $(\omega\to0)$ ⟹ $\angle H(j\omega) \simeq 0°$; high freq $(\omega\to\infty)$ ⟹ $\angle H(j\omega) \simeq 90°$

∴ impulse response $h(t)$ corresponds to 1^{st} order LPF





Bode Plot

* used to plot and analyze filters

* uses logarithmic base 10, eg: 1 Hz → 10 Hz → 100 Hz

spans 2 decades

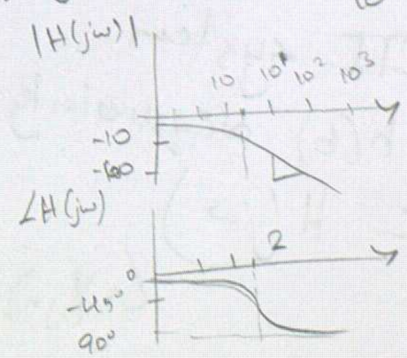
* then scales by 20 dB (or decibels)

* Note: $H(j\omega) = H_1(j\omega)H_2(j\omega)$

$$\to 20\log_{10}|H(j\omega)| = 20\log_{10}|H_1(j\omega)| + 20\log_{10}|H_2(j\omega)|$$

⟹ $H(j\omega) = \frac{H_1(j\omega)}{H_2(j\omega)}$

$$20\log_{10}|H(j\omega)| = 20\log_{10}|H_1(j\omega)| - 20\log_{10}|H_2(j\omega)|$$



-20 dB per decade

* Compute RC ckt by FT method

$x(j\omega) = \frac{1}{C+j\omega}$ and $H(j\omega) = \frac{1}{1+RCj\omega}$

$Y(j\omega) = X(j\omega)\,H(j\omega)$

$= \frac{1}{C+j\omega} \cdot \frac{1}{1+RCj\omega} = \frac{1}{(C+j\omega)(1+RCj\omega)}$ PFD

$= \frac{A}{C+j\omega} + \frac{B}{1+j\omega RC}$ take $\underline{\text{IFT}}$

$y(t) = Ae^{-ct}u(t) + Be^{-\frac{t}{RC}}u(t)$

* must know aliasing and LPF

$y(t) = \frac{e^{-ct}}{1-RC_c}u(t) - \frac{RC\,e^{-t/RC}}{1-RC_c}u(t)$

Filter

* is a system/defice that modifies the frequency content of signal allowing certain frequencies to pass while others are attenuated

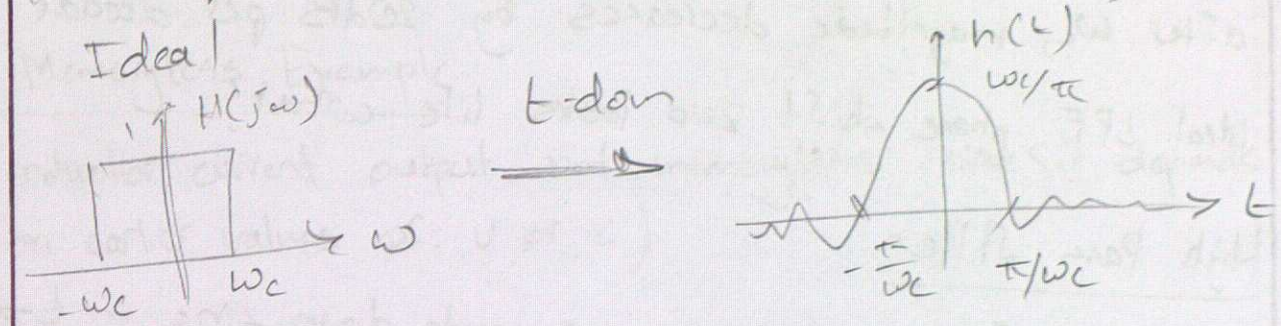
* can be designed using LTI system through impulse response $h(t)$ or equivalently by its frequency response $H(j\omega)$ $(\to Y(j\omega) = X(j\omega)H(j\omega))$

Low Pass Filter (LPF)

Sinc Function

* low frequ pass, high frequ attenuated

* Ideal



Frequency response $H(j\omega)$

~~Switch to Hamid Notes~~

Week 9: Lecture 2 — Mar 12, 2025

First order filter (LPF)

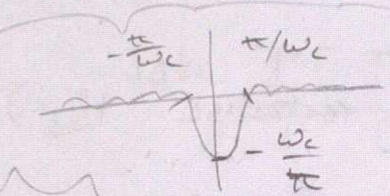
* at ω_c, $20\log_{10}|H(j\omega)|$ is -3dB and phase is $-45°$
* after ω_c, magnitude decreases by 20dB per decade

Q. ideal LPF, phase shift zero looks like what?

High Pass filter

* high pass frequency response → upside down sinc in t-dom

$$h(t) = \delta(t) - \frac{\sin(\omega_c t)}{\pi t}$$



* take output accross resistor
  - ↳ high frequ ≡ cap shorts ⟹ output gets input
  - ↳ low frequ ≡ cap opens ⟹ output zero (blocked low freq)

Q. where did delta come from

Bandwidth (BW)

* $BW = f_{max} - f_{min}$

Sampling Review

* $s(t) = \sum_{k=-\infty}^{\infty} \delta(t - nT_s) \longleftrightarrow S(j\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega - k\omega_s)$

Week 9: Lecture 3 — Mar 14, 2025

Sampling Example

Q: where is 2π factor for product ↔ convolution?

Memoryless Example

* inductor current output not memoryless (since it depends on earlier values of v or i)
* $x(t) \longrightarrow \boxed{\times 3}_{T} \longrightarrow y(t)$ memoryless : $T[x(t)] = 3x(t) = y(t)$

Linearity Example

$$T[a x_1(t) + b x_2(t)] \overset{?}{=} aT[x_1(t)] + bT[x_2(t)]$$

Time Invariant Example

* L example, if from $-\infty$, TI, but if from 0, not TI

Week 10: Tutorial 8 Mar 17, 2025

Laplace Transform

$x(t) \xrightarrow{L} X(s)$ $\qquad s = \sigma + j\omega$

$X(s) = \int_{0^-}^{+\infty} x(t) e^{-st}\, dt$ $\qquad$ ROC: region for which $X(s)$ is valid

- as in HW 7, comp. exp. input to LTI gives $y(t) = H(j\omega_0)x(t)$
+ but for Laplace, you only get steady-state response

Q1) i) $H(s) = \frac{1}{s+1}$, $\mathrm{Re}\{s\} > -1$ (LTI, BIBO, Causal)

$x(t) = 2\cos(t - \frac{\pi}{4})u(t) \longrightarrow$ apply formula from notes

$x(t) = A\cos(\omega_0 t + \phi)u(t) \longrightarrow y_{ss}(t) = A|H(j\omega_0)|\cos(\omega_0 t + \phi + \measuredangle H(j\omega_0))u(t)$

$H(j\omega) = \frac{1}{j\omega + 1}$ and $\omega_0 = 1$

$|H(j\omega)| = \frac{1}{\sqrt{\omega^2+1}}$ and $\measuredangle H(j\omega) = -\arctan(\omega)$

plug $\omega = \omega_0 = 1 \Longrightarrow |H(j\omega_0)| = \frac{1}{\sqrt{2}}$, $\measuredangle H(j\omega_0) = -\frac{\pi}{4}$

$\therefore y_{ss}(t) = \frac{2}{\sqrt{2}}\cos(t - \frac{\pi}{4} - \frac{\pi}{4})u(t)$

ii) $Y(s) = X(s)\cdot H(s) \longrightarrow \frac{2se^{-\frac{\pi}{4}s}}{s^2+1}\cdot\frac{1}{s+1}$ $\qquad$ can ignore for now since just t-dom shift

$X(s) = 2\left(\frac{s}{s^2+1}\right)\left(e^{-\frac{\pi}{4}s}\right)$

$\rightarrow \frac{2s}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$

$\frac{2s}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$

$A(s^2+1) + (s+1)(Bs+C) = 2s \Longrightarrow$ A = , B = , C =

$Y(s) = e^{-\frac{\pi}{4}s}\left(\frac{A}{s+1} + \frac{Bs+C}{s^2+1}\right)$

$= e^{-\frac{\pi}{4}s}\left(\frac{A}{s+1} + B\frac{s}{s^2+1} + \frac{C}{s^2+1}\right)$ $\qquad \omega_0 = 1$

$y_1(t) = Ae^{-t} + B\cos(t) + \frac{C}{\omega_0}\sin(\omega_0 t)$ $u(t)$ $\quad u(t)$

$\downarrow$ apply t-shift

$y(t) = Ae^{-(t-\frac{\pi}{4})}u(t-\frac{\pi}{4}) + B\cos(t-\frac{\pi}{4})u(t-\frac{\pi}{4})$
$+ C\sin(t-\frac{\pi}{4})u(t-\frac{\pi}{4})$

Q.2 i) $F(s) = \frac{1}{(s^2+a^2)^2} = \underbrace{\frac{1}{(s^2+a^2)}}_{G(s)} \cdot \underbrace{\frac{1}{(s^2+a^2)}}_{G(s)}$

$L^{-1}\{G(s)\} = \frac{1}{a}\sin(at)u(t) = g(t)$

$L^{-1}\{F(s) = (G(s))^2\} = g(t) * g(t) = \int_{-\infty}^{\infty} \frac{\sin a(t-\tau)}{a} u(t-\tau) \cdot \frac{\sin(a\tau)u(\tau)}{a}\, d\tau$

$= \frac{1}{a^2}\int_0^t \sin(a(t-\tau))\sin(a\tau)\, d\tau$

$\vdots$

next page

$-5^2 - t + 5^2 + 2s + t$

$\sin(a(t-\tau)) = \sin(at)\cos(a\tau) - \cos(at)\sin(a\tau)$

$$\rightarrow \frac{1}{a^2}\int_0^t \left(\sin(at)\cdot\frac{\sin(2a\tau)}{2} - \cos(at)\cdot\underbrace{\sin^2(a\tau)}_{\frac{1-\cos(2a\tau)}{2}}\right)d\tau$$

$= \dots$

Q.5 i) $\ddot{\theta}(t) + 10\theta(t) - \delta(t) = 0$

$\theta(t)$ — angle of pendulum

$\theta(t) = ?$ find based on Laplace Transform

$L\{\ddot{\theta}(t)\} = s^2\theta(s) - s\cancel{\theta(0)}^{\to 0} - \cancel{\dot{\theta}(0)}^{\to 0}$

$L\{10\theta(t)\} = 10\,\theta\{s\}$

$\Rightarrow s^2\theta(s) + 10\theta(s) - 1 = 0$

$L\{\delta(t)\} = 1$

$\Rightarrow \theta(s) = \frac{1}{s^2+10} = \frac{1}{s^2+(\sqrt{10})^2}$

$\theta(t) = L^{-1}\{\theta(s)\}$

$\theta(t) = \frac{1}{\sqrt{10}}\sin(\sqrt{10}\,t)\,u(t)$

ii) $\ddot{\theta}(t) + 2\dot{\theta}(t) + 10\theta(t) - \delta(t) = 0$ go to s-dom

$s^2\theta(s) + 2s\theta(s) + 10\theta(s) - 1 = 0$

$\theta(s) = \frac{1}{s^2+2s+10} = \frac{1}{(s+1)^2+9} \Rightarrow \theta(t) = \frac{e^{-t}\sin(3t)}{3}u(t)$

Week 10: Lecture 1 — Mar 20, 2025

Laplace Transform $\longrightarrow$ take Hamid Notes

Two Sided (Bilateral) LT

$s = \sigma + j\omega$

$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$ where both past and future values are considered

One Sided (Unilateral) LT

$H(s) = \int_0^{\infty} h(t)e^{-st}\,dt$ used for causal signals/systems

Region of Convergence

- refers to a set of values that results in convergence of LT integral
- procedure: given a 1-sided (right-sided) LT as

  $X(s) = \int_0^{\infty} x(t)e^{-st}\,dt$

- compute LL: express $X(s)$ in its factored form and identify poles where $X(s) \longrightarrow \infty$
- apply one-sided ROC Rule (region of convergence)

ROC: $Re(s) > s_0$ $\leftarrow$ sometimes 2 diff. might have the same LT, so we can differentiate them based on their ROC

Week 10: Lecture 2 — Mar 21, 2025

* $e^{-at}\cos(\omega_0 t) \xleftrightarrow{LT}$ question might come

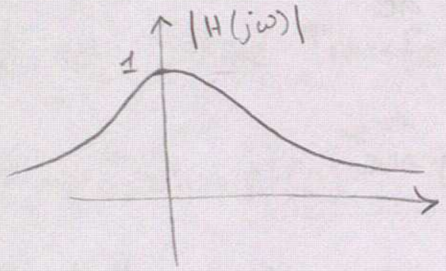
Q: is the ROC always to the right of rightmost pole?

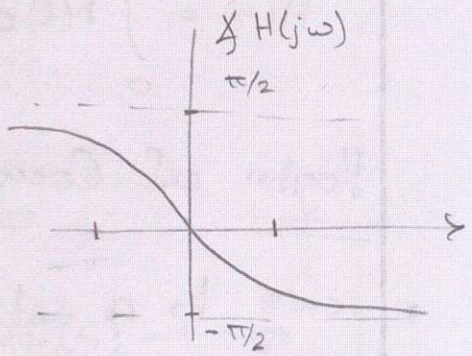
* existence of CTFT can be proved if ROC of LT include the Im axis ($j\omega$-axis)

Week 11: Tutorial 8 part 2 — Mar 24, 2025

4) i) $H(s) = \dfrac{k}{1+\frac{s}{\omega_c}}$, $k = 1$, $\omega_c = 2$

$$H(j\omega) = \frac{1}{1+\frac{j\omega}{2}}$$





$$|H(j\omega)| = \frac{1}{\sqrt{1+\frac{\omega^2}{4}}} \implies H(j\omega) = \frac{1 - j\frac{\omega}{2}}{1+\frac{\omega^2}{4}}, \quad \measuredangle H(j\omega) = -\arctan\left(\frac{\omega}{2}\right)$$

ii) $x(t) = (\cos t + \cos 4t)\cdot u(t)$, and since LTI, $y(t) = x(t) * h(t)$

$\implies Y(s) = X(s)\cdot H(s)$, and $X(s) = \dfrac{s}{s^2+1} + \dfrac{s}{s^2+16}$

$$Y(s) = \left(\frac{s}{s^2+1} + \frac{s}{s^2+16}\right)\cdot\frac{1}{1+\frac{s}{2}}$$

$$= \frac{\overbrace{2\left(s(s^2+16)+s(s^2+1)\right)}^{\text{deg } 3}}{\underbrace{(s^2+1)(s^2+16)(s+2)}_{\text{deg } 5}} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+16} + \frac{F}{s+2}$$

$$y(t) = \left(A\cos t + B\sin t + C\cos 4t + \frac{D}{4}\sin 4t + Fe^{-2t}\right)u(t)$$

iii) input is $x(t) = \cos t + \cos 40t$

$\therefore y(t) \longrightarrow$ frequency of output is different

* the larger freq. components of output have smaller coefficients $\implies \therefore$ LPF attenuates higher frequencies

6) i) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT)\,u(t)$

$$P(s) = \int_{0^-}^{+\infty} p(t)e^{-st}\,dt = \int_{0^-}^{+\infty}\sum_{n=-\infty}^{+\infty}\delta(t-nT)u(t)e^{-st}\,dt$$

$$= \sum_{n=-\infty}^{+\infty}\int_{0^-}^{+\infty}\delta(t-nT)\,u(t)\,e^{-st}\,dt$$

$$= \sum_{n=-\infty}^{+\infty}\begin{cases}0, & n<0\\ e^{-snT}, & n\geq 0\end{cases}$$

geometric series

$$= \sum_{n=0}^{+\infty} e^{-snT} = \sum_{n=0}^{+\infty}\left(e^{-sT}\right)^n = \begin{cases}+\infty, & |e^{-sT}|\geq 1\\ \dfrac{1}{1-e^{-sT}}, & |e^{-sT}|<1\end{cases}$$

$\therefore$ ROC for LT is $|e^{-sT}|<1$

ii) $|e^{-sT}|<1 \implies |e^{-(\sigma+j\omega)T}|<1 \implies |e^{-\sigma T}\underbrace{e^{-j\omega T}}_{\text{magnitude}=1}|<1 \implies |e^{-\sigma T}|<1$

* $\sigma T > 0$ so that $|e^{-\sigma T}|<1$

and since T is >0, our restriction is that $\sigma>0 \implies \operatorname{Re}\{s\}>0$

$\therefore$ ROC is $\operatorname{Re}\{s\}>0$

Week 11 Tutorial 9 Mar 24, 2025

1) i) $y[n] = x[n]\cos(0.2\pi n)$ $\therefore$ causal, linear, not TI

TI: $x[n] \to y[n]$

$x[n-n_0] \xrightarrow{} y[n-n_0]$ if TI

$y'[n] = x[n-n_0]\cos(0.2\pi n)$

$\neq y[n-n_0] = x[n-n_0]\cos(0.2\pi(n-n_0))$

ii) $y[n] = x[n] - x[n-1]$ $\therefore$ causal, linear, TI

iii) $y[n] = |x[n]|$ $\therefore$ causal, TI, not linear ($|\cdot|$ not linear operator)

iv) $y[n] = Ax[n] + B$ $\therefore$ causal, TI, not linear (unless $B=0$)

2) i) $y[n] = \alpha x[n] + \beta\left(x[n-n_0]\right)^2$

linear: if $\beta = 0$

ii) causal: if $n_0 > 0$

iii) TI: TI for any α, β, n_0

3) iv) $y[n] = \begin{cases} x[n+1], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

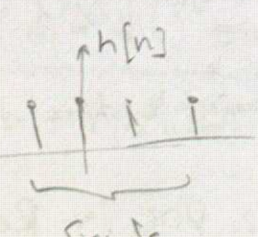
a) not selected (sys. linear)

d) not selected (sys. not causal)

b) not selected (sys. not TI since cond. diff.)

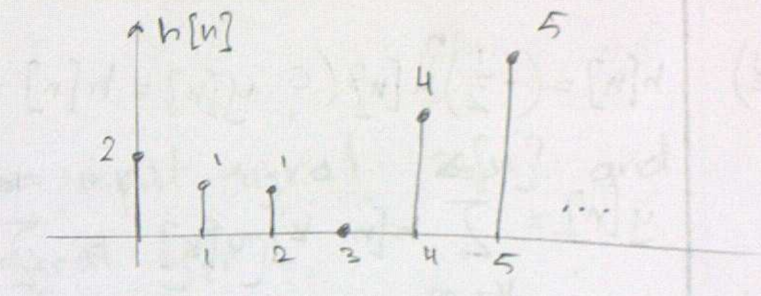
$\therefore$ c) system is BIBO stable

vi) FIR :



4) $y[n] = \sum_{k=0}^{M} b_k x[n-k]$

$h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$



from graph: $b_0 = 2$, $b_1 = b_2 = 1$, $b_3 = 0$, $b_4 = 4$, $b_5 = 5$

and $M = 5$ since rest of b_k values for $k > 5$ is zero

5) i) $x[n] \to \underbrace{\boxed{h_1} \to \boxed{h_2} \to \boxed{h_3}}_{h} \to y[n]$

S_1: $y_1[n] = x_1[n] - x_1[n-1] = x_1[n] * h_1[n]$

S_2: $y_2[n] = x_2[n] + x_2[n-2] = x_2[n] * h_2[n]$

S_3: $y_3[n] = x_3[n-1] + x_3[n-2] = x_3[n] * h_3[n]$

can find easily by plugging in $\delta[n]$ as input

$h_1[n] = \delta[n] - \delta[n-1]$

$h_2[n] = \delta[n] + \delta[n-2]$

$h_3[n] = \delta[n-1] + \delta[n-2]$

$h[n] = (h_1[n] * h_2[n]) * h_3[n]$

ii) $y[n] = x[n] * (h_1[n] * h_2[n] * h_3[n])$

$h[n] = (\delta[n] + \delta[n-2] - \delta[n-1] - \delta[n-3]) * (\delta[n-1] + \delta[n-2])$

$= \delta[n-1] + \delta[n-3] - \delta[n-2] - \delta[n-4]$

$+ \delta[n-2] + \delta[n-4] - \delta[n-3] - \delta[n-5]$ $= \delta[n-1] - \delta[n-5]$

+ cascaded systems can be combined by convoluting their impulse responses

7) i) $h[n] = \left(-\frac{1}{2}\right)^n u[n]$, $y[n] = h[n] * u[n]$

$$y[n] = \sum_{k=-\infty}^{+\infty} h[n-k]\, u[k] = \sum_{k=-\infty}^{+\infty} \left(-\tfrac{1}{2}\right)^{n-k} u[n-k]\, u[k]$$

$$= \begin{cases} 0 & , n<0 \\ \sum_{k=0}^{n} \left(-\frac{1}{2}\right)^{n-k} & , n \geq 0 \end{cases}$$

$$= \begin{cases} 0 & , n<0 \\ \left(-\frac{1}{2}\right)^n \sum_{k=0}^{n} \left(-\frac{1}{2}\right)^{-k} & , n \geq 0 \end{cases} \qquad \left(\tfrac{1}{2}\right)^{-k} = (2)^k$$

$$= \begin{cases} 0 & , n<0 \\ \left(-\frac{1}{2}\right)^n \sum_{k=0}^{n} (-2)^k & , n \geq 0 \end{cases} \qquad \sum_{k=0}^{n} q^n = \frac{1-q^{n+1}}{1-q}$$

$$= \begin{cases} 0 & , n<0 \\ \left(-1/2\right)^n \cdot \dfrac{\left(1-(-2)^{n+1}\right)}{1-(-2)} & , n \geq 0 \end{cases}$$

10)

$x * (y * z) = (x * y) * z$

$x[n] = u[n]$, $y[n] = \delta[n] - \delta[n-1]$, $z[n] = 1$

Week 11: Lecture 1 Mar 26

DT Systems

- a process that takes an input signal $x[n]$ and produces an output signal $y[n]$

$$x[n] \longrightarrow \boxed{T} \longrightarrow y[n] = T\{x\}[n]$$
$$= T\{x[n]\}$$

eg: $y[n] = x[n-1]$ delays input by 1

$$x[n] \longrightarrow \boxed{T} \longrightarrow y[n]$$

$\{0, 1, 2, 3, \dots\}$ $\{0, 0, 1, 2, 3, \dots\}$

DT system Properties

- Linearity : additivity and homogeneity
- memoryless :
- invertibility
- stability (BIBO stable)
- linear time-invariant (LTI)

Running Average Filter (moving average filter)

$$y[n] = \frac{x[n] + x[n-1] + x[n-2]}{3}$$

- smooths a DT signal by averaging a few consecutive samples
- here, at each time, the output is averaged among current and past 2 input samples
- this helps smooth out noise, or sudden changes in input signal

Let $x[n] = \{1, 2, 4, 6, 3, 0, 1, 0, \ldots\}$

and $x[n] = 0$ for $n < 0$ init. cond.

n	$x[n]$	$x[n-1]$	$x[n-2]$	$y[n]$
0	1	0	0	0.33
1	2	1	0	1.
2	4	2	1	2.33
3	6	4	2	4
4	3	6	4	4.33
5	0	3	6	3
6	10	0	3	4.33

$T\{x[n]\}$	
memory ✓	uses past values
causal ✓	past + present (no future)
stable ✓	BIBO stables
LTI ✓	

- preserves slow changes
- attenuates fast changes

$\}$ L.P.F

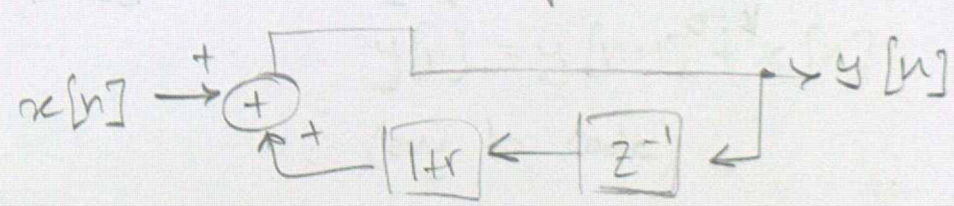
Bank Account (savings account)

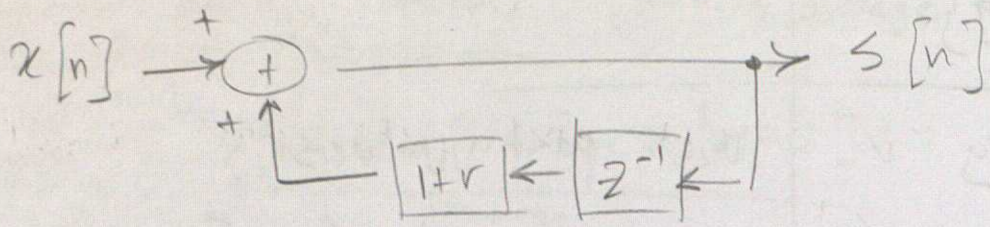
$x[n]$: deposit (input) at month n

$s[n]$: balance (output) " "

r : monthly interest rate ($1\% \rightarrow 0.01 = r$)

- a typical model is : $\boxed{s[n] = \underbrace{(1+r)\overbrace{s[n-1]}^{\text{bal. from last month}}}_{\text{old balance plus interest}} + \underbrace{x[n]}_{\text{new deposit}}}$
- first order linear DT system:





but z^{-1} : $s[n-1]$

* recursive def. of a DT system
* since output defined in terms of prev. val's

* Convert recursive → pointwise

$s[0] = x[0]$

$s[1] = (1+r)s[0] + x[1] = (1+r)x[0] + x[1]$

$s[2] = (1+r)s[1] + x[2]$

$= (1+r)\left[(1+r)x[0] + x[1]\right] + x[2]$

$= (1+r)^2 x[0] + (1+r)x[1] + x[2]$

$s[3] = (1+r)^3 x[0] + (1+r)^2 x[1] + (1+r)x[2] + x[3]$

$\vdots$

$s[n] = \sum_{k=0}^{n} (1+r)^{n-k} x[k]$ ← pointwise def of a DT system

Recursive Def.

* efficient: each output only need current input and previous output values
  - ⇒ faster for real-time applications
  - ⇒ no need to store all the old values
* many physical systems are recursive (eg filters, control loops, bank account)

Pointwise Def.

* shows any potential dependence on initial values
  → great for analyzing system behavior (eg. decay/growth, convergence)
* great for understanding how input builds up to current value

Sum System (Accumulator)

$y[n] = \sum_{k=-\infty}^{n} x[k]$ ← could also use recursive definition, for example

$y[n] = y[n-1] + x[n]$

(easy to code)

Nature of Accumulators (and integrators)

- inherently have infinite memory and can grow forever if not controlled
- suppose $x[n] = 1$ $\longrightarrow$ must be careful

n	$y[n]$
0	$0+1=1$
1	2
2	3
3	4
4	5
$\vdots$	
n	$n+1$

Impulse Response of DT LTI Systems

- is output of sys. when input is $\delta[n]$

$$x[n]=\delta[n] \longrightarrow \boxed{LTI} \longrightarrow y[n] = h[n]$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

- Consider $h[n] = \sum_{k=-\infty}^{n} \delta[k]$

$\delta[k] = \begin{cases} 1 & k=0 \\ 0 & \text{otherwise} \end{cases}$, then $\sum_{k=-\infty}^{n} \delta[k]$ has only one term contributing $\rightarrow$ the one at $k=0$ if $n \geq 0$

$$\Rightarrow h[n] = \begin{cases} 0 & n<0 \\ 1 & n \geq 0 \end{cases} \Rightarrow u[n]$$

$$\therefore h[n] = u[n] \Rightarrow u[n] = \sum_{k=-\infty}^{n} \delta[k], \quad n \geq 0$$

n		
-2	0	no impulse yet
-1	0	no " "
0	1	
1	1	
2	1	
$\vdots$		

Ex: Find impulse response of $y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$

Recall $h[n]$ is output when input is

$$\delta[n] \Rightarrow x[n] = \delta[n]$$

$$h[n] = \frac{1}{3}(\delta[n-1] + \delta[n] + \delta[n+1])$$

eg: $y[n] = x[n-2]$, so $h[n] = \delta[n-2]$

Consider:

$$x = \sum_{k=-\infty}^{\infty} x[k]\delta_k \rightarrow \boxed{\begin{matrix} LTI \\ T \end{matrix}} \rightarrow y = T\{x\}$$

* since linear

$$y = T\{x\} = T\left\{ \sum_{k=-\infty}^{\infty} x[k]\delta_k \right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, T\{\delta_k\}$$

note that $T\{\delta_k\} = h_k$

* since system is TI (time inveriant), can write

$$T\{\delta_k\}[n] = h[n-k]$$

* this leads to convolution sum

$$y[n] = T\{x[n]\} = \sum_{k=-\infty}^{\infty} x[k]\, h_k[n]$$

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]}$$

Ex: Find $y[n]$ for $x[n]$ if $h[n] = \delta[n-n_0]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[\underbrace{n-k-n_0}] \rightarrow k = n-n_0$$

$$\Longrightarrow y[n] = x[n-n_0]$$

Ex: Find $y[n]$ given $x[n]$ and $h[n] = \delta[n] - \delta[n-1]$

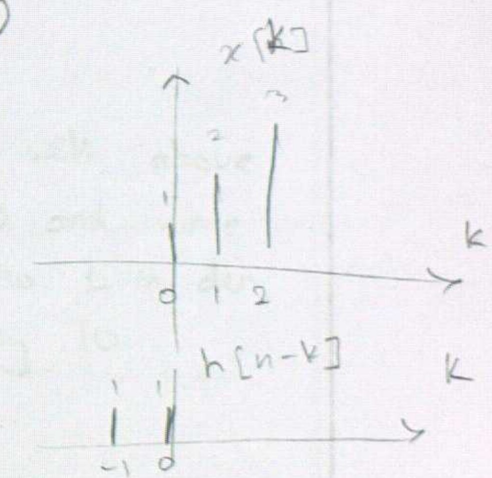
$$y[n] = x[n] * h[n]$$

$$y[n] = x[n] - x[n-1]$$ ← this system is a difference system

* analagous to derivative
* shows how fast signal is changing

Ex: $x[n] = \{1, 2, 3\}$ $h[n] = \{1, 1\}$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$



n		y[n]
0	$y[0] = 1\cdot 1 = 1$	1
1	$y[1] = 1\cdot 1 + 1\cdot 2 = 3$	3
2	$y[2] = 1\cdot 2 + 1\cdot 3 = 5$	5
3	$y[3] = 3\cdot 1 = 3$	3

$$y[n] = \{1, 3, 5, 3\}$$

Ex: $x[n] = u[n]$, $h[n] = \left(\frac{1}{2}\right)^n u[n]$

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{u[k]}_{\to k=0} \left(\frac{1}{2}\right)^{n-k} \underbrace{u[n-k]}_{\to k=n}$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{n-k}$$

change of var.

$m \triangleq n-k$

upper: $k=n \Rightarrow m=0$

lower: $k=0 \Rightarrow m=n$

$$= \sum_{m=0}^{n}$$

Week 12: Tutorial 10 Mar 31, 2025

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k]$$

1) i) $h[n] = \frac{1}{L}\sum_{k=0}^{L-1} \delta[n-k]$

ii) DTFS on $h[n] \longrightarrow H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n}$

$$\left\{ \begin{array}{l} \delta[n] \xrightarrow{\text{DTFT}} 1 \\ \delta[n-k] \xrightarrow[\text{DTFT}]{} e^{-j\omega k} \end{array} \right\} \quad H(e^{j\omega}) = \frac{1}{L}\sum_{k=0}^{L-1} e^{-j\omega k} = \frac{1}{L}\sum_{k=0}^{L-1} \underbrace{\left(e^{-j\omega}\right)^k}_{\text{geometric series}}$$

$$H(e^{j\omega}) = \frac{\left(1-\left(e^{-j\omega}\right)^L\right)}{L\left(1-e^{-j\omega}\right)} = \frac{1}{L} \cdot \frac{e^{-j\frac{\omega L}{2}}\left(e^{j\omega\frac{L}{2}} - e^{-j\frac{\omega L}{2}}\right) \times \frac{1}{2j}}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right) \times \frac{1}{2j}}$$

$$H(e^{j\omega}) = \frac{1}{L} e^{-j\frac{\omega}{2}(L-1)} \cdot \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

iii) $H(e^{j\omega}) = e^{-j4.5\omega} \dfrac{\sin(5\omega)}{\sin(\omega/2)}$, and comparing with above, we see $L=10$, and since this one has no L in den, we scale it by 10

$$\therefore h[n] = \frac{1}{L}\sum_{k=0}^{L-1} \delta[n-k]$$

$$= \frac{1}{10}\sum_{k=0}^{9} \delta[n-k] \times 10$$

$$h[n] = \sum_{k=0}^{9} \delta[n-k]$$

2\. i) $y[n] = \frac{1}{4}\sum_{k=0}^{3} x[n-k]$

info from Q 1)

$h[n] = \frac{1}{4}\sum_{k=0}^{3} \delta[n-k] \longleftrightarrow H(e^{j\omega}) = \frac{1}{4} e^{-j\omega 1.5} \frac{\sin(2\omega)}{\sin(\omega/2)}$

ii) $|H(e^{j\omega})| = \frac{1}{4} \underbrace{\left|e^{-j\omega 1.5}\right|}_{\text{mag}=1} \cdot \left|\frac{\sin(2\omega)}{\sin(\omega/2)}\right|$

$|H(e^{j\omega})| = \frac{1}{4}\left|\frac{\sin(2\omega)}{\sin(\omega/2)}\right|$

remember that $e^{j\theta} = |e^{j\theta}| \angle\theta$

iii) $\angle H(e^{j\omega}) = \angle e^{-j\omega 1.5} + \angle \frac{\sin(2\omega)}{\sin(\omega/2)}$

always real since real ÷ real = real and $\angle$ real = 0 or π or $-\pi$ ...

$= -1.5\omega + \angle \frac{\sin(2\omega)}{\sin(\omega/2)}$

$$\text{sign}\left\{\left(\frac{\sin(2\omega)}{\sin(\omega/2)}\right)\right\} = \begin{cases} (-)\text{ve}, & -\pi \le \omega \le -\frac{\pi}{2} \quad ① \\ (+)\text{ve}, & -\frac{\pi}{2} \le \omega \le 0 \quad ② \\ (+)\text{ve}, & 0 \le \omega \le \frac{\pi}{2} \quad ③ \\ (-)\text{ve}, & \frac{\pi}{2} \le \omega \le \pi \quad ④ \end{cases}$$

when sign of sin changes

$\sin(2\omega)$: $2\omega = \pi$, $\omega = \frac{\pi}{2}$

$\sin(\frac{\omega}{2}) \Rightarrow \frac{\omega}{2} = \pi$, $\omega = 2\pi$

smaller

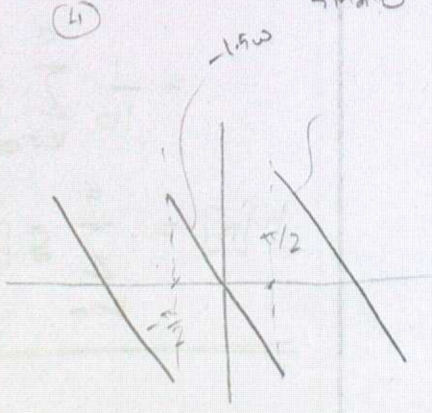
(+)

① $-2\pi \le 2\omega \le -\pi$

$-\frac{\pi}{2} \le \frac{\omega}{2} \le -\frac{\pi}{4}$

$\therefore \frac{(+)}{(-)} = (-)$ (-)

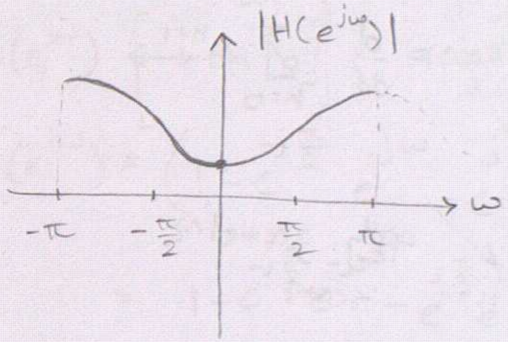
$$\angle H(e^{j\omega}) = -1.5\omega + \begin{cases} \pm\pi, & -\pi \le \omega \le -\frac{\pi}{2} \\ 0, & \\ \pm\pi & \end{cases}$$



3\. $H(e^{j\omega}) = \frac{1}{1+\frac{1}{2}e^{-j\omega}} = \frac{1}{\underbrace{1+\frac{1}{2}\cos\omega}_{\text{Re}} \underbrace{-\frac{1}{2}j\sin\omega}_{\text{Im}}} = \frac{1+\frac{1}{2}\cos\omega+\frac{1}{2}j\sin\omega}{(1+\frac{1}{2}\cos\omega)^2+\frac{\sin^2\omega}{4}}$

$|H(e^{j\omega})| = \frac{1}{(1+\frac{1}{2}\cos\omega)^2+(-\frac{1}{2}j\sin\omega)^2}$

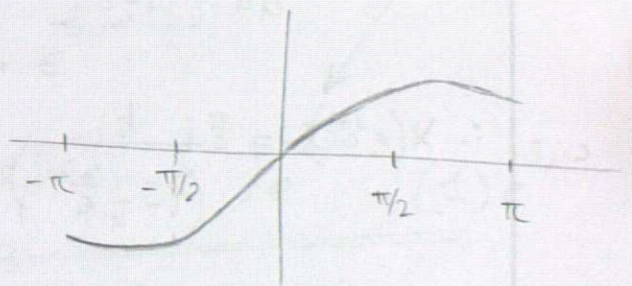
$= \frac{1}{\sqrt{1+\cos\omega+\frac{1}{4}}} = \frac{1}{\sqrt{\frac{5}{4}+\cos\omega}}$



$\omega = 0 \Rightarrow \frac{1}{\sqrt{\frac{5}{4}+1}} = \frac{1}{\sqrt{\frac{9}{4}}} = \frac{2}{3}$

and as ω goes to $\pi/-\pi$ the denom decreases and $\therefore |H(e^{j\omega})|$ increases

$\angle H(e^{j\omega}) = \arctan\left(\frac{\frac{1}{2}\sin\omega}{1+\frac{1}{2}\cos\omega}\right)$



$\frac{\text{odd}}{\text{even}}$ = odd ← since argument odd, shape like that

4\. i) $h[n] = (\frac{1}{2})^n u[n]$

$H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$

ii) $x[n] = (\frac{3}{4})^n u[n] \longleftrightarrow X(e^{j\omega}) = \frac{1}{1-\frac{3}{4}e^{-j\omega}}$

$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{1}{1-\frac{3}{4}e^{-j\omega}} \cdot \frac{1}{1-\frac{1}{2}e^{-j\omega}}$

iii) $x[n] = (n+1)\left(\frac{1}{4}\right)^n u[n]$ $\qquad a \triangleq \frac{1}{4}e^{-j\omega} < 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} (n+1)\left(\tfrac{1}{4}\right)^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{+\infty} (n+1)\left(\tfrac{1}{4}\right)^n e^{-j\omega n}$$

$$= \sum_{n=0}^{+\infty} (n+1)\left(\tfrac{1}{4}e^{-j\omega}\right)^n$$

$$\sum_{n=0}^{+\infty} (n+1) a^n$$

trick

$$= \sum_{n=0}^{+\infty} \frac{d}{da}\left[a^{n+1}\right]$$

$$= \frac{d}{da}\left[\sum_{n=0}^{+\infty} a^{n+1}\right]$$

apply geometric series

$$= \frac{d}{da}\left[\frac{a}{1-a}\right] = \frac{1(1-a) - a(-1)}{(1-a)} = \frac{1}{(1-a)^2}$$

$$\therefore X(e^{j\omega}) = \frac{1}{\left(1 - \frac{1}{4}e^{-j\omega}\right)^2}$$

5. $H(e^{j\omega}) = \left(1 - e^{j\frac{\pi}{2}} e^{-j\omega}\right)\left(1 - e^{-j\frac{\pi}{2}} \cdot e^{-j\omega}\right)\left(1 + e^{-j\omega}\right)$

$$x[n] = \underset{x_1}{5} + \underset{x_2}{20\cos(0.5\pi n + 0.25\pi)} + \underset{x_3}{10\,\delta(n-3)}$$

$$e^{j\omega_0 n} \longrightarrow \boxed{LTI}_{\substack{h[n] \\ H(e^{j\omega})}} \longrightarrow H(e^{j\omega})\big|_{\omega=\omega_0} \times e^{j\omega_0 n}$$

$$y_1[n] = 5 \cdot H(e^{j\omega})\big|_{\omega=0} = 5\underbrace{\left(1 - e^{j\frac{\pi}{2}}\right)}_{1-j}\underbrace{\left(1 - e^{-j\frac{\pi}{2}}\right)}_{1+j} \times 2 = 20$$

$$x_2[n] = 10\left(e^{j(0.5\pi n + 0.25\pi)} + e^{-j(0.5\pi n + 0.25\pi)}\right)$$

$$y_2[n] = 10\,e^{0.5\pi n} \cdot e^{0.25\pi} \cdot H(e^{j0.5\pi}) + 10\,e^{-j0.5\pi n}\, e^{-j0.25\pi}\, H(e^{-j0.5\pi})$$

$$x_3[n] = 10\,\delta[n-3] \longrightarrow y_3[n] = 10\,h[n-3]$$

$H(e^{j\omega}) \longleftrightarrow h[n]$: recall $\boxed{H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n}}$ / use "compare coefficients" method for $n = 0, \pm1, \pm2 \ldots$

$$H(e^{j\omega}) = \left(1 - e^{-j\frac{\pi}{2}} e^{-j\omega} - e^{j\frac{\pi}{2}} e^{-j\omega} + e^{-2j\omega}\right)\left(1 + e^{-j\omega}\right)$$

$$= 1 - e^{-j\frac{\pi}{2}} e^{-j\omega} - e^{j\frac{\pi}{2}} e^{-j\omega} + e^{-2j\omega}$$

$$+ e^{-j\omega} - e^{-j\frac{\pi}{2}} e^{-2j\omega} - e^{j\frac{\pi}{2}} e^{-2j\omega} + e^{-3j\omega}$$

$$= 1 + \left(-e^{-j\frac{\pi}{2}} - e^{j\frac{\pi}{2}} + 1\right) e^{-j\omega} + \left(1 - e^{-j\frac{\pi}{2}} - e^{j\frac{\pi}{2}}\right) e^{-2j\omega} + (1) e^{-3j\omega}$$

☆ how to shift if I write like

$$\cancel{h[n] = \begin{cases} 1 & , n = 0, 3 \\ -e^{-j\frac{\pi}{2}} - e^{j\frac{\pi}{2}} + 1 & , n = 1, 2 \\ 10 & , \text{otherwise} \end{cases}}$$

now $y_3[n]$ is this shifted 3

$$h[n] = h[0]\,\delta[n] + h[1]\,\delta[n-1] + h[2]\,\delta[n-\ ] + h[3]\,\delta[n-3]$$

$$y_3[n] = 10\,h[n-3]$$

and $\underline{y[n] = y_1[n] + y_2[n] + y_3[n]}$

7. i) $1\ 1\ 1\ 1\ 5 \leftarrow x[n],\ N_0 = 5$

$1\ 1\ 1\ 3\ 3 \leftarrow y[n],\ N_0 = 5$

$$\alpha_k = \frac{1}{N_0}\sum_{n=0}^{4} x[n]\, e^{-j\frac{2\pi}{N_0}kn} \quad \leftarrow x[n] \text{ FS coeff.}$$

$$\beta_k = \frac{1}{N_0}\sum_{n=0}^{4} y[n]\, e^{-j\frac{2\pi}{N_0}kn} \quad \leftarrow y[n] \text{ FS coeff.}$$

ii) $h[n] = \alpha\,\delta[n] + \beta\,\delta[n+1]$

$y[n] = x[n] * h[n]$

$\quad = \alpha\, x[n] + \beta\, x[n+1]$

$n=0 \Rightarrow 1 = \alpha\cdot 1 + \beta \cdot 1$

$n=1 \Rightarrow 1 = \alpha\cdot 1 + \beta\cdot 1$

$\vdots$

$n=2 \Rightarrow 1 = \alpha\cdot 1 + \beta 1$

$n=3 \Rightarrow 3 = \alpha\cdot 1 + \beta\cdot 5$

$n=4 \Rightarrow 3 = \alpha\cdot 5 + \beta\cdot 1$

$n=0,1,2 \Rightarrow 2 = 4\beta \Rightarrow \beta = \frac{1}{2}$

rest $\quad \alpha = \frac{1}{2}$

$\therefore h[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n+1]$

iii) $H(e^{j\omega}) = \frac{1}{2} + \frac{1}{2}e^{+j\omega}$

Week 12: Lecture 1

Bank Account Model

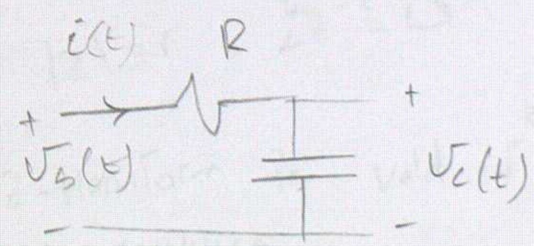
$y[k] = (1+r)\, y[\cancel{k}]$

New topic: Backward and Forward Euler Approx.

+ methods in DT
+ these are techniques to approx. and convert CT derivatives into DT difference equations

Backward	Forward
$\frac{dy(t)}{dt} \simeq \frac{y[n]-y[n-1]}{T_s}$	$\frac{dy(t)}{dt} \simeq \frac{y[n+1]-y[n]}{T_s}$

Ex: Discretize RC ckt → first do in CT



$$RC\frac{dV_c(t)}{dt} + V_c(t) = V_s(t)$$

$$\frac{dV_c(t)}{dt} = \frac{1}{RC}V_s(t) - \frac{1}{RC}V_c(t)$$

Backward Euler:

$$\frac{dV_c(t)}{dt} \simeq \frac{V_c[n]-V_c[n-1]}{T_s}$$

$$\Rightarrow \frac{V_c[n]-V_c[n-1]}{T_s} = \frac{1}{RC}V_s[n] - \frac{1}{RC}V_c[n]$$

$$V_c[n] = V_c[n-1] + T_s\left(\frac{1}{RC}V_s[n] - \frac{1}{RC}V_c[n]\right)$$

$$V_c[n]\left(1+\frac{T_s}{RC}\right) = V_c[n-1] + \frac{T_s}{RC}V_s[n]$$

$$V_c[n] = \frac{1}{1+\frac{T_s}{RC}}V_c[n-1] + \frac{T_s/RC}{1+\frac{T_s}{RC}}V_s[n]$$

difference

Let $\beta = \frac{1}{1+\frac{T_s}{RC}}$ recursive equ.

$$\Rightarrow V_c[n] = \beta V_c[n-1] + (1-\beta)V_s[n]$$

eg: $R = 1k\Omega$, $C = 1\mu F$ $T_s = 1ms$

$$\beta = \frac{1}{1+\frac{0.01}{10^{-3}\cdot 10^{-6}}} \simeq 0.09$$

$$V_c[n] = 0.09V_c[n-1] + 0.90V_c[n]$$

Z-transform

* is a powerful mathmatical tool used to analyze DT signals and systems
* is the counter part to Laplace Transform
* Transforms DE to algebraic eq.

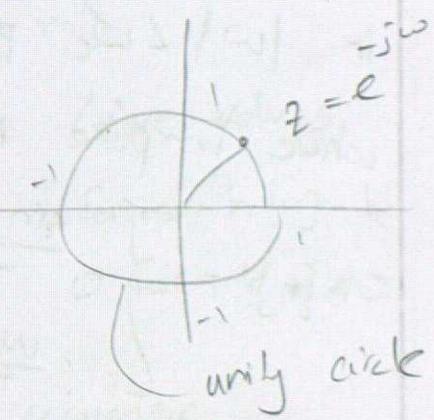
Given by $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ ← bilateral z-transform (2-sided)

one sided (unilateral) $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$

* z is a complex num. that represents a point on the complex plane

$$z = re^{j\omega} = r(\cos\omega + j\sin\omega)$$

$$|z| = r,\ \angle z = \omega$$



* Z-transform is vald if infinite series converges
* Regin of convergence (ROC) is the set of val

Week 12 : Lecture 2

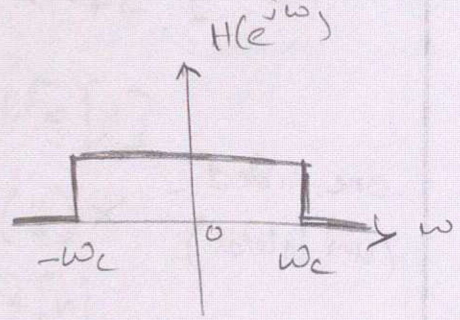
Non-Ideal DT Low-Pass Filter

* in DT, filters can be implemented using difference equations ; due to practical limitations, these filters are non-dieal

Ideal LPF

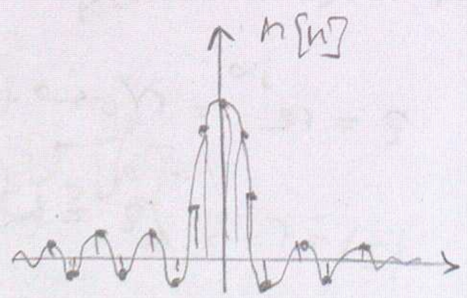
* has properties :

$$H(e^{j\omega}) = \begin{cases} 1 & , -\omega_c \le \omega \le \omega_c \\ 0 & , \text{otherwise} \end{cases}$$



* this means any signal with frequencies in $|\omega| < \omega_c$ passes through [slide 9-408]
* where impulse response

$$h[n] = \begin{cases} \frac{\sin(\omega_c n)}{\pi n} & , n \neq 0 \\ \frac{\omega_c}{\pi} & , n = 0 \end{cases}$$



* note that $h[n] \neq 0$ for $n < 0$ as sinc function extends to infinity $(-\infty, \infty)$

$\rightarrow$ $h[n]$ is non-causal (since future val's exist)

* this is because ideal LPF has perfect sharpen to cut off at ω_c that requires $h[n]$ to be infinitely long
* this is problem $\longrightarrow$ practical filters do not have access to infinite future values
* turning an ideal non-causal LPF to practical causal LPF

1) * consider ideal LPF

$$h_{ideal}[n] = \frac{\sin(\omega_c n)}{\pi n} \qquad n \in (-\infty, \infty)$$

2) shift to the right to make it causal

$$h[n] = h_{ideal}[n - n_0] = \frac{\sin(\omega_c(n-n_0))}{\pi(n-n_0)}$$

3) multiply by $u[n]$ to $\rightarrow$ truncate (-)ve values

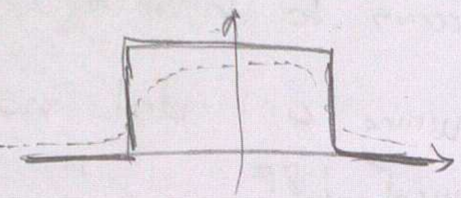
$$h[n] = \frac{\sin(\omega_c(n-n_0))}{\pi(n-n_0)} u[n] \longleftarrow$$ forces $h[n]$ be 0 for $\forall n < 0$

4) Compute DTFT to find frequency response

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n}$$

$\Rightarrow H(e^{j\omega}) = \sum_{n=0}^{\infty} \frac{\sin(\omega_c(n-n_0))}{\pi(n-n_0)} e^{-j\omega n}$

- causal LPF and can be implemented in real time
- results in distortions and ripples in frequency ~~do~~ response due to truncation:



☆ Ex: $y[n] = (1-\omega_c)x[n] + \omega_c y[n-1]$

Derive frequency response

→ Apply DTFT

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$F\{y[n]\} = (1-\omega_c)F\{x[n]\} + \omega_c F\{y[n-1]\}$

$Y(e^{j\omega}) = (1-\omega_c)X(e^{j\omega}) + \omega_c Y(e^{j\omega})e^{j\omega}$

$H(e^{j\omega}) = \frac{1-\omega_c}{1-\omega_c e^{-j\omega}}$

Ex: Given $h[n] = \frac{1}{4}\delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2]$

$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$

$= \frac{1}{4}e^{-j\omega(0)} + \frac{1}{2}e^{-j\omega(1)} + \frac{1}{4}e^{-j\omega(2)}$

$= \frac{1}{4} + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j\omega 2}$

* Filters (FIR, IIR, non-ideal filters)
* DT systems (difference equations etc.)
* Laplace Transform
  - ROC
  - if $\sigma = 0$, only $j\omega$ remains
* LTI with comp. exp. (RC ckt.)
  - phase and magnitude
  - filters sinc function
* feedback systems LTI
* convolution properties

back → Sampling

* 5 questions → 20 each

mining bld 101

ECE216 Course Summary

Basics

* given $x(t)$ and you want to graph $y(t) = x(\alpha t + \beta)$
  - ↳ first apply time shift β
  - ↳ then scale t-axis by α (divide all t's by α) $\longrightarrow \begin{cases} |\alpha| > 1: \text{ horiz. "wider"} \\ |\alpha| < 1: \text{ horiz. "squished"} \end{cases}$
* is $y(t) = x_1(t) + x_2(t)$ periodic and with what T_0?
  - ↳ yes if $\left.\frac{T_1}{T_2} = \frac{l}{k}\right\}$ and l/k is rational
  - ↳ then $T_0 = lT_2 = kT_1 \Longrightarrow T_0 = LCM(T_1, T_2)$
* DT comp. expo. periodic iff ω is a rational multiple of 2π
  - ↳ $\omega_0 = k\frac{2\pi}{N_0}$ ← ω_0 must satisfy for smallest integer k

3. Fourier Series

* trigonometric FS: $x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(\omega_0 n t) + b_n\sin(\omega_0 n t)\right]$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt \quad a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(\omega_0 n t)\,dt \quad b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(\omega_0 n t)\,dt$$

* CTFS v.s. DTFS (CTFS generally aperiodic, but DTFS is N_0-periodic)

$x(t)$ periodic $\begin{cases} \rightarrow \text{CTFS represents } x(t) \text{ as infinite discrete sum of CT comp. exp.} \\ \rightarrow \text{DTFS } \text{〃} \ x[n] \text{ 〃 finite discrete sum of DT 〃 〃} \end{cases}$

* FS repr. of $x(t)$ is $x(t)$ repr. by sum of comp. exp. and lets us analyze frequency content of $x(t)$ by examining α_k

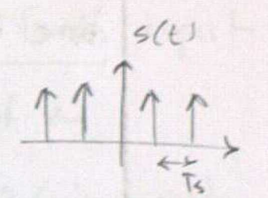
4. Fourier Transform

* FT is extension of FS to aperiodic signals
* idea: aperiodic signal is periodic signal with infinite period
* $x(t) \xleftrightarrow{F} X(j\omega)$ ← called "FT" or "spectrum" of x
* $u(t+a) - u(t-a) \xleftrightarrow{FT} \frac{2\sin(a\omega)}{\omega}$
* $\frac{\sin(at)}{\pi t} \xleftrightarrow{FT} u(\omega+a) - u(\omega-a)$
* if $x(t)$ very concentrated, $X(j\omega)$ very spread out
* $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega_0 t} \xleftrightarrow{FT} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \alpha_k \delta(\omega - k\omega_0)$
* CTFT of a periodic signal is a sum of impulse functions located at multiples of the fundamental frequency ω_0
* CTFT v.s. DTFT
  → CTFT represents aperiodic $x(t)$ as continuous sum of CT comp. exp. and in general is not periodic (spectrum)

  → DTFT " " $x[n]$ " continuous sum of DT comp. exp. " " " is 2π-periodic (spectrum)

  → both $X(j\omega)$ and $X(e^{j\omega})$ capture the "amount" of frequency ω contained in x

5. Sampling, Aliasing, Interpolation



* sampling function : $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$ ← sampling period T_s
* weighted sum of impulses : $\underbrace{s(t)\,x(t)}_{x_s(t)} = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)\,x(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)\,x(nT_s)$
  $x_s(t)$ ← sampled function
* CTFS of sampling func. : $\alpha_k = \frac{1}{T_s} \Rightarrow s(t) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$
* CTFT of sampling func. : $s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} e^{jk\omega_s t} \xleftrightarrow{FT} S(j\omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$
* The CTFT of the sampling function is a sampling function! (impulse in t-dom spaced by T_s, in $j\omega$-dom spaced by ω_s)
* The spectrum of the sampled signal is a periodized version of spectrum of the original signal, with period ω_s in $j\omega$-dom
* $\omega_s \geq 2\omega_{max}$ or else will have aliasing (Nyquist Rate)

6. CT Systems

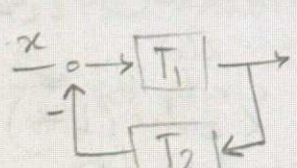
* Linearity : it holds that $T\{\alpha x_1(t) + \beta x_2(t)\} = \alpha T\{x_1(t)\} + \beta T\{x_2(t)\}$
  → can apply superposition
  → input of 0 always results in 0
  → can apply to sums and integrals : $T\{\int \alpha x_1(t)dt\} = \alpha \int T\{x_1(t)\}dt$

* time invariance: if it holds that $y(t-a) = T\{x(t-a)\}$
  → to check, compute $y(t-a)$ and $T\{x(t-a)\}$ and check if they are equal → if yes, then T.I.
  → deals with only inputs and outputs (if you have say $y(t) = x(t) \cos t$, inputting $x(t-a)$ doesn't change cost ... but shifting output does → ∴ not TI)
  → for integral systems, can apply change of variables
* Causality: if it holds that $y(t)$ only depends on past and present values of t (if future, then not causal)
  → if two inputs agree up to some time t_0, then the corresponding outputs must also agree up to that time t_0.
* memoryless: only the current time matters (not future or past)
  → derivative and RC ckt systems not memoryless
* invertibility: you can "undo" the operation of T
  → delay system invertible, but squaring system not
* BIBO stability: bounded inputs produce bounded outputs
  → to show system is not BIBO stable, find example bounded input that leads to unbounded output

* impulse response is system's response to a delta func. input
  → $h(t) = T\{\delta(t)\}$
  → once $h(t)$ known, can compute $y(t)$ for any $x(t)$ input
* output of any LTI system given by:
  $y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$
* a CT LTI system with $h(t)$ is
  1) Causal iff $h(t) = 0$ for $\forall t < 0$
  ☆ 2) Memoryless iff $h(t) = \alpha\, \delta(t)$ for $\alpha \in \mathbb{C}$
  3) BIBO stable if h has finite action: $\int_{-\infty}^{\infty} |h(t)|\, dt < +\infty$
* changing order of series LTI systems still gives same result
* the inverse of an LTI sys. (if ∃ one) is also LTI sys. ($\delta * y = y$)
* the impulse responce of parallel LTI's is $h_1(t) + h_2(t)$
* linear, inhomogeneous, constant-coefficient ODE's can define causal LTI systems.
  → if we restrict ourselves to right-sided inputs $x(t)$
  → " " " " " " solutions $y(t)$
  then
  → syst. is LTI and causal
  RHS ... $x(t)$ $y''(t) + y'(t)$ ...
  → syst. is BIBO stable iff order(input) ≤ order(LHS)
  and poles of transfer function (s-dom) have (-)ve real part

7. CT System Analysis with FT

- FT of impulse response $h(t)$ is $H(j\omega)$ frequency response
- LTI system with comp. exp. input : $x(t) = e^{j\omega_0 t}$
  - $\rightarrow y(t) = H(j\omega)|_{\omega_0} \times e^{j\omega_0 t} = H(j\omega_0) e^{j\omega_0 t}$
  - $\rightarrow$ due to eigenfunction property, LTI system amplitude-scales and phase-shifts any comp. exp. input
  - $\rightarrow$ meaning if input is comp. exp. then output is also comp. exp. with same frequency but scaled by $H(j\omega)$
- LTI response to periodic input : $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega_0 t}$
  - $\rightarrow y(t) = \sum_{k=-\infty}^{\infty} \alpha_k H(jk\omega_0) e^{jk_0 \omega_0 t}$
  - $\rightarrow$ if we know α_k of $x(t)$, we can get $X(j\omega)$ and we know $Y(j\omega) = X(j\omega) H(j\omega)$ and tak IFT of that
  - $\rightarrow$ output $y(t)$ is $T_0 = \frac{2\pi}{\omega_0}$ periodic just like input
- convolution in t-dom means multiplication in $j\omega$-dom
- An LTI system is invertible iff $H(j\omega) \neq 0$ for $\forall \omega \in \mathbb{R}$
- The above lets you find $H(j\omega)$ for feedback systems:



$y = (x - y * h_2) * h_1$

$y = x * h_1 - y * h_2 * h_1$

$y + y * h_2 * h_1 = x * h_1$

$(\delta + h_2 * h_1) y = x * h_1$

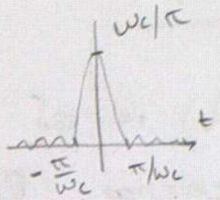
$y = (\delta + h_2 * h_1)^{-1} x * h_1$

iff $(1 + H_2(j\omega) H_1(j\omega)) \neq 0$

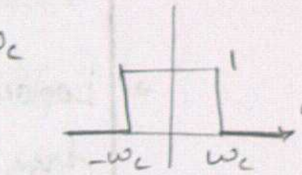
then $H(j\omega) = \dfrac{H_1(j\omega)}{1 + H_1(j\omega) H_2(j\omega)}$

- The ideal LPF

$h(t) = \dfrac{\sin(\omega_c t)}{\pi t}$

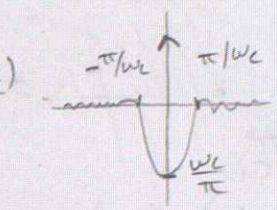


$\overset{FT}{\longleftrightarrow}$ $H(j\omega) = \begin{cases} 1, & -\omega_c < \omega < \omega_c \\ 0, & \text{oth.w.} \end{cases}$



- The ideal HPF

$h(t) = \delta(t) - \dfrac{\sin(\omega_c t)}{\pi t}$



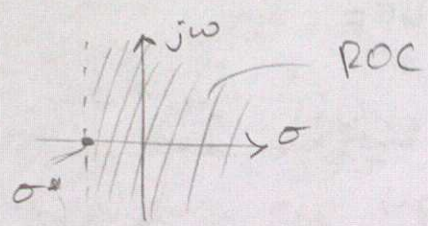
$\overset{FT}{\longleftrightarrow}$ $H(j\omega) = \begin{cases} 1, & |\omega| > \omega_c \\ 0, & \text{oth.w.} \end{cases}$

- Ideal non-causal filters v.s. non-ideal causal filters
  - $\rightarrow$ ideal filter $h(t) \neq 0$ for $\forall\, t < 0$, so not causal and cannot be used for real-time application
  - $\rightarrow$ ideal filters anticipate the input change (since not causal) and perfectly removes/rejects frequencies
  - $\rightarrow$ non-ideal filters react only after input changes (since causal) and can only attenuate (not completely remove for eg.) signal
  - $\rightarrow$ non-ideal : LPF, output @ C and HPF, output @ L $\rightarrow$ for RC ckts
- FT method v.s. LT methods
  - $\rightarrow$ FT : * 2-sided CTFT-able inputs
    - * causality not required
    - * used in signal processing ...
  - $\rightarrow$ LT : * LT-able inputs which are right-sided from time 0
    - * causal systems only
    - * used in control, energy, robotics, aerospace ...

8. CT System Analysis with LT

- Laplace Transform defined if $x(t)$ is of exponential order/class and on a region of convergence (ROC)

$$X(s) = \int_{0^-}^{\infty} x(t) e^{-st} \, dt$$



$s = \sigma + j\omega$

- if LT ROC contains imaginary ($j\omega$) axis, the FT exists and can be found by subbing $s = j\omega$ into LT
- for causal BIBO systems, freq. response $H(j\omega)$ quatifies the steady-state

9. Fundamentals of DT Systems

- properties from CT systems generally hold
- impulse response of summer : $h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n]$ $n \geq 0$